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BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

RUDOLPH ERNEST LANGER
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36

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JANUARY TO DECEMBER, 1947

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THE MATHEMATICS PROGRAM OF THE OFFICE OF NAVAL RESEARCH

The Mathematics Branch of the Office of Naval Research (ONR) has now been in operation for over a year. During that time support of mathematical research in the United States has been extended primarily to universities; and it seems appropriate to give to the members of the American Mathematical Society some account of the philosophy which has determined the mathematical research projects which ONR is sponsoring. The effectiveness of the program in strengthening mathematical research in the United States is largely dependent upon the cooperation of the mathematical community. This cooperation has been generously offered by many mathematicians during the past year; and it is hoped that the present account will inform a wider mathematical public of ONR's activities. It should perhaps be explained immediately that ONR has no statutory power to make grants, or award fellowships; so that all the activity which is here reported takes the form of contracts between the Navy and the university or other research institution.

The Office of Naval Research is committed primarily to the support of fundamental research in the sciences, as contrasted with development, or with applications of known scientific results—the types of activity in which scientists were largely engaged during the war. This is a basic policy decision which recognizes that the United States must be strong scientifically if it is to be strong militarily. It is natural, however, that the most obvious types of mathematical research which would seem to warrant Navy support would be research in applied directions such as those set forth in the recent report of the Society's Special Committee on Applied Mathematics.[1] And it is a fact that over 4/5 of the annual mathematics expenditure is in support of research in "applied mathematics," mathematical statistics, numerical analysis and computing devices. This last category includes a rather large expense for the construction of an electronic digital computer, and for the partial support of the National Applied Mathematics Laboratories which are currently being set up by the National Bureau of Standards. It includes also almost the entire support of the computing group formerly known as Mathematical Tables Project, which has been assimilated into the National Applied Mathematics Laboratories, and which will, in the future, derive its principal support from other sources. Mention is made of these special activities in an attempt to put ONR's financial commitments in perspective. For, although appropriations for theoretical studies in mathematics represent less than 1/5 of the annual total, the number of contracts with theoretical objectives is more than 1/3 of the entire group.

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[1] *Instruction and research in applied mathematics*, prepared by the Special Committee on Applied Mathematics and approved by the Council of the American Mathematical Society, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 639–640.

Early in the course of the operation of the Mathematics Branch, it was decided as a matter of policy that sound support of mathematical research in this country must include support of work in pure mathematics, and a portion of the budget was accordingly allocated for that purpose. Thus one of the earliest projects supported by this office was a project in conformal mapping, some aspects of which were reported to the Society at its 1946 Los Angeles meeting by Professor D. C. Spencer of Stanford University. Other projects which are now on the books of ONR include work on spectral theory for normal operators in function spaces; research on free groups; a systematic study of relationships between analytic and measure theoretical concepts of surface area with a view to applications in the theory of the calculus of variations; work on the analytic number theory of prime number distribution, with special reference to the representation of primes by polynomials; investigation of random chemical, physical and biological processes; and a study of deformations in topology. Certain of these contracts were entered into on the basis of the recommendations of a committee (consisting of Professors J. von Neumann, G. C. Evans, H. M. Morse, M. H. Stone, H. Whitney, O. Zariski) set up by the National Academy of Sciences to act as advisers to the Mathematics Branch on questions connected with projects in pure mathematics, and to suggest the most promising contracts in support of post-doctoral research by young mathematicians. Others represent partial support of research undertaken by relatively senior mathematicians during their sabbatical leaves of absence. There are several contracts which involve research on differential equations, both linear and nonlinear, but with special emphasis on problems connected with nonlinear partial differential equations. It is hardly necessary to point out that in every case the research proposal originates with the principal investigator. In every case, the principal criteria which determine the Navy's willingness to enter into a contract are the research records of the senior workers under the contract, and the judgment of mathematicians who have been outstanding contributors to the field of research in question as to the potential mathematical significance of the proposed investigations. Mathematicians have been exceedingly generous in giving the Mathematics Branch the benefit of their advice in connection with proposals for the support of mathematical research whenever such advice has been sought, and in assisting in the formulation of general policy. An aspect of this general policy which is of special interest to many readers of the Bulletin is ONR's desire, within the general framework of its budget and in accordance with the criteria already mentioned, to provide assistance to universities which are taking steps to strengthen their research activity in either pure or applied mathematics, and particularly in parts of the country which lack strong research centers.

Activity in the applied phases of mathematics which occupied a relatively minor place in the research life of this country in 1940 is now drawing the interest of an increasing number of our ablest scholars. Mathematical statistics is playing an ever more important role in our scientific and industrial life; and many competent people believe that the scope and power of applied mathematics will be affected in an important way by the high speed digital computers which are now under development. Activities in mathematical statistics and in computer theory and development, along with research in the classical fields of applied mathematics which in the past have engaged the interest of some of the greatest figures in mathematical history, are deemed by the Office of Naval Research to be of the greatest importance in the development of the scientific strength of the country. Thus, in addition to supporting research in pure mathematics, the Mathematics Branch is attempting to (1) assist the work of able groups which are active in applied fields, (2) facilitate the formation of strong new research groups in these fields, (3) provide for the training, by leaders in applied mathematics, and on the level of basic mathematical research, of promising young men who are interested in these activities, and (4) make the results of work in these fields available generally to the Navy and to the scientific workers of the country.

It may be remarked parenthetically that ONR encourages publications of research results in the usual mathematical periodicals and in books. In the case of two translations of Russian books which have been prepared under ONR contracts, one, *Interior problems of the theory of schlicht functions*, by G. M. Golusin, has been printed in a limited edition by ONR because no other form of publication seemed feasible; while the other, *Theory of oscillations*, by Andronow and Chajkin is to be published by the Princeton University Press. In general, dissemination through established channels is greatly preferred to private distribution.

A vigorous development of applied mathematics has begun abroad, particularly in Russia, Germany, England, and Holland; and ONR is attempting to assist a similar development in this country. It may be of interest to indicate some of the types of research which are supported under this heading. Under a broad contract with New York University, a principal field of research has been gas-dynamics (the so-called Shock Wave Manual which was prepared during the war has undergone a complete revision and should be published shortly). Another related topic studied systematically under this contract is the mathematical theory of gravity waves in water. Previous research in the theory of underwater explosions has been carried forward. Other topics in mathematical physics have been studied ranging from the theory of two-dimensional free jets to various topics in wave propagation and to perturbation theory of linear operators in a continuous spectrum. At Brown, ONR sponsored research is concerned with a critical survey of recent contributions in the field of elastic-plastic and purely plastic deformations of solids, particularly research carried on during the war and in the post-war years. A comprehensive review of the entire field of plasticity has been made which resulted in the determination of five subfields for immediate investigation—crystal plasticity, the plastic behavior of quasi-isotropic materials, stress-strain relations for quasi-isotropic materials in the plastic range, theories of integration, and a critical survey of the classical theory of plasticity and its applications. Another project in applied mathematics in which a number of striking results have been obtained is a study of the dependence of the electrostatic capacity of a conductor or system of conductors upon its geometrical configuration. ONR is also assisting in the establishment of a review journal, Applied Mechanics Reviews, to succeed the defunct Zentralblatt für Mechanik. This new journal is jointly sponsored by the American Mathematical Society, the American Society of Civil Engineers, the Society for Experimental Stress Analysis, the American Society of Mechanical Engineers, the American Institute of Physics, and the Institute for Aeronautical Sciences; and is to be published by the American Society of Mechanical Engineers. It is expected that the first issue will appear in January 1948.

In mathematical statistics, some of the research fields covered are Small Sample Theory, at Princeton; Factorial Design of Experiments, at Iowa State; Multivariate Analysis, and Statistical Analysis of Groups of Experiments, at the University of North Carolina; and Statistical Theory of Multi-Valued Decisions at Columbia. In computing and computing equipment, some of the activities have already been mentioned. One contract of interest which has not been mentioned supports the development of an analog machine to provide for two-dimensional Fourier Analysis for use in studies of crystal structure.

In the projects which were in force in June 1947, there were 36 senior mathematicians and 46 graduate students in applied mathematics, 16 senior research workers and 26 graduate students in mathematical statistics, 42 senior scientists and 38 graduate students on computing projects, and 36 senior mathematicians and 10 graduate students on pure mathematics projects. It will be seen that one of the most important by-products of this program is the training of promising young mathematicians. Thus the inclusion of able Ph.D. candidates as junior research workers under ONR contracts is encouraged. Since there is no restriction on the publication of research results, it is possible to prepare a Ph.D. thesis as part of the research performed under the contract.

It is believed that the program currently supported by ONR is enabling many productive mathematicians to expand and strengthen their normal research activities, and to assist in the accelerated training of promising young research workers; while able junior mathematicians are, in several cases, being given the opportunity to carry on their investigations for an additional year, before assuming their university duties.

It is hoped that the impact of this program on the mathematical life of the United States will be in the direction of stimulating significant mathematical research, and of enabling the faculties of universities in all parts of the country to encourage the interest in mathematical research of their most promising students. It is expected that this activity will contribute toward the production of a gradually increasing corps of able and experienced mathematical research workers.

In its broad outline, the ONR program reflects the Navy's recognition that intensive research in certain fields of applied mathematics may be counted upon to yield specific results directly applicable to engineering and other scientific problems and therefore of use to industry and research laboratories as well as to Naval establishments; but it reflects also the recognition that a lively activity in mathematical research and a sustained growth in the number of mathematically trained personnel are necessary to enhance the scientific life of this country and to maintain the position of the United States relative to scientific progress abroad.

MINA REES,
Head, Mathematics Branch,
Office of Naval Research

ORDER STATISTICS

S. S. WILKS

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An address delivered before the Summer meeting of the Society in New Haven on September 4, 1947, by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors October 21, 1947.

1. Introduction. Within the past twenty-five years, a large body of statistical inference theory has been developed for samples from populations having normal, binomial, Poisson, multinomial and other specified forms of distribution functions depending on one or more unknown population parameters. These developments fall into two main categories: (i) statistical estimation, and (ii) the testing of statistical hypotheses. The theory of statistical estimation deals with the problem of estimating values of the unknown parameters of distribution functions of specified form from random samples assumed to have been drawn from such populations. The testing of statistical hypotheses deals with the problem of testing, on the basis of a random sample, whether a population parameter has a specified value, or more generally whether one or more specified functional relationships exist among two or more population parameters.

All of this theory has now been placed on a foundation of probability theory through the work of R. A. Fisher, J. Neyman, E. S. Pearson, A. Wald, and others. It has been applied to most of the common distribution functions which occur in statistical practice. Many statistical tables have been prepared to facilitate application of the theory.

There are many problems of statistical inference in which one is unable to assume the functional form of the population distribution. Many of these problems are such that the strongest assumption which can be reasonably made is continuity of the cumulative distribution function of the population. An increasing amount of attention is being devoted to statistical tests which hold for all populations having continuous cumulative distribution functions. Problems of this type in which the distribution function is arbitrary within a broad class are referred to as *non-parametric* problems of statistical inference. An excellent expository account of the theory of non-parametric statistical inference has been given by Scheffé [60].[1] In nonparametric problems it is being found that *order statistics*, that is, the ordered set of values in a random sample from least to greatest, are playing a fundamental rôle. It is to be particularly noted that the term *order* is not being used here in the sense of arrangement of sample values in a sequence as they are actually drawn.

There are both theoretical and practical reasons for this increased attention to nonparametric problems and order statistics. From a theoretical point of view it is obviously desirable to develop methods of statistical inference which are valid with respect to broad classes of population distribution functions. It will be seen that a considerable amount of new statistical inference theory can be established from order statistics assuming nothing stronger than continuity of the cumulative distribution function of the population. Further important large sample results can be obtained by assuming continuity of the derivative of the cumulative distribution function. From a practical point of view it is desirable to make the statistical procedures themselves as simple and as broadly applicable as possible. This is indeed the case with statistical inference theory based on order statistics. Order statistics also permit very simple "inefficient" solutions of some of the more important *parametric* problems of statistical estimation and testing of hypotheses. The solutions are inefficient in the sense that they do not utilize all of the information contained in the sample as it would be utilized by "best possible" (and computationally more complicated) methods. But this inefficiency can be offset in many practical situations where the size of the sample can be increased by a trivial amount of effort and expense.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

It is the purpose of this paper to present some of the more important results in the sampling theory of order statistics and of functions of order statistics and their applications to statistical inference together with some reference to important unsolved problems at appropriate places in the paper. The results will be given without proofs, since these may be found in references cited throughout the paper. Before proceeding to the technical discussion it may be of interest to make a few historical remarks about order statistics.

One of the earliest problems on the sampling theory of order statistics was the *Galton difference problem* studied in 1902 by Karl Pearson [51]. The mathematical problem here, which was solved by Pearson, was to find the mean value of the difference between the rth and $(r+1)$th order statistic in a sample of n values from a population having a continuous probability density function. In 1925 Tippett [72] extended the work of Pearson to find the mean value of the sample *range* (that is, the difference between least and greatest order statistics in a sample). In the same paper Tippett tabulated, for certain sample sizes ranging from 3 to 1000, the cumulative distribution function of the largest order statistic in a sample from a normal population having zero mean and unit variance. In 1928 Fisher and Tippett [14] determined by a method of functional equations and for certain restricted regularity conditions on the population distribution the limiting distribution of the greatest (and also least) value in a sample as the sample size increases indefinitely. R. von Mises [35] made a precise determination of these regularity conditions. Gumbel has made further studies of these limiting distributions and has made various applications to such problems as flood flows [19] and maximum time intervals between successive emissions of gamma rays from a given source [18]. In 1932 A. T. Craig [4] gave general expressions for the exact distribution functions of the median, quartiles, and range of a sample size of n. Daniels [7] has recently made an interesting application of the sampling theory of order statistics to develop the probability theory of the breaking strength of bundles of threads.

One of the simplest and most important functions of order statistics is the sample cumulative distribution function $F_n(x)$, the fraction of values in a sample of n values not exceeding x. In 1933, Kolmogoroff [30] established a fundamental limit theorem in probability theory which enables one to set up from $F_n(x)$ a confidence band (for large n) for an unknown continuous cumulative distribution function $F(x)$ of the population from which the sample is assumed to have been drawn. Smirnoff [66] extended Kolmogoroff's results to a treatment of the probability theory of the difference between two sample cumulative distribution functions $F_{n_1}(x)$ and $F_{n_2}(x)$ for large samples. Wald and Wolfowitz [76] have developed a method for determining exact confidence limits for $F(x)$ from a sample of n values.

In 1936 Thompson [70] showed how confidence limits for the median and other quantities of a population having a continuous cumulative distribution function could be established from order statistics in a sample from such a population. His result was discovered independently by Savur [59] in 1937. In 1941 the author [85] showed how the probability function of the portion of the distribution (continuous) of a population lying between two order statistics could be used to set up *tolerance limits* for the population from which the sample is assumed to have been drawn. These ideas were extended by Wald [75] to the determination of rectangular tolerance regions for populations having distribution functions of several variables. More recently Tukey [73] has extended Wald's ideas to the determination of more general tolerance regions. Tukey's extensions give promise of a variety of applications in statistical inference.

One of the most important properties of the probability distribution for a sample of n values from a population having a continuous cumulative distribution function is that the probabilities associated with the $n!$ different permutations of sample values are equal. Fisher [12, 13] initiated the idea of utilizing this property to develop the *randomization method* for constructing statistical tests and illustrated his ideas in several examples. Friedman [16], Hotelling and Pabst [26], Pitman [55, 56, 57], Olmstead and Tukey [45] and Welch [83, 84] have used the randomization method and its extensions to several samples for developing various statistical tests valid for populations with continuous cumulative distribution functions. Wald and Wolfowitz [77] used the idea to develop a test of the statistical hypothesis that two samples have come from populations having identical continuous cumulative distribution functions. Wolfowitz [88] has proposed an extension of the Neyman-Pearson likelihood ratio method (a standard method for determining test criteria in parametric problems) for systematically determining test criteria in nonparametric problems, making use of the randomization principle.

2. Notation and preliminary definitions.

Throughout this paper we shall consider only populations with continuous cumulative distribution functions. At certain points in the paper, which will be indicated, we shall consider cumulative distribution functions (cdf) with derivatives which are continuous except possibly for a set of points of measure zero; we shall refer to such derivatives as continuous with this understanding.

A one-dimensional continuous cdf $F(x)$ satisfies the following conditions:

(a) $F(x)$ is continuous on the entire one-dimensional x space.
(b) $F(+\infty)=1$.
(c) $F(-\infty)=0$.
(d) $F(x)$ is nondecreasing.

If X is such that the probability that $X\leqq x$ is $F(x)$, or briefly if

$$\Pr(X \leqq x) = F(x),$$

then we say that X is a *random variable* which has the cdf $F(x)$. If $F(x)$ has a continuous derivative $f(x)$, then $f(x)dx$ is called the *probability element* of X, and $f(x)$ the *probability density function* (pdf) of X.

Since we are considering continuous cdf's it should be noted that $\Pr(X\leqq x)=\Pr(X<x)=F(x)$.

Similarly, a k-dimensional continuous cdf $F(x_1, x_2, \cdots, x_k)$ satisfies the following conditions:

(a) $F(x_1, x_2, \cdots, x_k)$ is continuous over the entire k-dimensional (Euclidean) space R_k.
(b) $F(+\infty, +\infty, \cdots, +\infty)=1$.
(c) $F(-\infty, x_2, \cdots, x_k)=F(x_1, -\infty, x_3, \cdots, x_k)=\cdots=F(x_1, x_2, \cdots, x_{k-1}, -\infty)=0$.

(d) $\Delta_{h_1}[\Delta_{h_2}\cdots[\Delta_{h_k}F(x_1, x_2, \cdots, x_k)]\cdots]\geq 0$ where $\Delta_{h_i}G(x_1, x_2, \cdots, x_k)=G(x_1, x_2, \cdots, x_{i-1}, x_i+h_i, x_{i+1}, \cdots, x_k)-G(x_1, x_2, \cdots, x_k)$, and $h_i>0$. If $X_1, X_2, \cdots, X_k$ are such that the probability that all the inequalities $X_1\leq x_1, X_2\leq x_2, \cdots, X_k\leq x_k$ hold is $F(x_1, x_2, \cdots, x_k)$, or more briefly if

$$\Pr(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_k \leq x_k) = F(x_1, x_2, \cdots, x_k),$$

then we say that $X_1, X_2, \cdots, X_k$ are *random variables* having cdf $F(x_1, x_2, \cdots, x_k)$. If $\partial^k F(x_1, x_2, \cdots, x_k)/\partial x_1\, \partial x_2 \cdots \partial x_k = f(x_1, x_2, \cdots, x_k)$, say, is continuous, $f(x_1, x_2, \cdots, x_k)dx_1dx_2\cdots dx_k$ is called the *probability element* of the k random variables, and $f(x_1, x_2, \cdots, x_k)$ is the *probability density function* (pdf) of the random variables.

In general, we shall denote random variables by capital letters and the variables in the cdf and pdf by the corresponding lower case letters.

If X is a random variable having cdf $F(x)$, and if $G(x)$ is a Borel-measurable function, then $G(X)$ is a random variable having a cdf $H(y)$ defined by the Lebesgue-Stieltjes integral

$$H(y) = \Pr(G(x) \leq y) = \int_S dF(x)$$

where S is the set of points for which $G(x)\leq y$. The *mean value* of $G(X)$ is defined by

$$E(G(X)) = \int_{-\infty}^{\infty} G(x)dF(x). \tag{1}$$

Similarly, in the case of k variables having cdf $F(x_1, x_2, \cdots, x_k)$ the cdf $H(y_1, y_2, \cdots, y_r)$ of r functionally independent Borel-measurable functions of the X's, say $\phi_i(X_1, X_2, \cdots, X_r)$, $i=1, 2, \cdots, r$ $(r\leq k)$, is defined by the Lebesgue-Stieltjes integral

$$H(y_1, y_2, \cdots, y_r) = \int_S dF(x_1, x_2, \cdots, x_k) \tag{2}$$

where S is the set of points in R_k for which

$$\phi_i(x_1, x_2, \cdots, x_k) \leq y_i \qquad (i = 1, 2, \cdots, r).$$

The mean value of $\phi_i(X_1, X_2, \cdots, X_k)$ is defined by

$$E(\phi_i) = \int_{R_k} \phi_i(x_1, x_2, \cdots, x_k)dF. \tag{3}$$

If $F(x)$ is a cdf and if $X_1, X_2, \cdots, X_n$ are random variables having cdf

$$F(x_1)\cdot F(x_2)\cdots F(x_n), \tag{4}$$

then $X_1, X_2, \cdots, X_n$ is said to be *a random sample* O_n *of size n from a population having* cdf $F(x)$. The n-dimensional Euclidean space R_n of the x's in the case of random sampling is called *sample space*. All sets of points and all functions considered in this paper are assumed to be Borel-measurable.

It should be noted that R_n is constructed as a product space from n one-dimensional spaces and has a cdf which is the product of the corresponding n one-dimensional (and identical) cdf's.

Similarly, one defines a random sample of size n from a population having a k-dimensional cdf. In this case the sample space would be kn-dimensional.

Suppose $X_1, X_2, \cdots, X_n$ is a sample of size n from a population having continuous cdf $F(x)$. It can be shown that the probability of two or more of the X's being equal is zero. Hence we shall always consider the X's in a sample as all having different values. Let the X's be arranged in increasing order of magnitude from least to greatest and denoted by $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. These ordered values of the X's are called *order statistics*, $X_{(r)}$ being the rth order statistic, $r=1, 2, \cdots, n$.

If I is any interval or set determined from the X's of a sample, the integral $C=\int_I dF(x)$ will be called the *population coverage* of I, or simply the *coverage* of I. For example, $F(X_{(j)})-F(X_{(i)})$, for $i<j$, is the coverage of the interval $(X_{(i)}, X_{(j)})$. (If this paper were written completely abstractly, that is, devoid of statistical interpretation, a better term for C would be *measure of I*.) The coverage of I is the fraction of the population or amount of probability in the population distribution function contained in I. Since I is a function of the sample X's, the coverage of I is a random variable. In the case of a population having a continuous k-dimensional cdf $F(X_1, X_2, \cdots, X_k)$ a similar definition holds for a coverage C_k over a k-dimensional region or set S_k in the variate space determined by the sample.

We shall find it convenient to introduce the following definition at this stage:

DEFINITION. *Suppose a sample of size n is drawn from a k-variate population having a continuous cdf* $F(x_1, x_2, \cdots, x_k)$, *and let* S_k *be a region (or set) in the* $x_1, x_2, \cdots, x_k$ *space having a boundary determined by the sample. If* C_k, *the coverage of* S_k, *has a distribution which does not depend on* $F(x_1, x_2, \cdots, x_k)$ *we shall say that* C_k *is distribution-free.*

If the cdf has a specified functional form but depends on certain parameters $\theta_1, \theta_2, \cdots, \theta_s$ and if S_k can be determined by the sample so that the coverage C_k has a distribution which does not depend on the θ's, we shall say that C_k is *parameter-free.* Such coverages have been determined by the author [85], and by Paulson [46], for the case of populations having a normal distribution. However, we shall not be interested in parameter-free coverages in this paper since interesting parameter-free coverages do not, in general, depend on order statistics for their determination. It is obvious, of course, that if a coverage is distribution-free it is also parameter-free.

3. Sampling distributions of coverages for the case of one dimension. One of the most important, useful, and well known properties of a continuous cdf $F(x)$ may be stated as follows:

If X is a random variable having a continuous cdf $F(x)$ then $F(X)$ is a random variable such that

$$\Pr\,(F(X) \leqq p) = p, \qquad 0 \leqq p \leqq 1.$$

Thus, if $X_1, X_2, \cdots, X_n$ is a sample of size n from a population having a continuous cdf, the probability element of $P_i = F(X_i)$, $i = 1, 2, \cdots, n$, is

$$dp_1 dp_2 \cdots dp_n, \tag{5}$$

the sample space of the P_i being the n-dimensional cube $0 \leqq p_i \leqq 1$ $(i = 1, 2, \cdots, n)$. Hence, it is seen that the random variables P_i are independently and uniformly distributed on the interval $(0, 1)$.

Now consider the order statistics $X_{(i)}$, $i = 1, 2, \cdots, n$, in a sample of size n from a population having a continuous cdf $F(x)$. Represent these order statistics as points on the x-axis, where x is the variate in the cdf $F(x)$. These points cut the x-axis into $n+1$ intervals $I_i = [X_{(i)}, X_{(i-1)}]$, $i = 1, 2, \cdots, n+1$, where $X_{(0)} = -\infty$, and $X_{(n+1)} = +\infty$. Let C_i be the coverage of $I_{(i)}$. These coverages are all random variables whose sum is unity. The probability element of any n of these coverages, for convenience the first n, is

$$n!\,dc_1 dc_2 \cdots dc_n, \tag{6}$$

where $c_i \geqq 0$ and $\sum_{i=1}^{n} c_i \leqq 1$. In other words, the pdf of the C_i is uniform over the n-dimensional simplex in the space of the c_i with corners at the $n+1$ points $(0, 0, \cdots, 0)$, $(1, 0, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 0, 1)$. If a set of $n+1$ coverages $C_1, C_2, \cdots, C_{n+1}$ for which $\sum_{i=1}^{n+1} C_i = 1$ have probability element (6), we shall say that this is a set of *elementary coverages.* The most important property of this distribution as far as statistical inference is concerned is that it *does not depend on* $F(x)$, which is usually unknown in statistical inference problems. In other words, *the coverages for the intervals I_i are distribution-free.*

The probability element given by (6), although very simple, is basic in the sampling theory of coverages and order statistics.

If we let U be the coverage composed of the sum of any r $(r \leqq n)$ of the $n+1$ elementary coverages C_i, we can verify from (6) that U has the probability element

$$\frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} u^{r-1}(1-u)^{n-r}du, \qquad 0 \leqq u \leqq 1. \tag{7}$$

If U is the sum of r elementary coverages and V is the sum of s elementary coverages $(r+s \leqq n)$ so that U and V contain no elementary coverages in common, then the probability element of U and V is

$$\frac{\Gamma(n+1)}{\Gamma(r)\Gamma(s)\Gamma(n-r-s+1)} u^{r-1}v^{s-1}(1-u-v)^{n-r-s}dudv, \tag{8}$$

where $u \geqq 0$, $v \geqq 0$, and $u+v \leqq 1$.

Similarly, one could set up the probability element of more than two disjoint sums of elementary coverages according to any partition scheme.

At this point it may be of interest to examine a few applications of the coverage distributions given by (7) and (8) and their immediate extensions.

4. Examples of direct applications of coverage distributions for the case of one dimension.

(a) *Confidence limits of medians, quartiles and other quantiles.* Suppose ξ_p is the *p-quantile*, that is, the value of x for which $F(\xi_p) = p$. If $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are the order statistics in a sample of size n from a population with continuous cdf $F(x)$, we can construct confidence limits for ξ_p from any two order statistics, $X_{(i)}$ and $X_{(j)}$, $i<j$. For we have

$$\begin{aligned} \Pr(X_{(i)} < \xi_p < X_{(j)}) &= \Pr[F(X_{(i)}) < p < F(X_{(j)})] \\ &= \Pr\{F(X_{(i)}) < p < F(X_{(i)}) + [F(X_{(j)}) - F(X_{(i)})]\} \\ &= \Pr(U < p < U + V), \end{aligned} \tag{9}$$

where U and V are distributed according to (8) with $r = i$ and $s = j - i$. Evaluating this probability, one finds

$$\Pr(X_{(i)} < \xi_p < X_{(j)}) = I_p(i, n-i+1) - I_p(j, n-j+1) \tag{10}$$

where $I_x(p, q)$ is the incomplete beta function defined by

$$(11) \qquad I_x(p, q) = \frac{\Gamma(p + q)}{\Gamma(p)\Gamma(q)} \int_0^x t^{p-1}(1 - t)^{q-1}dt.$$

Thus, the probability can be calculated from the incomplete beta function tables [54] that the *confidence interval* $(X_{(i)}, X_{(j)})$ covers the unknown p-quantile ξ_p. In setting up confidence intervals for practical purposes one would choose a *confidence coefficient* α (say $\alpha = .95$) and then for each value of n select the pair of values of i and j closest together (or one of these pairs in accordance with practical considerations if there are more than one pair), such that the expression on the right of (10) is not less than α.

If $p = 0.5$, $\xi_{0.5}$ is the *median* of the cdf $F(x)$. In this case one would choose $i = k$ and $j = n - k + 1$, $k < (n+1)/2$ (that is, the kth largest and the kth smallest order statistics) and then select a value of k for each n in accordance with the method described above. Expression (10) becomes, in this case,

$$(12) \qquad \Pr\left(X_{(k)} < \xi_{0.5} < X_{(n-k+1)}\right) = 1 - 2I_{0.5}(k, n - k + 1).$$

Similarly, one could deal with the *lower quartile* $\xi_{0.25}$ and *upper quartile* $\xi_{0.75}$ or any other quantile. Expression (12) was established in 1936 by Thompson [70], and discovered independently by Savur [59]. Nair [42] has tabulated the largest value of k for which $\Pr(X_{(k)} < \xi_{0.5} < X_{(n-k+1)}) \geqq 0.95$ and also $\geqq 0.99$ for $n = 6, 7, 8, \cdots, 81$.

(b) *Population tolerance limits.* The following problem, pointed out by Shewhart [63], arose in the statistical aspects of a mass production problem. Suppose a mass production operation is yielding a sequence of product pieces such that measurements on a given dimension are behaving so they can be idealized as values of a random variable from a population having some (unknown) continuous cdf $F(x)$. Can one find a sample size n and two *tolerance limits* $L_1(X_1, X_2, \cdots, X_n)$, $L_2(X_1, X_2, \cdots, X_n)$ from the sample values $X_1, X_2, \cdots, X_n$ so that no matter what continuous cdf $F(x)$ is, the probability is α that the fraction of the population covered by the interval (L_1, L_2) is at least β, that is,

$$(13) \qquad \Pr\left[(F(L_2) - F(L_1)) > \beta\right] = \alpha?$$

That is, what functions L_1 and L_2 will produce an interval (L_1, L_2) such that the population coverage for (L_1, L_2) is *distribution-free?* Values of α and β of practical interest are high (that is, 0.90 to 0.99). It was shown by the author [86] that this problem could be solved by choosing L_1 and L_2 as order statistics. For example, if $L_1 = X_{(k)}$ and $L_2 = X_{(n-k+1)}$ (in practice k would be taken as 1, 2, 3 or some small value compared with n) then it follows immediately from (7) that $F(X_{(n-k+1)})-F(X_{(k)})=U$ has the probability element

$$\frac{\Gamma(n+1)}{\Gamma(2k)\Gamma(n-2k+1)}u^{n-2k}(1-u)^{2k-1}du. \tag{14}$$

In this case the problem reduces to the determination of the smallest value of n for which

$$1-I_\beta(n-2k+1,\,2k)\geq\alpha. \tag{15}$$

Note that this solution is approximate. In general there exists no value of n for which the probability in (13) is exactly equal to α for fixed α and β and specified order statistics, but there is always a smallest n for which (15) holds.

A question which naturally arises here is whether there are functions L_1 and L_2 of the sample $X_1, X_2, \cdots, X_n$ other than order statistics for which the problem of tolerance limits can be solved if nothing is assumed about the population cdf $F(x)$ beyond continuity. This is still an open question. However, if $F(x)$ has a derivative, Robbins [58] has proved that the only (distribution-free) tolerance limits which are symmetric in the sample values $X_1, X_2, \cdots, X_n$ are order statistics.

5. **Distribution of single order statistics.** The problem of determining the sampling theory of an order statistic consists of applying an appropriate transformation to the coverage distribution (7). In this section we shall consider some special cases of interest.

(a) *Exact distribution of* $X_{(r)}$. Suppose $X_1, X_2, \cdots, X_n$ is a sample from a population having a continuous cdf $F(x)$ which has a derivative (pdf) $f(x)$. The probability element $\phi(x_{(r)})dx_{(r)}$ for the rth order statistic $X_{(r)}$ is given by applying the transformation $u=F(x_{(r)})$ to (7), thus yielding

$$\begin{aligned}&\phi(x_{(r)})dx_{(r)}\\ &\quad=\frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)}[F(x_{(r)})]^{r-1}[1-F(x_{(r)})]^{n-r}f(x_{(r)})dx_{(r)}.\end{aligned} \tag{16}$$

It should be noted that as soon as we go from (7) to (16) via the transformation $u=F(x_{(r)})$ we pass from consideration of nonparametric problems to parametric ones. That is, we cannot calculate probabilities of $X_{(r)}$ falling in specified intervals from (16) without knowing $F(x)$, whereas we can calculate similar probabilities pertaining to $F(X_{(r)})$ from (7).

A. T. Craig [4] gave special cases of (16) in 1932, namely, for the median and the two quartiles of the sample. For samples of size $2m+1$, $X_{(m+1)}$ is the *sample median*, and for samples of size $4k+3$, $X_{(k+1)}$ and $X_{(3k+3)}$ are the *lower* and *upper quartiles of the sample.*

(b) *Limiting distribution of largest* (*or smallest*) *value in a sample.* Limiting forms of the distribution (16) for $r=n$ (or 1) as $n\to\infty$ and properties of these limiting forms for various classes of cdf's $F(x)$ have been considered by a number of authors, particularly Dodd [10], Smirnoff [65], Fisher and Tippett [14], Fréchet [15], Gumbel [17], and von Mises [35]. One of the earliest studies of these extreme values was made in 1923 by Dodd [10], who investigated stochastic limits of greatest values in samples of size n as $n\to\infty$. Dodd's results, although of interest in the theory of probability, are not particularly useful in statistical inference, and therefore no attempt will be made to summarize them in detail here. It will, perhaps, be sufficient to state a typical one of Dodd's theorems:

If $f(x)$ *is the pdf of* X, *such that* $0<k_1<x^{1+\alpha}f(x)<k_2<\infty$ *for* $x>0$, *and* $f(x)=0$ *for* $x<0$, *where* $\alpha>0$, *then as the sample size* $n\to\infty$,

$$\lim_{n\to\infty} \Pr\left(n^{1/\alpha-\epsilon} < X_{(n)} < n^{1/\alpha+\epsilon}\right) = 1 \tag{17}$$

for any $\epsilon>0$.

Dodd established five other theorems similar to this one, depending on various assumptions regarding the behavior of $f(x)$, particularly for large values of x.

One of the most important items in Dodd's paper is his remark to the effect that if $\xi_{0.5}^{(n)}$ is the median of the pdf $\phi(x_{(n)})$ of the largest sample value $X_{(n)}$ in a sample from a population having cdf $F(x)$ and pdf $f(x)$, then $\xi_{0.5}^{(n)}$ may be determined by solving the equation $[F(\xi_{0.5}^{(n)})]^n=0.5$. The immediate extension of this is that if $\xi_p^{(n)}$) is the p-quantile of the cdf of $X_{(n)}$ then the probability of all values in the sample being less than $\xi_p^{(n)}$ is equal to p, that is,

$$[F(\xi_p^{(n)})]^n = p, \tag{18}$$

from which $\xi_p^{(n)}$ is given by solving the equation

$$F(\xi_p^{(n)}) = p^{1/n}. \tag{19}$$

In 1925 Tippett [72], using the immediate extension (18) of Dodd's results, tabulated the cdf $\int_{-\infty}^{x}\phi(x_{(n)})dx_{(n)}$ of the largest value in samples of size n from a normal population for which

$$F(x) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{x} e^{-t^2/2}dt. \tag{20}$$

Thus, the basic formula used by Tippett in his tabulation was

$$\int_{-\infty}^{x} \phi(x_{(n)})dx_{(n)} = \left[\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{x} e^{-t^2/2}dt\right]^n. \tag{21}$$

Tippett's tabulation was made for $n=3$, 5, 10, 20, 50, 100, 200, 300, $\cdots$, 1000 and for x ranging from -2.5 to 6.5 by intervals of 0.1.

In 1928 Fisher and Tippett [14] obtained limiting distributions, as $n \to \infty$, of the greatest (and least) values in a sample from a population having a cdf $F(x)$ and satisfying certain regularity conditions for large values of x. They used an ingenious method of functional equations, based on the fact that the largest value in a sample of size mn has the same distribution as the largest value in a sample of size n from a population of largest values of samples of size m. They arrived at three types of limiting distributions. Almost simultaneously, Fréchet [15] arrived at one of the types. R. von Mises [35] has set forth more explicitly the regularity conditions on which the Fisher-Tippett results depend. Fisher and Tippett obtained three important classes of limiting distributions of $X_{(n)}$, the largest value in a sample of size n from a population having cdf $F(x)$, which may be stated as follows:

CASE I. *Suppose $F(x)$ is a cdf which is less than* 1 *for every finite value of x, and which is continuous and has two derivatives for all x greater than some finite value x_0, and that*

$$\lim_{x \to \infty} \frac{d}{dx}\left(\frac{1 - F(x)}{F'(x)}\right) = 0.$$

Then

$$\lim_{n \to \infty} \Pr\left[(X_{(n)} - l_n)nF'(l_n) \leq u\right] = e^{-e^{-u}} \tag{22}$$

for any u on the interval $(-\infty, +\infty)$, where l_n satisfies the condition

$$F(l_n) = (n-1)/n.$$

CASE II. *Suppose $F(x)$ is a cdf which is less than* 1 *for every finite value of x, and has a derivative $F'(x)$ for x greater than some finite value x_0 such that*

$$\lim_{x \to \infty} \frac{xF'(x)}{1 - F(x)} = k$$

where $k>0$. Then

$$\lim_{n\to\infty} \Pr\left(\frac{X_{(n)}}{l_n} \leqq u\right) = e^{-u^{-k}} \tag{23}$$

for any u on the interval $(0, \infty)$, *where l_n is defined in Case* I.

CASE III. *Suppose $F(x)$ is a cdf such that $F(a) = 1$, and that the first $k-1$ derivatives of $F(x)$ exist and are 0 at $x = a$. Suppose $F^{(k)}(a) = (-1)^{k+1}c$, where $c > 0$, and that $F^{(k+1)}(x)$ is bounded on the interval $(a-\epsilon, a)$ for some $\epsilon > 0$. Then*

$$\lim_{n\to\infty} \Pr\left((X_{(n)} - a)\left(\frac{nc}{k!}\right)^{1/k} \leqq u\right) = e^{-(-u)^k} \tag{24}$$

for any u on the interval $(-\infty, 0)$.

In 1927 Fréchet [15] obtained the limiting distribution in Case II in a slightly different form. The issue of the journal in which his result was published was not printed until 1928, thus indicating that Fréchet's results and those by Fisher and Tippett actually appeared almost simultaneously.

To take an important example of Case I, suppose $F(x)$ is the normal cdf given by (20). After some calculation, neglecting terms of order l_n^{-2} and $(\log n)^{-1}$, the expression $(X_{(n)} - l_n)nF'(l_n)$ reduces to

$$\left[X_{(n)}(2\log n)^{1/2} - \log\left(\frac{n^2}{(4\pi)^{1/2}\log n}\right)\right]. \tag{25}$$

Thus, for large n, the probability element of this expression is approximately

$$e^{-u-e^{-u}}du. \tag{26}$$

If one were interested in the largest value $X_{(n)}$ in a sample from a normal population with mean m and variance σ^2, he would merely replace $X_{(n)}$ in (25) by $(X_{(n)} - m)/\sigma$.

A rather direct approach to the problem of obtaining the limiting distribution of the largest or smallest values in a sample can be made from (16). Consider the case of the largest value. We have $r = n$ in (16). The probability element of Y, where $Y = n[1 - F(X_{(n)})]$ and where $X_{(n)}$ has probability element (16) is

$$(1 - y/n)^{n-1}dy \qquad (0 \leqq y \leqq n) \tag{27}$$

and hence the limiting distribution as Y as $n \to \infty$ has probability element

$$e^{-y}dy. \tag{28}$$

The problem of finding the approximate distribution of $X_{(n)}$ for large

n is a matter of making the transformation $Y=n[1-F(X_{(n)})]$ on (28).

(c) *Limiting distribution of* $X_{(r)}$. Results similar to these given in the preceding paragraphs can be written down for the limiting distribution for the rth largest (or rth smallest) order statistic for fixed r. For example, in the case of the rth largest statistic $X_{(n-r+1)}$, the limiting distribution of $Y=n[1-F(X_{(n-r+1)})]$ has probability element

$$\frac{y^{r-1}e^{-y}}{\Gamma(r)}dy \tag{29}$$

which plays the fundamental role in deriving the limiting distribution of $X_{(n-r+1)}$ for any particular cdf $F(x)$ and pdf $f(x)$ satisfying suitable regularity conditions.

If, however, one allows r to increase with n so that $r/n=p$, and if $f(x)$ is continuous and not equal to 0 at the p-quantile ξ_p, then the order statistic $X_{(r)}$ is asymptotically normally distributed with mean ξ_p and variance $p(1-p)/n[f(\xi_p)]^2$, that is,

$$\lim_{n\to\infty} \Pr\left[\frac{(X_{(r)}-\xi_p)f(\xi_p)n^{1/2}}{(p(1-p))^{1/2}} \le u\right] = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{u} e^{-t^2/2}dt. \tag{30}$$

Of the authors who have considered the limiting distributions of order statistics, Smirnoff [65] made a rather systematic and comprehensive study of them in 1935, although there is no evidence that he was aware of the results on largest values which had been reached earlier by Fisher and Tippett [14] and Fréchet [15].

6. Joint distributions of several one-dimensional order statistics and applications.

There are problems in parametric statistical inference which involve the sampling theory of two or more order statistics. Determination of the sampling distributions of the sample *range* $(X_{(n)}-X_{(1)})$ and *midrange* $(X_{(1)}+X_{(n)})/2$ are examples of such problems. The exact sampling theory of two or more order statistics in samples from a population having a cdf $F(x)$ with a continuous derivative $f(x)$ is straightforward and presents no particular difficulties. (See [87].) For example, consider two order statistics $X_{(r)}$ and $X_{(r')}$ $(r<r')$. The probability element $\phi(x_{(r)}, x_{(r')})dx_{(r)}dx_{(r')}$ is obtained by applying the transformation

$$u = F(x_{(r)}), \qquad v = F(x_{(r')}) - F(x_{(r)}) \tag{31}$$

to (8) and setting $s=r'-r$. Setting $r'=r+1$, we have the joint distribution of $X_{(r+1)}$ and $X_{(r)}$, from which Karl Pearson [51] obtained the solution to the Galton difference problem, namely

$$E(X_{(r+1)} - X_{(r)}) = \frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)} \int_{-\infty}^{\infty} F(x)^{n-r}[1-F(x)]^r dx. \tag{32}$$

(a) *Distribution of sample range.* For $r=1$, and $r'=n$, we have the probability element for the two extremes in a sample from which it follows that the probability element $g(r)dr$ of the exact distribution of the *sample range* $R=X_{(n)}-X_{(1)}$ is

$$g(r)dr = n(n-1)\left\{\int_{-\infty}^{\infty}[F(r+x_{(1)})-F(x_{(1)})]^{n-2}f(r+x_{(1)})f(x_{(1)})dx_{(1)}\right\}dr. \tag{33}$$

A similar expression holds for the probability element of the mid-range $M=(X_{(n)}+X_{(1)})/2$.

The problem of the distribution theory of the sample range was originally discussed by von Bortkiewicz [1] in 1921. In 1925 Tippett [72] showed that the mean value of R is given by

$$E(R) = \int_{-\infty}^{\infty}\{1-[F(x)]^n-[1-F(x)]^n\}dx. \tag{34}$$

He tabulated $E(R)$ and also the variance of R for samples from size 2 to 1,000 from a normal population in which $F(x)$ is given by (20). In 1933 McKay and E. S. Pearson [33] determined explicit expressions for the distributions of ranges in samples of size 2 and 3 from a normal population. More recently, Hartley [23] has tabulated the cdf of the range R for samples of size 2, 3, 4, $\cdots$, 20 from a normal population with zero mean and unit variance.

Setting up new random variables Y and Z defined by

$$Y = nF(X_{(1)}), \qquad Z = n[1-F(X_{(n)})] \tag{35}$$

one finds Y and Z independently distributed in the limit as $n\to\infty$, which means that $X_{(1)}$ *and* $X_{(n)}$ *are asymptotically independently distributed in large samples*, and hence the problem of determining the limiting distribution of the range R as $n\to\infty$ is one of dealing with the difference between two asymptotically independently distributed random variables. This limiting distribution has been discussed by Gumbel [20], who has also dealt with what he calls mth *ranges* $(X_{(m)}-X_{(n-m+1)})$ and mth *midranges* $(X_{(m)}+X_{(n-m+1)})/2$. In particular, he shows that for fixed m, the order statistics $X_{(m)}$ and $X_{(n-m+1)}$ are independently distributed in the limit, as $n\to\infty$. More recently Gumbel [22] has studied the *reduced range* $R^* = g_n[R-2l_n]$ where l_n is given by $F(l_n)=(n-1)/n$ and g_n is the ratio

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$F'(l_n)/[1-F(l_n)]$. He has made some tabulations for the case of samples of size n from a normal population. Carlton [3] has given the asymptotic distribution of the range of a sample from a population having probability element dx/θ on the interval $(0, \theta)$.

(b) *Limiting distribution of two or more order statistics in large samples.* Smirnoff [65] has shown that if n, n_1, and n_2 increase indefinitely so that $n_1/n = p_1$, and $n_2/n = p_2$ $(0 < p_1, p_2 < 1)$, then the order statistics $X_{(n_1)}$ and $X_{(n_2)}$ in a sample of size n from a population having a pdf $f(x)$ (which is greater than 0 and continuous at the quantiles ξ_{p_1} and ξ_{p_2}) are asymptotically jointly normally distributed with means ξ_{p_i} $(i = 1, 2)$, variances $p_i(1-p_i)/n[f(\xi_{p_i})]^2$ $(i = 1, 2)$, and covariance $p_1(1-p_2)/nf(\xi_{p_1})f(\xi_{p_2})$. It should be remarked that these variances and covariance were derived as early as 1920 by Karl Pearson [53]. However, he did not establish the fact that the limiting distribution of $X_{(n_1)}$ and $X_{(n_2)}$, as $n \to \infty$, is normal. Mosteller [39] has extended Smirnoff's result to a set of k order statistics, obtaining, of course, a normal k-variate limiting distribution.

(c) *Estimation of population parameters by order statistics.* Mosteller [39] has investigated the efficiency of various linear combinations of several order statistics in large samples for estimating the mean and variance of a normal distribution function. To pick one simple case, he shows that by choosing $p_1 = .2702$ and $p_2 = .7298$, the average of the two order statistics $x_{(np_1)}$ and $x_{(np_2)}$ have an efficiency 0.81 in estimating the population mean for large n. We are using efficiency here in the sense of R. A. Fisher, briefly and roughly stated as follows: if the "most efficient" method of estimating the population mean had been used (the arithmetic mean of all sample values in this case) a sample of size $0.81n$ would produce an estimate with the same variance, to terms of order $1/n$, as that for the estimate based on the two given order statistics. To state the matter another way, this means that the use of these two order statistics for estimating the population mean "utilized about 81% of the information contained in the sample." Among other results, Mosteller has shown that by using the average of 10 properly spaced order statistics one can obtain an estimate of the mean with an efficiency of more than .97. He considered the cases of $2, 3, 4, \cdots, 10$ symmetrically spaced order statistics in the estimation of the mean of a normal population. He also considered order statistics in the estimation of the variance σ^2 of a normal population. The use of the range for this purpose in small samples was investigated by Davies and Pearson [8]. They found that for samples of size 10 or less, the range (multiplied by a suitable factor) is almost as accurate for estimating σ as the customary method based on the sum of squares of deviations of sample values from their mean.

The use of order statistics promises to furnish a simple and effective practical method for estimating parameters of the normal and other populations having continuous pdf's for both large and small samples—especially in situations where large samples are easily available. The development of this field will depend on a vast amount of computation, but the reward will be extremely simple statistical methods useful for both large and small samples under certain practical conditions.

(d) *Application of order statistics to breaking strength of thread bundles.* An interesting application of the joint distribution theory of a set of one-dimensional order statistics has been made by Daniels [7] to the statistical theory of breaking strength of bundles of threads, which illustrates a type of application of order statistics which may be important in the critical study of other physical phenomena of a similar nature.

Suppose the breaking strength of a population of thread lengths has a continuous cdf $F(x)$. Let the ordered breaking strengths (from least to greatest) of a sample be $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$. Now suppose these threads are made into a bundle of n parallel threads and let a load S be applied and consider the probability that S will break the bundle. It is evident that the bundle will break if the following conditions are satisfied:

$$\begin{aligned} &0 < X_{(1)} < S/n, \\ &X_{(i-1)} < X_{(i)} < S/(n+1-i) \qquad (i = 2, 3, \cdots, n). \end{aligned} \tag{36}$$

If we denote by B the region in sample space for which these inequalities hold then the probability that S will break the bundle is

$$p_n(S) = n! \int_B dF(x_{(1)})dF(x_{(2)}) \cdots dF(x_{(n)}). \tag{37}$$

But, setting $p_i = F(x_{(i)})$ $(i = 1, 2, \cdots, n)$, this is equivalent to the following integration

$$p_n(S) = n! \int_0^{b_n} \int_{p_1}^{b_{n-1}} \int_{p_2}^{b_{n-2}} \cdots \int_{p_{n-1}}^{b_1} dp_n dp_{n-1} \cdots dp_1 \tag{38}$$

where $b_i = F(S/i)$, $i = 1, 2, \cdots, n$, which yields

$$p_n(S) = \sum_r \frac{n!(b_1 - b_2)^{r_1}(b_2 - b_3)^{r_2} \cdots (b_{n-1} - b_n)^{r_{n-1}} b_n^{r_n}}{r_1! r_2! \cdots r_n!} \tag{39}$$

where $\sum_r$ denotes summation over all values of $r_1, r_2, \cdots, r_n$ (positive integers or zero) for which

$$\sum_{j=1}^{i} r_j \leqq i \qquad (i = 1, 2, \cdots, n-1)$$

and $\sum_1^n r_i = n$.

For a specified sample of size n, let S^* be the largest value of S for which the inequalities (36) are not all true. Then S^* is a random variable representing the *breaking load* of the bundle of n threads for this sample. Daniels has shown that if, in addition to the continuity of $F(x)$, we assume that $\operatorname{Lim}_{x\to\infty} x[1-F(x)]=0$ and $x[1-F(x)]$ has a maximum for $x=x^*$, then

$$(40) \qquad \operatorname*{Lim}_{n\to\infty} \Pr\left\{\frac{S^* - nx^*[1 - F(x^*)]}{x^*(nF(x^*)[1 - F(x^*)])^{1/2}} \leqq u\right\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{u} e^{-t^2/2}dt.$$

In other words, S^* is approximately normally distributed in large samples with mean $nx^*[1-F(x^*)]$ and variance $nx^{*2}F(x^*)[1-F(x^*)]$.

7. Confidence bands for the cdf $F(x)$. A fundamental problem in nonparametric statistical inference is that of finding confidence bands for a continuous cdf $F(x)$ from a sample of size n. The natural approach to this problem is to make use of the sample cdf $F_n(x)$ in determining these confidence bands. $F_n(x)$ is a function of order statistics defined as follows:

$$(41) \qquad \begin{aligned} F_n(x) &= 0 & x &< X_{(1)}, \\ F_n(x) &= i/n, & X_{(i)} \leqq x &< X_{(i+1)}, \\ F_n(x) &= 1, & x &\geqq x_{(n)}. \end{aligned}$$

The sample cdf $F_n(x)$ has been referred to by Cramér [5] as the *statistical image* of the population cdf $F(x)$.

Consider the transformation $T_{(1)}=F(X_{(1)})$, $T_{(2)}=F(X_{(2)})$, $\cdots$, $T_{(n)}=F(X_{(n)})$. It is clear that $T_{(1)}, \cdots, T_{(n)}$ are order statistics in a sample of size n from a population having probability element dt on the interval (0, 1). The cdf of T is t. Denoting the sample cdf by $G_n(t)$, it is seen that $\operatorname{Sup}_{(t)} [t-G_n(t)] = \operatorname{Sup}_{(x)} [F(x)-F_n(x)]$.

By working with the simplified random variable T and using the fact just stated, Kolmogoroff [30] showed in 1933 that

$$(42) \qquad \operatorname*{Lim}_{n\to\infty} \Pr\left\{\sup_{(x)} [F(x) - F_n(x)]n^{1/2} \leqq \lambda\right\} = \Phi(\lambda)$$

where

$$\Phi(\lambda) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2\lambda^2}. \tag{43}$$

Thus, for a given confidence coefficient α, approximate confidence bands for $F(x)$ are furnished by the two functions $F_n(x) \pm \lambda_\alpha/n^{1/2}$, where λ_α is determined so that $\Phi(\lambda) = \alpha$. Thus, we can say that for large n, the probability is approximately α that the curve $y = F(x)$ will be contained in the band bounded by the four graphs: $y = F_n(x) \pm \lambda_\alpha/n^{1/2}$, $y = 0$ and $y = 1$. Kolmogoroff tabulated values of λ_α for $\alpha = .95$, .98, .99, .995, and .999.

A result closely related to Kolmogoroff's and obtained by Smirnoff [66] relates to the difference between two sample cdf's. Suppose $F_m(x)$ and $F_n(x)$ are sample cdf's for samples of sizes m and n respectively from a population having a continuous cdf $F(x)$, and the ratio m/n remains between two fixed positive numbers a_1 and a_2. Then Smirnoff showed that

$$\lim_{n\to\infty} \Pr\left\{\sup_{(x)} \frac{[F_m(x) - F_n(x)]}{(1/m + 1/n)^{1/2}} \leq \lambda\right\} = \Phi(\lambda). \tag{44}$$

Wald and Wolfowitz [76] have considered the problem of setting up exact confidence bands for $F(x)$. They have shown, for confidence coefficient α and constant d_α, how to construct confidence bands for $F(x)$ of the form $F_n(x) \pm d_\alpha$. Thus, the probability is α that the region or band bounded by the graphs of $y = F_n(x) \pm d_\alpha$, $y = 0$, and $y = 1$ contains the curve $y = F(x)$. There are many ways of determining the d_α, but their evaluation is rather laborious. It would be very desirable to have an investigation of sets of values of the d_α which would produce "best" confidence bands, perhaps of smallest average vertical width weighted in some sense, and to have suitable tabulations to facilitate their application.

8. **Sampling distributions of coverages for the case of two or more dimensions.** In §3 we discussed the sampling distribution of the set of elementary coverages produced by the order statistics in a sample of size n from a population having a one-dimensional continuous cdf $F(x)$. We recall that these elementary coverages have the extremely simple probability element (6), which does not depend on the population cdf $F(x)$, from which one can readily determine the distribution of two or more sums of elementary coverages. We shall now deal with the problem of coverages for the case of two or more dimensions.

For the sake of simplicity, we shall deal with the case of two dimensions. Extension of the results to more than two dimensions is straightforward. Thus, suppose we have a sample O_n of size n, say $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$, from a population having a continuous cdf $F(x, y)$. This sample can be represented as n points in the xy-plane. The basic question is this: *What kinds of two-dimensional intervals or regions S determined by the sample O_n are there, if any, which have distribution-free coverages?*

We shall discuss several methods of constructing such regions.

(a) *For regions determined by one-dimensional order statistics.* It is obvious from the one-dimensional case that the *slices* produced by the n lines parallel to the y axis passing through the n sample points are regions having distribution-free coverages. In fact, the coverages (two-dimensional) for these regions have the probability element given by (6). This is evident from the fact that this particular slicing operation is equivalent to considering only the X's of the sample in precisely the same manner as we did in arriving at (6). Entirely similar results hold, of course, if the slicing is carried out with the n lines through the sample points parallel to the x-axis. In this case we would be considering only the Y's of the sample.

The preceding considerations suggest a more general method of slicing the xy-plane into $n+1$ regions or sets. Suppose $h(x, y)$ is a function such that if X and Y are random variables with continuous cdf $F(x, y)$, the cdf of $h(X, Y)$ $(=Z$, say) is continuous. For the n sample points there will be n values of Z, namely $Z_i = h(X_i, Y_i)$. Suppose $Z_{(i)}$ $(i=1, 2, \cdots, n)$ are the ordered values of the Z_i. Now there will be $n+1$ regions or sets of points in the xy-plane, $S_1, S_2, \cdots, S_{n+1}$, where S_1 is the set of points for which $h(x, y) < Z_{(1)}$, S_2 the set for which $Z_{(1)} < h(x, y) < Z_{(2)}$ and so on. Let $C_1, C_2, \cdots, C_{n+1}$ be the coverages associated with $S_1, S_2, \cdots, S_{n+1}$, respectively. Then these coverages are distribution-free and any n of them have probability element (6). This follows at once from the fact that under the conditions stated, $Z = h(X, Y)$ is a random variable having a continuous cdf and that we are simply considering the elementary coverages determined by the order statistics $Z_{(1)}, Z_{(2)}, \cdots, Z_{(n)}$.

But there are more general regions than these which are determined by the sample and for which the coverages are distribution-free as we shall see in the following paragraphs.

(b) *For rectangular regions in case of independent random variables.* In case X and Y are independent in the probability sense, that is, when $F(x, y) = F_1(x) \cdot F_2(y)$ ($F_1(x)$ and $F_2(y)$ being continuous cdf's of X and Y respectively), one can construct rectangular regions having distribution-free coverages by setting up a product distribution in the product space of X and Y as follows: Suppose the sample O_n is ordered with respect to the X's. The order statistics $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ determine a set of one-dimensional elementary coverages having probability element (6). The sum U of any r ($r \leqq n$) of these elementary coverages has probability element (7). Similarly, if the sample is ordered with respect to the Y's, a set of one-dimensional elementary coverages is determined such that the sum V of any s ($s \leqq n$) of them has probability element (7) with u replaced by v and r by s. Since U and V are functions of independently distributed random variables, they are themselves independently distributed. If I_x is the set of x intervals belonging to U and I_y is the set of y intervals belonging to V, then $I_x \cdot I_y$ is a two-dimensional product set of rectangles, say S. The sum of the two-dimensional coverages over S is simply the product UV. Hence the probability element of the two-dimensional coverage $UV = W$, say, can be calculated from the joint distribution of U and V,

$$(45) \qquad \frac{\Gamma^2(n+1)}{\Gamma(r)\Gamma(s)\Gamma(n-r+1)\Gamma(n-r+1)} u^{r-1}v^{s-1}(1-u)^{n-r}(1-u)^{n-s}du\,dv$$

by integrating u and v over the region for which $w < uv < w + dw$ and neglecting terms of order $(dw)^2$ and higher.

To take a case in which the two-dimensional region S is very simple, suppose U is taken as the sum of the elementary coverages for the intervals $(0, X_{(1)}), (X_{(1)}, X_{(2)}), \cdots, (X_{(r-1)}, X_{(r)})$, then $U = F_1(X_{(r)})$. Similarly, let $V = F_2(Y_{(s)})$. Then the two-dimensional set S is the "corner" for which $x < X_{(r)}$, $y < Y_{(s)}$, and the two-dimensional coverage over S is $F_1(X_{(r)}) \cdot F_2(Y_{(s)})$. This two-dimensional coverage is the product of two independently distributed random variables U and V which have distributions independent of $F_1(x)$ and $F_2(y)$. Consequently, the distribution of $F_1(X_{(r)}) \cdot F_2(Y_{(s)})$ is independent of $F_1(x)F_2(y)$.

This product method of constructing two-dimensional regions having distribution-free coverages can readily be extended to any number of dimensions. It was devised by the author [86] in connection with the problem of tolerance limits for two independent variables. To take a simple case, the rectangular region S for which $X_{(1)} < x < X_{(n)}$ and $Y_{(1)} < y < Y_{(n)}$ is such that coverage W over it has probability element

$$(46) \qquad n^2(n-1)^2w^{n-2}[2(w-1) - (w+1)\log w]dw.$$

By integrating this expression from β to 1 and setting the result equal to α one has an equation involving the three variables α, β

and n. By fixing two of them one may solve for the third. For instance if $\alpha=0.95$ and $n=500$ one finds $\beta=0.991$. Thus, we may say that the probability is 0.95 that, in a sample of size 500, at least 99.1% of the population will be included in the rectangle determined by the range of the X's and the range of the Y's in the sample.

By using the product method one can find the joint distribution of one or more X order statistics and one or more Y order statistics in a sample from a two-dimensional population in which the two random variables are independent, and similarly for higher dimensions. But in the case of dependence this problem is extremely complicated. Mood [37] has given some results on the joint distribution of the median of X's and the median of Y's in a sample from a population in which x and y are not independent. He showed that the asymptotic joint distribution of the medians for the multivariate case is normal and determined the matrix of the variances and covariances for the two and three dimensional cases.

(c) *Wald's results for rectangular regions in case the random variables are not independent.* The product method of constructing regions described above is severely restricted by the requirement of independence, that is, that $F(x, y)=F_1(x)\cdot F_2(y)$. The problem of constructing rectangular regions having distribution-free coverages for the case of samples from populations having *any* continuous cdf in two or more dimensions was solved by Wald [75] in his study of the problem of tolerance limits, when the random variables for the population are not assumed to be independent.

Let us consider the two-dimensional case. We have n sample pairs represented as n points in the xy-plane. Consider the X order statistics $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$. The lines $x=X_{(r_1)}$ and $x=X_{(s_1)}$ $(r_1<s_1)$ cut the xy-plane into three slices. There are s_1-r_1-1 sample points lying within the middle slice. Let the Y order statistics for the s_1-r_1-1 points of the middle slice be denoted by $Y'_{(r_1+1)}$, $Y'_{(r_1+2)}, \cdots, Y'_{(s_1-1)}$. Consider the region S consisting of the portion of the middle slice which lies between the lines $y=Y'_{(r_2)}$ and $y=Y'_{(s_2)}$ $(r_2<s_2)$. S is therefore a rectangle defined by $X_{(r_1)}<x<X_{(s_1)}$, and $Y'_{(r_2)}<y<Y'_{(s_2)}$. Within S there are s_2-r_2-1 sample points. Wald's basic result is as follows:

The region S has a distribution-free coverage W which has probability element

$$\frac{\Gamma(n+1)}{\Gamma(s_2-r_2)\Gamma(n-s_2+r_2+1)}\, w^{s_2-r_2-1}(1-w)^{n-s_2+r_2}dw. \tag{47}$$

The proof of this statement depends on an ingenious application of conditional probability theory.

Wald extended this result so that the region S could consist of several rectangles rather than a single one. More specifically, suppose k $(<n)$ vertical lines $x = X_{(m_i)}$ $(i = 1, 2, \cdots, k)$ are drawn. The xy-plane will be cut into $k+1$ slices. Within the left-most slice there will be $m_1 - 1$ sample points. Within the next slice there will be $m_2 - m_1 - 1$ sample points, and so on. Now suppose we draw a horizontal line through each of the $m_1 - 1$ sample points within the left-most slice, each horizontal line being drawn completely across this slice but not extending into the other slices. We thus cut the first slice into m_1 rectangular blocks. (Note that two of the blocks in this slice have two infinite boundaries each, while $m_1 - 2$ of the blocks have one infinite boundary each.) We do a similar operation on each of the remaining slices. The total number of rectangular blocks into which the xy-plane has been cut is $n+1$. Let R_{ij} be the jth block in the ith slice, and let S^* be a set consisting of s of these rectangular blocks. Let W^* be the sum of the coverages for these s blocks, that is, $W^* = \int_{S^*} dF(x, y)$.

Wald's extended result is this:

W^* is a distribution-free coverage with probability element

$$\frac{\Gamma(n+1)}{\Gamma(s)\Gamma(n-s+1)} (w^*)^{s-1}(1 - w^*)^{n-s} dw^*. \tag{48}$$

Actually, it can be shown that any n of the coverages for the $n+1$ blocks R_{ij} have probability element (7). The extension of Wald's two-dimensional results to the k-dimensional case is straightforward, and, in fact, has been carried out by Wald himself.

(d) *Tukey's generalization of Wald's results.* Tukey [73] has generalized Wald's ideas to a stage which shows the full possibilities of the use of order statistics in cutting up the xy-plane into regions having distribution-free coverages. Although Tukey has generalized Wald's results for any finite number of dimensions we shall again, for the sake of simplicity, confine ourselves to the two-dimensional case.

Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ be a sample of size n from a population having cdf $F(x, y)$, and consider the n points in the xy-plane which represent the sample. Suppose, for each value of i $(i = 1, 2, \cdots, n)$, that $h_i(x, y)$ is a function such that $h_i(X, Y)$ is a random variable having a continuous cdf. Two or more of these functions may be identical. Thus, $h_i(x, y)$ may be used to order the sample points so that the values $h_i(X_j, Y_j) = Z_{ij}$ $(j = 1, 2, \cdots, n)$ will be represented as n points on some interval (a_i, b_i). The Z_i order statistics will be $Z_{i(1)}, Z_{i(2)}, \cdots, Z_{i(n)}$. The "curve" $h_1(x, y) = Z_{1(n)}$ cuts the xy-plane into two regions or sets (we exclude boundaries;

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the probability contained in them is zero since the cdf's of the $h_i(X, Y)$ are continuous):

$$S_1: \quad \text{for which } h_1(x, y) > Z_{1(n)},$$

and

$$S_1': \quad \text{for which } h_1(x, y) < Z_{1(n)}.$$

The "curve" $h_2(x, y) = Z_{2(n-1)}$ cuts S_1' into two sets

$$S_2: \quad \text{for which } h_2(x, y) > Z_{2(n-1)},$$

and

$$S_2': \quad \text{for which } h_2(x, y) < Z_{2(n-1)}.$$

Similarly, we cut S_2' into two sets, S_3 and S_3', by the "curve" $h_3(x, y) = Z_{3(n-2)}$, and so on. The xy-plane therefore will be cut up into $n+1$ disjoint sets, $S_1, S_2, \cdots, S_{n+1}$ which are completely defined as follows:

$$\begin{array}{ll}
S_1: & \{h_1(x, y) > Z_{1(n)}\} \\
S_2: & \{h_1(x, y) < Z_{1(n)},\ h_2(x, y) > Z_{2(n-1)}\} \\
S_3: & \{h_1(x, y) < Z_{1(n)},\ h_2(x, y) < Z_{2(n-1)},\ h_3(x, y) > Z_{3(n-2)}\} \\
\cdot & \\
\cdot & \\
S_n: & \{h_1(x, y) < Z_{1(n)}, \cdots, h_{n-1}(x, y) < Z_{n-1(2)},\ h_n(x, y) > Z_{n(1)}\} \\
S_{n+1}: & \{h_i(x, y) < Z_{i(n+1-i)},\ i = 1, 2, \cdots, n\}.
\end{array} \tag{49}$$

If $C_1, C_2, \cdots, C_{n+1}$ are the coverages (two-dimensional) for $S_1, S_2, \cdots, S_{n+1}$, respectively, we have $\sum_{i=1}^{n+1} C_i = 1$. It is to be noted that there are $n!$ ways in which the xy-plane can be sliced up into $n+1$ sets of type $S_1, S_2 \cdots, S_{n+1}$, depending on the order in which the functions $h_i(x, y)$ are selected. Tukey's basic *slicing principle* is as follows:

For any given order in which the sets $S_1, S_2, \cdots, S_{n+1}$ *are determined, the coverages* $C_1, C_2, \cdots, C_{n+1}$ *associated with these sets are distribution-free and have probability element* (6).

This principle is more general than it may first appear, for we could have cut the xy-plane into two sets by using a "cutting curve" determined by a general order statistic $Z_{1(t_1)}$, that is, $h_1(x, y) = Z_{1(t_1)}$, at the first step. The xy-plane is thus cut into two sets containing $t_1 - 1$ and $n - t_1$ sample points respectively. Similarly, each of these sets can be cut into two sets by using a "cutting curve" determined by a general order statistic $Z_{2(t_2)}$, that is, $h_2(x, y) = Z_{1(t_1)}$, and so on, until the xy-plane has been cut into $n+1$ disjoint sets.

The possibilities of applying Tukey's slicing principle to statistical inference problems look promising but are still to be explored. One of the most immediate applications is that the method provides a more flexible "trimming" procedure for constructing tolerance regions than one can construct by Wald's method of combining rectangles discussed earlier. As an example, consider n sample points represented in the xy-plane as shown in Fig. 1.

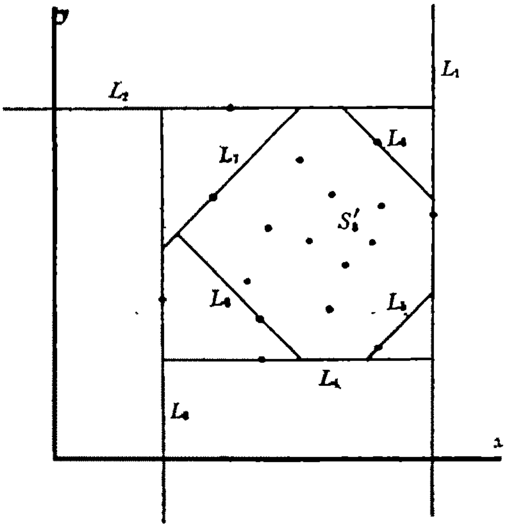

Fig. 1

Now suppose the lines (or line segments) $L_1, L_2, \cdots, L_8$, as shown in the figure, are drawn in that order. Lines L_1 and L_3 are vertical, L_2 and L_4 horizontal, L_5 and L_7 are 45° with respect to the x axis and L_6 and L_8 are 135°. Note that eight sample points have been used in constructing these eight lines, and that $n-8$ points remain in the interior "residual" region S_8'. It follows from Tukey's results that the coverage W for S_8' is distribution-free and has probability element

$$\text{(50)} \qquad \frac{\Gamma(n+1)}{\Gamma(n-7)\Gamma(8)} w^{n-8}(1-w)^7 dw.$$

Thus, the probability that S_8', when used as a tolerance region, contains at least $100\beta\%$ of the population is obtained by integrating (50) from β to 1. Note that by continuing the slicing of S_8' in any manner in accordance with Tukey's principle one would finally cut S_1' into $n-7$ regions, thus making a total $(n-7)+8$ or $n+1$ regions having distribution-free coverages with probability element (6).

(e) *Extensions of distribution theory of coverages and order statistics to the discrete case.* It should be noted that all of the results concerning coverages for any number of dimensions which have been discussed heretofore depend on the continuity of the population cdf. It is important to have analogous results for the case in which the cdf is not continuous. This importance stems from the fact that, because of limitations of making measurements, the observational data, more strictly idealized, must be considered as a sample from a *discrete* population, that is, one having a step-function cdf rather than a continuous one. In actual applications ties *do* occur among the values of a sample.

Tukey, in some work not yet published, has shown how the probability theory of coverages and order statistics can be extended to the discrete case in any number of dimensions by substituting inequalities for equalities in the probability statements for the continuous case. Scheffé and Tukey [62] discussed the problem for the one-dimensional case in 1945.

9. Application of multi-dimensional coverages and order statistics to estimation problems. In §6c it was pointed out that some studies had been made of the problem of making "inefficient" estimates of the mean and variance of a normal population by means of order statistics. Analogous problems exist for the case of normal distributions of two or more variables. A few results have been obtained on these problems but much remains to be done.

Let us consider the case of two dimensions. Suppose (X_1, Y_1), (X_2, Y_2), $\cdots$, (X_n, Y_n) is a sample from a normal bivariate distribution having means m_x and m_y, variances σ_x^2, σ_y^2 and correlation coefficient ρ. Estimates of the mean and variance of x (or y) may be made from the X (or Y) order statistics along lines described in §6c. But the problem of estimating ρ from order statistics is one which must be considered in terms of using both X and Y order statistics. Mosteller [39] and Hotelling and Pabst [26] have made some contributions to this problem. Mosteller considered the numbers of sample points falling into four regions S_{11}, S_{12}, S_{21}, and S_{22} of the xy-plane defined as follows. Draw the vertical lines $x=X_{(m+1)}$ and $x=X_{(n-m)}$ $(2m<n)$. There will be m sample points to the left of the first line and m to the right of the second. Draw any horizontal line $y=y'$ such that m of these $2m$ points lie above this line and m below. Consider the four "corner" regions:

$$
\begin{aligned}
&S_1: \quad x > X_{(n-m)}, \qquad y > y', \\
&S_2: \quad x > X_{(n-m)}, \qquad y < y', \\
&S_3: \quad x < X_{(m+1)}, \qquad y > y', \\
&S_4: \quad x < X_{(m+1)}, \qquad y < y'.
\end{aligned} \tag{51}
$$

Let m^* be the number of sample points falling in S_1. Then considering only the numbers of sample points falling into these four regions, the maximum likelihood estimate of ρ, say R^*, is given by solving the equation,

$$\frac{m^*}{m - m^*} = \frac{np^*}{m - np^*} \tag{52}$$

where

$$p^* = \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_0^\infty \int_k^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2+y^2-2\rho xy\right]\right\} dxdy \tag{53}$$

and k is given by

$$\frac{m}{n} = \frac{1}{(2\pi)^{1/2}} \int_k^\infty e^{-t^2/2} dt. \tag{54}$$

The variance of R^* for large samples is

$$\sigma_{R^*}^2 = \frac{p^*(m - np^*)}{2nmp^*} \tag{55}$$

and the efficiency of this method of estimating ρ is about 51.5% (the Pearson product-moment correlation being 100% efficient), in the case of samples from a normal population in which the true correlation ρ between X and Y is zero. Mosteller has given a graph to facilitate the solution of equation (52) for ρ in practical applications.

Suppose the sample pairs are ordered with respect to the X's, that is, $(X_{(1)}, Y_{(\alpha_1)})$, $(X_{(2)}, Y_{(\alpha_2)})$, $\cdots$, $(X_{(n)}, Y_{(\alpha_n)})$. Then the α's will be some permutation of the integers $1, 2, \cdots, n$. Let R' be the *rank correlation* coefficient in the sample between X's and Y's, that is, defined as the ordinary Pearsonian correlation coefficient between the *ranks* $1, 2, \cdots, n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ (see (75)). Pursuing a lead suggested by the work of Karl Pearson [52] on rank correlation, Hotelling and Pabst [26] have proposed an estimate R' of ρ defined by

$$R' = 2 \sin \pi r'/6 \cdot \tag{56}$$

The variance of R' for the case of $\rho=0$ is $\pi^2/9n$, and the efficiency of ρ' is 91.2%.

It would be desirable to have further studies of the problem of estimating ρ by other schemes depending on order statistics and simple functions of order statistics. There are probably other very simple "inefficient" methods of estimating ρ which would be useful in situations where large samples can be obtained with very little effort. This is true in such fields as psychological testing where large amounts of data are available on punch cards which have already been used for operational purposes.

In normal multivariate statistical theory it would be desirable to have similar studies made of the problem of estimating correlation coefficients from samples in which only ranks are given for sample values of some of the variables.

10. Order statistics in the testing of statistical hypotheses—the method of randomization.

The use of order statistics in the construction of tests for nonparametric statistical hypotheses (valid for populations having any continuous cdf) have been receiving an increasing amount of attention during recent years. A considerable number of results have already been obtained and many more can be expected within the next few years. The main value of nonparametric tests of statistical hypotheses lies in the fact that they are free from the frequently uncomfortable assumption that a population distribution is normal or has some other specific functional form.

One of the simplest and most elegant results in nonparametric statistical inference is the confidence interval theory of the median and other quantiles of a continuous cdf based on order statistics, as discussed in §4(a). Confidence band theory for a continuous cdf $F(x)$ provided by Kolmogoroff's theorem (for large n) and by the Wald-Wolfowitz method (for small n), discussed in §7, furnishes a satisfactory solution to the problem of testing the hypothesis that a sample is from a population with a specified continuous cdf $F_0(x)$.

Beyond these results, there is yet to be developed a satisfactory general theory of nonparametric statistical inference, particularly test theory. The only nonparametric tests which have thus far been developed are rather special cases in which intuition and experience with the well-established parametric test theory have served as guides. What is needed is a general theory of constructing nonparametric tests and a satisfactory method of characterizing the power of these tests. Wolfowitz [88] has made a step in this direction by proposing an extension of the likelihood ratio method, used in deriving parametric statistical tests. He has illustrated this method with several examples, but more remains to be done before the method can be considered generally useful.

If there is a single property of order statistics which has played the greatest role in the development of nonparametric tests thus far it may be stated as follows:

Suppose $X_1, X_2, \cdots, X_n$ *is a sample from a population having a continuous cdf* $F(x)$. *Consider the region* $S_{\{\alpha_i\}}$ *in the sample space* R_n *for which the inequality*

$$X_{\alpha_1} < X_{\alpha_2} < \cdots < X_{\alpha_n} \tag{57}$$

holds, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ *is any one of the* $n!$ *permutations of the integers* 1, 2, 3, $\cdots$, n. *Then the amount of probability contained in* $S_{\{\alpha_i\}}$ *is* $1/n!$.

The truth of this statement is clear from the fact that

$$\begin{aligned} &\Pr(X_{\alpha_1} < X_{\alpha_2} < \cdots < X_{\alpha_n}) \\ &= \Pr\left(\int_{-\infty}^{X_{\alpha_1}} dF(x) < \int_{-\infty}^{X_{\alpha_2}} dF(x) < \cdots < \int_{-\infty}^{X_{\alpha_n}} dF(x)\right) \\ &= \Pr(P_{\alpha_1} < P_{\alpha_2} < \cdots < P_{\alpha_n}) \end{aligned} \tag{58}$$

where $P_{\alpha_1}, P_{\alpha_2}, \cdots, P_{\alpha_n}$ are defined in §3, and have probability element (5). Integrating (5) over the region for which $0 < p_{\alpha_1} < p_{\alpha_2} < \cdots < p_{\alpha_n} < 1$ gives the desired result $1/n!$.

A method of devising nonparametric tests of statistical hypotheses on the basis of order relations among the values of a sample known as the *method of randomization* has been suggested by Fisher [12, 13]. The general idea here is to construct a *rejection region* W in sample space R_n by taking enough regions in the set $\{S_{\{\alpha_i\}}\}$ (each of which has probability $1/n!$) so that the probability contained in W will be a specified amount ϵ. This method has been used in a number of special problems, the method of selecting regions from $\{S_{\{\alpha_i\}}\}$ being more or less intuitive in each case. We should note, however, that the randomization method cannot be used to test the hypothesis that $F(x)$ has a specified form $F_0(x)$, against the class of alternatives consisting of all continuous cdf's different from $F_0(x)$. For, suppose we construct a rejection region W for H_0 in sample space R_n by taking the sum of enough of the regions in the set $\{S_{\{\alpha_i\}}\}$ to make the probability contained in W equal to ϵ (or approximately so if ϵ is not a multiple of $1/n!$) when $F(x) \equiv F_0(x)$. Such a region will contain exactly the same probability ϵ for any other continuous cdf $F(x)$, and the test would have no *power* for testing the hypothesis that $F(x) \equiv F_0(x)$ over any of the alternatives $F(x) \not\equiv F_0(x)$. In other words, the probability of the rejection of an alternative cdf will always be ϵ no matter which alternative it is. The method of randomization for constructing statistical tests therefore must be applied to situations in which it is desired to test whether certain subsets of values in a sample come from populations having continuous cdf's differing in some respect.

The construction of rejection regions for nonparametric tests by the method of randomization has not yet been placed on a satisfactory general foundation as has been done in the case of parametric tests. The best way to discuss the method is perhaps to consider some examples. These examples will show how intuition plays an important role in each case and will emphasize the need of a general theory.

Before considering applications of the method of randomization to the construction of nonparametric statistical tests, it should be emphasized that the method of randomization has other interesting applications. It furnishes a way of dealing with any problem of determining probabilities of specified order relations among values in two or more samples which have been drawn from the same population having a continuous cdf.

For example we may ask what is the probability that a pair of samples O_m (first sample) and O_n (second sample) from a population with continuous cdf $F(x)$ will be such that all the values of O_n will fall within the range established by O_m? What is the probability that the pair of samples will be such that the largest value in O_n will exceed the largest value in O_m? Suppose the values in O_n are drawn one by one until a value exceeds the largest value in O_m. For pairs of samples drawn in this way what is the probability function of n, the size of the second sample which must be drawn to satisfy the conditions of the problem? There are many problems of this type which can be dealt with by the randomization method. They all reduce to combinatorial analysis of the set of equally probable permutations of the $m+n$ values of the two samples, enumerating the subset which satisfies the conditions of the problem. Similarly, one could deal with order relations problems for three or more samples. Also, one can raise analogous questions for the case of two or more samples from populations having continuous cdf's of two or more dimensions.

11. Examples of nonparametric statistical tests for one dimension based on the method of randomization.

(a) *Two-sample tests.* First, let us consider the two-sample problem treated originally in a very specific example by Fisher [13] and more generally treated by Pitman [55]. Let $X_1, X_2, \cdots, X_m$ and $X_{m+1}, X_{m+2}, \cdots, X_{m+n}$ be two independent samples O_m and O_n of sizes m and n from populations having continuous cdf's $F(x)$ and $G(x)$ respectively. Let the values in O_m and O_n be pooled together into a single sample $Z_1, Z_2, \cdots, Z_{m+n}$ of size $m+n$ and let the order statistics in the pooled sample be denoted by $Z_{(1)}, Z_{(2)}, \cdots, Z_{(m+n)}$. If $F(x) \equiv G(x)$ then the probabilities associated with all permutations of the sample values are all equal to $1/(m+n)!$. Let a *separation* be a division of the pooled sample into two sets, consisting of m values and n values respectively. There are $C_{m+n,n}$ possible separations. For any particular separation let $Z_1, Z_2, \cdots, Z_m$ and $Z_1', Z_2', \cdots, Z_n'$ be the two sets of values. Let $\bar{Z}$ and $\bar{Z}'$ be the means of the sets. Let $S = \sum_{i=1}^{m} (Z_i - \bar{Z})^2$ and $S' = \sum_{j=1}^{n} (Z_j' - \bar{Z}')^2$. Pitman proposed

$$V = \frac{mn(\bar{Z} - \bar{Z}')^2}{(m + n)(S + S') + mn(\bar{Z} - \bar{Z}')^2} \tag{59}$$

as a criterion for testing the difference between the two means $\bar{Z}$ and $\bar{Z}'$, large values of V being *significant*. He determined the first three moments of the distribution of V over the set of all separations (equally weighted) and for large m and n he showed the probability element of the cdf of V to be approximately

$$\frac{\Gamma((m + n + 1)/2)}{\Gamma(1/2)\Gamma((m + n)/2)} v^{-1/2}(1 - v)^{(m+n)/2-1}dv. \tag{60}$$

It is to be noted that the equal weighting of the separations comes from the fact that when $F(x) \equiv G(x)$ (that is, when the *null hypothesis* is true) the $(m+n)!$ regions $\{S_{(\alpha_i)}\}$ corresponding to the $(m+n)!$ permutations of the type $Z_{\alpha_1} < Z_{\alpha_2} < \cdots < Z_{\alpha_{m+n}}$ all have probabilities equal to $1/(m+n)!$.

In this example it is to be noted that a rejection region W in the sample space R_n is constructed by taking those regions from the set $\{S_{(\alpha_i)}\}$ for which V has its largest values, and enough of them to produce a total probability as close as possible to ϵ, the significance level. More precisely if v_ϵ is the *critical* value of V at significance level ϵ, then W is the region in sample space constructed of sets from $\{S_{(\alpha_i)}\}$ for which $V \geq v_\epsilon$, where v_ϵ is the smallest number for which $\Pr(V \geq v_\epsilon) \leq \epsilon$, when $F(x) \equiv G(x)$.

If we are to subject the test criterion V and the rejection region W to a more penetrating analysis we must ask for what class Ω of pairs of continuous cdf's $\{F(x), G(x)\}$, including the null hypothesis $F(x) \equiv G(x)$, is the test (consisting of V and the rejection region W) "satisfactory"? This question has not been considered. Some might consider as a minimum requirement for the test to be satisfactory that it be *unbiased*, that is, that the probability (approximately equal to ϵ) associated with the rejection region W be smaller when $F(x) \equiv G(x)$ than for any other pair of cdf's in Ω. Actually, the question of whether a test is satisfactory or not has to be determined from a consideration of the whole *power function* of the test, that is, the probability contained in W expressed as a function of the pair $\{F(x), G(x)\}$. If W and W' are two regions such that the power functions are equal when $F(x) \equiv G(x)$ and the power function of W' is greater than that for W for all pairs $F(x)$, $G(x)$ which are different, then the test based on W' is "better" than that based on W.

The Pitman test is unbiased if $G(x) = F(x+m)$, that is, if Ω consists of all pairs of continuous *cdf*'s of the form $\{F(x), F(x+m)\}$. In other words, the test would be satisfactory as a *slippage* or *location* test.

A second rather simple and elegant test for the problem of two samples has been proposed by Wald and Wolfowitz [77]. To describe their results we again consider two samples, O_m and O_n of m and n values from populations having continuous cdf's $F(x)$ and $G(x)$ respectively. There is no loss of generality in assuming $m \leqq n$. As before, let the pooled sample be $Z_1, Z_2, \cdots, Z_{m+n}$ and the ordered values of the Z's be $Z_{(1)}, Z_{(2)}, \cdots, Z_{(m+n)}$. This sequence of order statistics will be a mixture of X's from the two samples. We shall define a *run* of length r from O_m in this sequence as an uninterrupted subsequence of X's from O_m, with a similar definition of a run of length s from O_n. The sequence of order statistics will then consist of a certain number of runs of X's from O_m and a certain number from O_n. Let U be the *total* number of runs. Wald and Wolfowitz proposed U as the criterion for testing the hypothesis $F(x) \equiv G(x)$, where small values of U are significant. This is an intuitively reasonable test since small values of U correspond to bunching or poor mixing of values from the two samples, and hence furnish an observational basis for doubt of the hypothesis that $F(x) \equiv G(x)$. More precisely, the rejection region W in the sample space R_n consists of all regions from $\{S_{\{\alpha_i\}}\}$ for which $U \leqq u_\epsilon$, where u_ϵ is the smallest integer for which $\Pr(U \leqq u_\epsilon) \geqq \epsilon$ when $F(x) \equiv G(x)$. The problem of determining u_ϵ is solved, of course, if the exact distribution of U is found when $F(x) \equiv G(x)$. This reduces to a completely combinatorial problem. The result obtained by Wald and Wolfowitz is

$$
\begin{aligned}
&\Pr(U = 2k) = \frac{2C_{m-1,k-1}C_{n-1,k-1}}{C_{m+n,m}} \qquad (k = 1, 2, \cdots, m), \\
(61)\qquad &\Pr(U = 2k-1) = \frac{C_{m-1,k-1}C_{n-1,k-2} + C_{m-1,k-2}C_{n-1,k-1}}{C_{m+n,m}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = 2, 3, \cdots, m+1).
\end{aligned}
$$

Swed and Eisenhart [69] have tabulated $\Pr(U \leq u')$ for $m \leq n \leq 20$, and for all values of u' in each case. They have also tabulated the smallest integer u_ϵ for which $\Pr(U \leq u_\epsilon) \geq \epsilon$ for $\epsilon = .005$, $.01$, $.025$, $.05$, and the largest integer u_ϵ for which $\Pr(U \leq u_\epsilon) \leq \epsilon$ for $\epsilon = .95$, $.975$, $.99$ and $.995$.

An important property of the U test established by Wald and Wolfowitz for certain regularity conditions on $F(x)$ and $G(x)$ is that the test is *consistent* for all pairs of cdf's $\{F(x), G(x)\}$ in the limit as m and $n \rightarrow \infty$ so that $m/n = r \neq 0$. This means that the probability of a sample point falling in the rejection region W in sample space when $F(x) \not\equiv G(x)$ approaches 1 as m and $n \rightarrow \infty$ in a constant ratio $m/n = r \neq 0$.

The results of Wald and Wolfowitz can be readily extended to the case of three or more samples. Such an extension would reduce to the theory of runs for three or more kinds of elements as developed by Mood [36].

Other simple tests of the two-sample problem using order statistics have been devised by Thompson [71], Dixon [9], and Mathisen [32]. The tests proposed by Thompson and Mathisen are not consistent for hypotheses involving all pairs of continuous cdf's $\{F(x), G(x)\}$, although it can be conjectured that they are consistent for the important case of a slippage test of unimodal cdf's, that is, for all pairs $\{F(x), F(x+m)\}$, when $F'(x)$ exists and has only one maximum.

(b) *Tests of independence or "randomness" in ordered sequences; run tests.* It was pointed out earlier that no satisfactory statistical test based on order relations among the values in a sample could be devised for testing the hypothesis that $F(x)$ has a specified form $F_0(x)$. This does not mean that there are not important hypotheses which can be tested by using order relations among the values in a single sample.

One of the most important problems in this category is the question of whether $F(x)$ is changing from drawing to drawing, or whether the sample values, $X_1, X_2, \cdots, X_n$, in the ordered sequence as drawn are "random." In this case we would consider the class Ω of all n-dimensional continuous cdf's $F(x_1, x_2, \cdots, x_n)$, or some subset of Ω, and the null hypothesis would state that

$$F(x_1, x_2, \cdots, x_n) = F(x_1) \cdot F(x_2) \cdot \cdots \cdot F(x_n). \tag{62}$$

This hypothesis of *independence* or "randomness" is basic to the whole theory of random sampling. The practical importance of this problem as one to be investigated before applying random sampling theory has been strongly emphasized by Shewhart [63] on the basis of his experience with sampling in industry. In dealing with this problem various tests of independence based on order statistics have been proposed. Several of them, now referred to as *run tests* and based on intuition have been studied. We shall briefly describe two of these tests: (i) ***runs above and below the median***, and (ii) ***runs up and down.***

First let us examine (i). Consider a sample of values $X_1, X_2, \cdots, X_{2n+1}$ drawn in the order indicated. Note that these are *not* order statistics. In this sequence let each X less than the median, $X_{(n+1)}$, be denoted by a, and let each X greater than $X_{(n+1)}$ be denoted by b. Then deleting the median from $X_1, X_2, \cdots, X_{2n+1}$ we have $2n$ X's left which are now replaced by some arrangement of n a's and n b's. There will be r_{1i} runs of a's of length i, and r_{2i} runs of b's of length i, $i=1, 2, \cdots, n$. Under the hypothesis of independence, the probability theory of r_{1i} and r_{2i} reduces to a consideration of the set of $(2n)!$ permutations of the n a's and n b's, all of which have equal probability $1/(2n)!$. The median is ignored as far as determining runs is concerned. In case of a sample of $2n$ values one can select any number between $X_{(n)}$ and $X_{(n+1)}$ as the value to separate X's into a's and b's, and the run theory is the same as that for the case just considered. Thus, the problem of finding the distribution function of the r_{1i} and r_{2i} is completely combinatorial, and has been solved by Mood [36]. He has also solved the analogous distribution problem for more than two kinds of elements, such as would arise if one used several arbitrary order statistics to cut up the set of sample values into more than two sets of elements.

Mosteller [38], using Mood's basic distribution theory of runs, has considered the length L of the longest run of a's as a criterion for testing randomness, large values of L being significant. The critical value of L for probability level ϵ is the smallest integer l_ϵ such that $\Pr(L \geqq l_\epsilon) \leqq \epsilon$. A similar criterion exists for the b's. He also considered a criterion L' defined as the length of the longest run of a's *or* b's. He tabulated critical values of l_ϵ and l'_ϵ for $2n=10, 20, 30, 40, 50$ and for $\epsilon=.01$ and $.05$.

It should be noted that the Wald-Wolfowitz U test may also be considered as a test of type (i) for testing the hypothesis of independence. In this case U would be the total number of runs of a's and b's.

Now let us consider a run test of type (ii) treated by Wolfowitz and Levene [89] and by Olmstead [44]. In this case let the sample values be $X_1, X_2, \cdots, X_n$ in the order drawn. We now set up a sequence of $n-1$ $+$'s and $-$'s defined as follows: If $X_i < X_{i+1}$ we write down a $+$, if $X_i > X_{i+1}$ we write down a $-$, $i=1, 2, \cdots, n-1$. We now consider runs of +'s and −'s. The test criterion L'' used in this test is the length of the longest run of +'s or −'s, large values of L'' being significant. The probability theory of L'' is much more complicated than that of L or L', and was worked out by Levene and Wolfowitz. Olmstead [44], making use of a recursion relation, has been able to tabulate the exact probability function of L'' for sample sizes ranging from 2 to 14 and has been able to find the approximate distribution for larger values of n. A criterion for testing independence in an ordered sequence proposed by Young [90] and based on the method of randomization is given by

$$C = 1 - \sum_{i=1}^{n-1} (X_i - X_{i+1})^2 \Big/ 2 \sum_{i=1}^{n} (X_i - \bar{X})^2. \tag{63}$$

Young found the first four moments of the distribution of C assuming independence as expressed by (62).

Wald and Wolfowitz [79] have considered a criterion similar to that studied by Young. Their criterion is defined as

$$D = \sum_{i=1}^{n-1} X_i X_{i+1} + X_1 X_n \tag{64}$$

and was proposed as a test for randomness, the null hypothesis being given by (62). The method of randomization was used in dealing with the distribution of D. The mean and variance of D was found when the null hypothesis stated by (62) is true and it was shown that D is asymptotically normally distributed for large n in this case.

It should be noted that these run tests have been devised from intuitive considerations for the purposes of detecting the presence of "causes" of slowly changing "secular changes" or slippage in the population cdf during the course of drawing a sample. The probability theory of all of these run tests is based on the assumption that the null hypothesis of independence expressed by (62) holds. It is likely that these run tests are satisfactory for alternatives to the hypothesis of random sampling in which the population cdf changes by slippage during the course of sampling, but this is yet to be explored.

The problem of testing for independence or for "randomness" in successive drawings from a population, which is fundamental in the theory of random sampling, deserves more theoretical attention than it has received. Possibly a consideration of the problem from the point of view of Wald's theory of sequential analysis would be profitable.

(c) *One-dimensional parametric tests involving order statistics.* A number of authors have investigated parametric statistical tests based on order statistics for samples from a normal population. E. S. Pearson and Hartley [50] have studied the "Studentized" range

$$q = \frac{(m-1)^{1/2}(X_{(n)} - X_{(1)})}{\left[\sum_{i=1}^{m} (X_i' - \bar{X}')^2\right]^{1/2}} \tag{65}$$

determined by two samples $X_1, X_2, \cdots, X_n$ and $X_1', X_2', \cdots, X_m'$ from a normal population having mean m and variance σ^2, where $\bar{X}'$ is the mean of the second sample. They have tabulated the 1% and 5% points of the distribution of q for all values of n from 3 to 20, and for $m = 11, 12, \cdots, 20, 24, 30, 40, 60$, and 120. The $100\alpha\%$ point in this case is defined as the value of q_α for which $\Pr(|q| > q_\alpha) = \alpha$ when both samples are from the same normal population.

Daly [6] has considered the quantity

$$G = \frac{\bar{X} - m}{X_{(n)} - X_{(1)}} \tag{66}$$

as an alternative to the Student t-test for testing the hypothesis that the population mean m in a normal population has a specified value m_0. He has found that for sample sizes less than 10, a test based on G is almost as powerful as the Student t-test. Walsh [82] has considered an even simpler alternative to the t-test for small samples from a normal population, namely.

$$H = \frac{(X_{(n)} + X_{(1)})/2 - m}{X_{(n)} - X_{(1)}}. \tag{67}$$

This test is almost as powerful as the Student t-test for values of n less than 10.

Walsh, in a paper not yet published, has devised very simple tests based on order statistics for testing the hypothesis that the median of a continuous cdf has a specified value. In the case of a normal population he has found the *power efficiency* of his tests to be at least 90%, when compared with the Student t-test, for samples of less than 15. The power efficiency of $E\%$ as used by Walsh means that the power function of the test for a sample of size n is approximately the same as that of a Student t-test for a sample of size $En/100$.

(d) *Analysis of variance tests by the method of randomization.* Among other one-dimensional nonparametric tests should be mentioned analysis of variance tests. In 1937 Friedman [16] devised an analysis of variance test based on ranks. In his test rs "observations" X_{ij} ($i=1, 2, \cdots, r$; $j=1, 2, \cdots, s$) are arranged in r rows and s columns. The sample values within each row are ordered and only the *ranks* $(1, 2, \cdots, s)$ of the ordered values in each row are used in the test. The hypothesis tested (null hypothesis) is that there are no column "effects," that is, that the sample values within each row are from a single population with some continuous cdf. This means that all possible permutations of ranks within a row are equally probable. If α_{ij} is the rank associated with the ith row and jth column, where $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{is}$ $(i=1, 2, \cdots, r)$ is some permutation of the integers $1, 2, \cdots, s$, the test criterion proposed by Friedman is

$$\chi_r^2 = \frac{12}{rs(s+1)} \sum_{j=1}^{s} \left(\sum_{i=1}^{r} \alpha_{ij} \right)^2 - 3r(s+1). \tag{68}$$

If there are no column "effects," the quantity χ_r^2 is, for large r, approximately distributed according to the chi-square law with $s-1$ degrees of freedom, that is, with probability element

$$\frac{(\chi_r^2/2)^{(s-3)/2}}{2\Gamma((s-1)/2)} e^{-\chi_r^2/2}\, d(\chi_r^2). \tag{69}$$

The test is a randomization test in which the null hypothesis is that the s values X_{ij} $(j=1, 2, \cdots, s)$ in the ith row $(i=1, 2, \cdots, r)$ is a sample from a population having some continuous cdf. We therefore have a sample of size s from each of r populations. Kendall and Smith [29] and also Wallis [80] have studied the same problem but used a criterion η_r^2 related to χ_r^2 by the expression $\chi_r^2 = r(s-1)\eta_r^2$.

In 1937 Pitman [57] investigated the standard analysis of variance test for testing column "effects" in a layout of r rows and s columns under the assumption that the observations in each row constitute a sample of size s from a population having a continuous cdf, the cdf's for the rows being identically the same except for slippage. Let X_{ij} be the observation for the ith row and jth column. Consider the ratio

$$W = \frac{\sum_{i,j} (\bar{X}_{\cdot j} - \bar{X})^2}{\sum_{i,j} (X_{ij} - \bar{X}_{\cdot j} - \bar{X}_{i\cdot} + \bar{X})^2 + \sum_{i,j} (\bar{X}_{\cdot j} - \bar{X})^2} \tag{70}$$

where $\sum_{i,j}$ denotes summation from $i=1$ to r and $j=1$ to s, $\bar{X}_{i\cdot}$, $\bar{X}_{\cdot j}$ are the averages of the X's in the ith row and jth columns respectively, and $\bar{X}$ is the average of all rs X's. Pitman, using the method of randomization, determined the first four moments of the distribution of W under the set of all permutations of the X's within rows, and he presented some evidence to the effect that the distribution of W could be reasonably well fitted by the beta distribution

$$(71) \qquad \frac{\Gamma(r(s-1)/2)}{\Gamma((s-1)/2)\Gamma((r-1)(s-1)/2)} w^{(s-3)/2}(1-w)^{(r-1)(s-1)/2-1}dw,$$

which is the same distribution as that obtained under the usual assumption of normality of the analysis of variance test. In 1938 Welch [84] made a study of the standard correlation ratio test for homogeneity from the point of view of the method of randomization. This problem is very similar to the analysis of variance problem treated by Pitman. Welch [83] also carried out an investigation of the analysis of variance test for randomized blocks and Latin squares from the point of view of the method of randomization.

All of these analysis of variance tests are essentially extensions of the usual analysis of variance test to the case of more general populations. One type of nonparametric test which needs consideration is a k-sample slippage or location test for testing the hypothesis that a sample which has the largest "observed slippage" to the right, let us say, as measured in some sense by order statistics, does, in fact, come from a population centered farther to the right than the populations from which the other samples come. Here it would be assumed that the cdf's of the k populations are of the form $F(x+m_i)$ $(i=1, 2, \cdots, k)$, and the null-hypothesis would be that the m_i are all equal. Mosteller [40] has proposed one simple test for this situation in the case of k samples, each of size n. To define his test we select from among the k samples that sample having the largest X. Let r be the number of X's in this sample which exceed the largest X in any of the remaining samples; r is the test criterion. The problem of determining the probability function of r under the null hypothesis (that is, when all m_i are equal) is an application of the method of randomization—it is completely combinatorial, and the probability that $r=r'$ under these conditions is

$$(72) \qquad \Pr(r=r') = \frac{k(n!)(kn-r')!}{(kn)!(n-r')!},$$

which is approximately $k^{1-r'}$ for large n. The important feature of this test is that one allows himself to select the sample with the largest "observed slippage" and then to ask whether there are enough values in it exceeding the largest value in the remaining samples to be "significant." A full analysis of this test—like other randomization tests—requires a study of its power function for all sets of cdf's of the form $\{F(x+m_1), F(x+m_2), \cdots, F(x+m_k)\}$. Mosteller has obtained a few inequalities on the power of the test.

12. Nonparametric tests for two or more dimensions based on the method of randomization. The principle of randomization can be extended to populations involving two or more variables. We shall not attempt to discuss the general problem here. Several examples of tests for the two-dimensional case will indicate the nature of such tests.

(a) *Tests of independence based on correlation coefficients.* Pitman [56], by using the principle of randomization, for two dimensions, developed a test for the independence of the two variables in a bivariate population having a continuous cdf $F(x, y)$. In this case the sample O_n is $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$. The hypothesis of independence is that $F(x, y) = F_1(x) \cdot F_2(y)$. If the hypothesis is true, one essentially has two samples: namely, $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$, in which the pairing of values has occurred "at random." In the X sample space there are $n!$ regions of the type $X_{\alpha_1} < X_{\alpha_2} < \cdots < X_{\alpha_n}$ where $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a permutation of the integers $1, 2, \cdots, n$, and similarly for the Y sample space. Therefore in the product space of the two samples there are $(n!)^2$ product regions in each of which an X inequality $X_{\alpha_1} < X_{\alpha_2} < \cdots < X_{\alpha_n}$ and a Y inequality $Y_{\beta_1} < Y_{\beta_2} < \cdots < Y_{\beta_n}$ hold simultaneously. The probability contained in each of these product regions is $(1/n!)^2$ when the hypothesis of independence is true. This means that the probability associated with any particular matching $(X_{\alpha_1}, Y_{\beta_1}), (X_{\alpha_2}, Y_{\beta_2}), \cdots, (X_{\alpha_n}, Y_{\beta_n})$ is $1/(n!)^2$. Pitman has considered the distribution of the ordinary correlation coefficient

$$R = \frac{\sum_{i=1}^{n} (X_{\alpha_i} - \bar{X})(Y_{\beta_i} - \bar{Y})}{\left(\sum_{i=1}^{n} (X_{\alpha_i} - \bar{X})^2 \cdot \sum_{i=1}^{n} (Y_{\beta_i} - \bar{Y})^2\right)^{1/2}} \tag{73}$$

over the $(n!)^2$ matchings of X's and Y's, where $\bar{X}$ and $\bar{Y}$ are the means of the X's and Y's respectively. He obtained the first four moments of R and for large n fitted a beta distribution to that of R, obtaining as the pdf of the fitted distribution

$$\frac{\Gamma(n/2 - 1/2)}{\Gamma(1/2)\Gamma(n/2 - 1)} (1 - t^2)^{n/2-2} dt, \tag{74}$$

for the interval $(-1, +1)$.

In 1936, Hotelling and Pabst [26], using the method of randomization, investigated the rank correlation coefficient as a criterion for testing independence of X and Y, defined by

$$(75)\qquad R' = \frac{\sum_{i=1}^{n}(\alpha_i - (n+1)/2)(\beta_i - (n+1)/2)}{\left(\sum_{i=1}^{n}(\alpha_i - (n+1)/2)^2 \cdot \sum_{i=1}^{n}(\beta_i - (n+1)/2)^2\right)^{1/2}}$$

which has the same approximate distribution as R', namely (74). Each of the quantities $n^{1/2}R$ and $n^{1/2}R'$ is asymptotically normally distributed with zero mean and unit variance for large n. Olds [43] has tabulated the distribution of a quantity S related to R' by the relation $R' = 1 - 6S/(n^3 - n)$ for values of n from 2 to 10.

(b) *A "corner" test of association.* Olmstead and Tukey [45] have recently devised a simple test for the independence of two random variables on the basis of a sample of size $2n+1$. Their test is as follows: consider the sample $(X_1, Y_1), (X_2, Y_2), \cdots, (X_{2n+1}, Y_{2n+1})$ as represented by $2n+1$ points in the xy-plane. Then draw the horizontal line $y = Y_{(n+1)}$ and the vertical line $x = X_{(n+1)}$. Designate the quadrants as $+, -, +, -$, beginning in the upper right-hand quadrant and moving counterclockwise. Now begin with the sample point farthest to the right and move toward the left as long as sample points are on the same side of the horizontal median line as the original one encountered. Let this number of points be P_1, attaching a $+$ or $-$ sign depending on whether the points are in a $+$ or $-$ quadrant. Do a similar counting operation moving from the top sample point downward, a similar one moving from the left-most to the right and a similar one moving from the bottom sample point upward. Let the numbers of points be P_2, P_3 and P_4, respectively, with attached $+$ or $-$ signs depending on whether the points in each case fall in a $+$ or $-$ quadrant. Let S be the algebraic sum of the four P's with their signs attached. Then if X and Y are independent, S is a function of only the order relations among the X's and Y's and the problem of determining the distribution function of S is therefore combinatorial. Olmstead and Tukey have shown that, for large n,

$$(76)\qquad \lim_{n\to\infty} \Pr(|S| \geq k) = \frac{9k^3 + 9k^2 + 168k + 208}{216(2)^k}.$$

They have also tabulated exact values of $\Pr(|S| \geq k)$ for $2n = 2, 4, 6, 8, 10, 14$, and for $k = 1, 2, 3, \cdots, 30$. An important feature of this test is that $\Pr(|S| \geqq k)$ approaches its limiting form very rapidly. This test has also been extended to the case of more than two dimensions.

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PRINCETON UNIVERSITY

THE OCTOBER MEETING IN NEW YORK

The four hundred twenty-eighth meeting of the American Mathematical Society was held at Hunter College, New York City, on Saturday, October 25. The attendance was over two hundred, including the following one hundred eighty-five members of the Society:

Milton Abramowitz, Leonidas Alaoglu, E. B. Allen, C. B. Allendoerfer, R. L. Anderson, R. G. Archibald, L. A. Aroian, W. A. Asprey, P. T. Bateman, F. P. Beer, Stefan Bergman, Lipman Bers, Gertrude Blanch, R. P. Boas, Salomon Bochner, G. L. Bolton, A. D. Bradley, H. W. Brinkmann, Paul Brock, A. B. Brown, Hobart Bushey, J. H. Bushey, S. S. Cairns, J. D. Campbell, K. Chandrasekharan, Herman Chernoff, J. A. Clarkson, I. S. Cohen, L. W. Cohen, T. F. Cope, Richard Courant, A. P. Cowgill, W. H. H. Cowles, A. R. Craw, J. E. Crawford, H. B. Curry, E. H. Cutler, M. D. Darkow, F. H. Davidson, A. S. Day, Bernard Dimsdale, M. P. Dolciani, C. H. Dowker, Y. N. Dowker, Arnold Dresden, R. J. Duffin, James Dugundji, W. F. Eberlein, Samuel Eilenberg, Paul Erdős, J. M. Feld, E. J. Finan, N. J. Fine, Emanuel Fischer, R. M. Foster, K. O. Friedrichs, Gerald Freilich, Abe Gelbart, B. P. Gill, O. E. Glenn, H. E. Goheen, V. D. Gokhale, A. W. Goodman, Saul Gorn, M. J. Gottlieb, W. H. Gottschalk, M. C. Gray, Harriet Griffin, E. J. Gumbel, F. C. Hall, P. R. Halmos, Carl Hammer, Gerald Harrison, K. E. Hazard, Alex Heller, Erik Hemmingsen, L. S. Hill, Einar Hille, Joseph Hilsenrath, W. M. Hirsch, A. J. Hoffman, T. R. Hollcroft, E. M. Hull, Witold Hurewicz, S. A. Joffe, Fritz John, R. A. Johnson, A. W. Jones, Aida Kalish, Hyman Kamel, E. L. Kaplan, Samuel Kaplan, M. E. Kellar, D. E. Kibbey, J. R. Kline, E. G. Kogbetliantz, E. R. Kolchin, B. O. Koopman, H. K. Kutman, A. W. Landers, M. K. Landers, F. X. Larkin, J. R. Lee, Solomon Lefschetz, Joseph Lehner, Howard Levi, M. A. Lipschutz, E. R. Lorch, Lee Lorch, A. W. McMillan, Brockway McMillan, L. A. MacColl, Saunders MacLane, Henry Malin, A. J. Maria, M. H. Maria, Imanuel Marx, W. S. Massey, F. I. Mautner, K. A. Morgan, F. J. Murray, D. S. Nathan, C. A. Nelson, Morris Newman, P. B. Norman, E. R. Ott, O. G. Owens, J. C. Oxtoby, J. S. Oxtoby, L. G. Peck, A. M. Peiser, Anna Pell-Wheeler, C. R. Phelps, Everett Pitcher, E. L. Post, Hans Rademacher, R. C. Rand, Irving Reiner, Daniel Resch, R. R. Reynolds, Moses Richardson, J. F. Ritt, S. L. Robinson, Saul Rosen, Herman Rubin, H. E. Salzer, Arthur Sard, Samuel Schechter, M. M. Schiffer, Lowell Schoenfeld, Pincus Schub, Abraham Schwartz, G. E. Schweigert, I. E. Segal, Max Shiffman, James Singer, P. A. Smith, J. J. Sopka, E. H. Spanier, George Springer, E. P. Starke, N. E. Steenrod, J. J. Stoker, Walter Strodt, M. M. Sullivan, Olga Taussky-Todd, C. B. Tompkins, Hing Tong, L. V. Toralballa, Annita Tuller, H. E. Vansant, W. G. Warnock, W. R. Wasow, G. C. Webber, Louis Weisner, A. M. Whelan, M. E. White, A. L. Whiteman, Norbert Wiener, John Williamson, G. M. Wing, J. W. Young, J. J. Zeig, H. J. Zimmerberg, Leo Zippin.

On Saturday morning there were two sections, one for papers in Analysis in which Professor Norbert Wiener presided, and one for papers in Algebra, Topology, and Logic in which Professor H. B. Curry presided.

On Saturday afternoon Professor C. B. Allendoerfer of Haverford

College gave an address on *Global theorems in Riemannian geometry*. President Einar Hille presided. At the beginning of this session a motion by Professor C. A. Nelson was passed unanimously authorizing the Secretary to express to President George N. Shuster of Hunter College the thanks and appreciation of the Society for the use of the facilities of Hunter College for this meeting.

Abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Professor Godement was introduced by Professor Claude Chevalley, Dr. Lorentz by Professor R. P. Agnew, and Professor Wright by Professor Saunders MacLane.

ALGEBRA AND THEORY OF NUMBERS

1*t*. Reinhold Baer: *The role of the center in the theory of direct decompositions.*

It has been noticed for some time that the center plays a fundamental part in the theory of direct decompositions of operator groups and loops. In particular it has been found that the existence of isomorphic refinements of direct decompositions can be assured by imposing conditions which refer solely to the center. It is the object of the author to give an explanation for these phenomena by proving that quite generally the validity of the refinement theory, properly stated, in an operator loop is a consequence of the validity of this theory in the center. (Received August 12, 1947.)

2*t*. Leonard Carlitz: *The singular series for sums of squares of polynomials.*

In several papers the writer had considered the problem of the number of representations of a polynomial in $GF[p^n, x]$ as the sum of first an even number of squares, and later as the sum of an odd number of squares. In a previous (unpublished) communication, the writer constructed an analog of the Hardy singular series for squares of rational integers. By this method it was possible to treat the even and odd cases simultaneously. The present paper is an improvement of the previous one in several respects. For one thing it is now shown that the singular series gives the correct result for one square. Secondly, in the even case it is shown to yield a theorem of Cohen (Duke Math. J. vol. 14 (1947) pp. 251–267). Finally, the evaluation of the series in the odd case is considerably simplified. This leads in particular to an elegant result on the number of *primitive* solutions. (Received August 22, 1947.)

3*t*. Leonard Carlitz: *Cauchy products of arithmetic functions in* $GF[p^n, x]$.

In a previous paper (L. Carlitz and Eckford Cohen, *Cauchy products of divisor functions in* $GF[p^n, x]$, Duke Math. J. vol. 14 (1947)) three kinds of Cauchy multiplication were defined; in particular the C_r-product of two arithmetic functions is defined by $\sum f(A)\, g(B)$, the summation extending over all polynomials A, B of degree less than r such that $A+B=M$. It follows that the set of functions $\{f\}$ forms a commutative ring R; the divisor functions of the previous paper form a proper sub-ring of R. It is proved that R is a direct sum of fields. Among the other results of the present paper may be mentioned the construction of a set of "orthogonal functions" e_{GH} which seem well-suited for applications. Application is made to certain problems on sums of squares. It is now possible to handle certain cases excluded by the "singular series" method. (See preceding abstract.) (Received August 22, 1947.)

4. O. E. Glenn: *Phases of the inverse theory of rational invariants.*

If an n-ary quantic $f=a_x^m=a_1x_1^m+\cdots$ has arbitrary coefficients, and a set $g_1(y)$, $\cdots$, $g_s(y)$ of rational, integral polynomials in variables $y_1, y_2, \cdots$ is cogredient with the set $(a_1, \cdots, a_s)$ under the induced or unspecified transformations on the x_i, then $\Delta=(g_1\partial/\partial a_1+\cdots+g_s\partial/\partial a_s)$ is an invariant operator. If c_x^r is a covariant of degree g in the a_i, then $F=\Delta^{g-1}c_x^r$ is a ground-quantic, that is, is of degree unity in the a_i. But F in general has three types of factors, universal covariants of linear transformations on functions of y_i (and of x_i), universal covariants of cognate transformations on x_i, and an irreducible invariant function whose coefficients are the a_i. The transformations can then be determined from these universal covariants and ultimately from c_x^r. Applied to a case in which $g_1=(y_3-y_2)^m$, $g_2=(y_1-y_3)^m$, $g_3=(y_2-y_1)^m$, and f is ternary, a factor of F is $d=(y_3-y_2)x_1+(y_1-y_3)x_2+(y_2-y_1)x_3$. Therefore the group is a certain formal group on x_i, in four parameters, under which any ternary quantic will have a fundamental system. The group can be generalized. (Received July 14, 1947.)

5t. V. L. Klee: *On a conjecture of Carmichael.*

Carmichael (Bull. Amer. Math. Soc. vol. 28 (1922) pp. 109–110) conjectured that for no integer N can the equation $\phi(x)=N$ have exactly one solution, ϕ denoting Euler's totient. In support of his conjecture, this note shows by elementary methods that each N for which there is a unique solution must satisfy various restrictions, one of which implies $N>10^{400}$. (Received August 28, 1947.)

6. Saunders MacLane: *The group of abelian group extensions.*

The structure of the group Ext (T, G) of all abelian group extensions of the abelian group G by a torsion group T is studied. Assume T denumerable and p-primary, let T_n be the subgroup of all elements of order at most p^n, ωT the subgroup of all elements of infinite height. Then Ext (T, G) may be mapped onto the group of all homomorphisms of the system of groups $T_n\leftrightarrow T_{n+1}$ into the system of groups $G/p^nG \leftrightarrow G/p^{n+1}G$. The kernel of the mapping is the set $\text{Ext}_f(T, G)$ of extensions trivial on finite subgroups of T; this kernel may be mapped onto the group of all homomorphisms of ωT into $G_\infty/\alpha G$, where G_∞ is the inverse limit of the sequence G/p^nG, and αG the natural image of G in G_∞. The kernel of this second mapping is isomorphic to $\text{Ext}(\omega T, \alpha G)$. To the latter group the same reduction may be applied again. (Received September 22, 1947.)

7t. F. I. Mautner: *The Peter-Weyl theorem for discrete groups.*

It is shown that any countable discrete group G has a system S of irreducible unitary representations (in Hilbert space) which is complete in $L_2(G)$ in the sense of the Parseval equation. In general S does not contain all irreducible unitary representations and the theory for L_2 can differ strongly from the theory of almost periodic functions on G. Use is made in the proofs of a decomposition theory for Hilbert spaces and rings of operators by von Neumann and recent results by Segal. The Murray-von Neumann theory of relative trace leads to a theory of group-characters, in particular their completeness in the space of class-functions in L_1. If the number of finite classes of conjugate elements of G is finite then S is finite (though G may have infinitely many inequivalent unitary irreducible representations) and orthogonality relations for group-characters hold. An improvement for discrete groups of the Gelfand-Raikov-Segal existence theorem for irreducible representations follows. (Received September 17, 1947.)

8t. C. E. Rickart: *One-to-one mappings of rings and lattices.*

Let $\mathfrak{R}$ and $\mathfrak{S}$ denote arbitrary rings and let $\phi(r)$ denote a one-to-one, multiplicative mapping of $\mathfrak{R}$ onto $\mathfrak{S}$. If $\mathfrak{R}$ is a Boolean ring, then $\phi(r)$ is additive as well as multiplicative. Also $\phi(r)$ is additive if $\mathfrak{R}$ contains a family $\mathfrak{J}$ of minimal right ideals with the following properties: (1) $Rr=(0)$, for each $R\in\mathfrak{J}$, implies $r=0$; (2) Each R is of dimension greater than one over the division ring of all endomorphisms of R which commute with each endomorphism induced in R through right multiplication by elements of $\mathfrak{R}$. It is also shown that one-to-one, meet preserving mappings of certain distributive lattices onto a distributive lattice are also join preserving. (Received September 15, 1947.)

9t. R. M. Robinson: *The critical numbers for unsymmetrical approximation.*

If ξ is an irrational number, the modulus of approximability from the right, $M^+(\xi)$, is defined as the least upper bound of the values of μ for which the inequality $0<A/B-\xi<1/\mu B^2$ has infinitely many solutions. In a similar way, $M^-(\xi)$ is defined measuring the approximability of ξ from the left. The number ξ is called *critical* if there is no other irrational number ξ' for which $M^+(\xi')<M^+(\xi)$ and $M^-(\xi')<M^-(\xi)$. A sequence of integers $r_1, r_2, r_3, r_4, \cdots$ will be called *derivable* if ultimately just two different numbers occur in the sequence, and these are consecutive integers k and $k+1$. The sequence $s_1, s_2, s_3, \cdots$ giving the number of $k+1$'s between consecutive k's is called the derivative. It is proved that *the number ξ is critical if and only if its expansion as a continued fraction has the form $\xi=[q_0, q_1, \cdots, q_{n-1}, 1, r_1, 1, r_2, 1, r_3, 1, \cdots]$, where the sequence $r_1, r_2, r_3, \cdots$ either has infinitely many derivatives, or its highest derivative is a sequence $t_1, t_2, t_3, \cdots$ such that $t_n\rightarrow k$, where k is a positive integer, or $t_n\rightarrow\infty$*. This paper is a continuation of *Unsymmetrical approximation of irrational numbers*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 351–361. (Received August 22, 1947.)

10. Lowell Schoenfeld: *On the function which enumerates the number of partitions of an integer into distinct kth powers.* Preliminary report.

Let $F_k(x)=\prod_{m=1}^{\infty}(1+x^{m^k})=1+\sum_{m=1}^{\infty}P_k(m)x^m$ and $f_k(x)=\prod_{m=1}^{\infty}(1-x^{m^k})^{-1}=1+\sum_{m=1}^{\infty}p_k(m)x^m$. Then $P_k(m)$ is the number of partitions of m into kth powers with distinct summands and $p_k(m)$ is the number of unrestricted partitions of m into kth powers. Wright (Acta Math. vol. 63 (1934) pp. 143–191) obtained an exact transformation equation for $f_k(x)$ which exhibits the behavior of $f_k(x)$ in the neighborhood of its singularities at the rational points of the unit circle. In this paper, the author obtains a similar equation for $F_k(x)$. This can be done in two ways: either derive this equation directly in a manner similar to that used by the author for $f_k(x)$ (Duke Math. J. vol. 11 (1944) pp. 873–887), or observe that $F_k(x)=f_k(x)/f_k(x^2)$ and use the transformation equation for $f_k(x)$. The second method, though formal in character, is not trivial since as simple an equation as possible is required. While most of Wright's work is applicable to the determination of an asymptotic series for $P_k(m)$, the oscillatory character of certain coefficients makes it difficult at the present time to ascertain the dominant term of this series. (Received October 8, 1947.)

11*t*. A. R. Schweitzer: *Remarks on groups of ordered dyads.*

The author constructs sets of postulates for groups of ordered dyads of which postulates 1, 2, 3 are the same as postulates 1, 2, 4 of a previous paper (Bull. Amer. Math. Soc. Abstract 52-11-352). Definition: $\alpha\beta\times\gamma\delta=\lambda\mu$ means: There exist ξ, η, ζ in S such that $\alpha\beta=\xi\eta$, $\gamma\delta=\eta\zeta$, $\lambda\mu=\xi\zeta$. This definition is effective relative to postulates 1, 2, 3, 4 and 1, 2, 3, 4′, 5′; 4. $\alpha\beta$, $\gamma\delta$ in T imply the existence of ξ, η in S such that $\alpha\beta=\eta\gamma$, $\gamma\delta=\beta\xi$, $\alpha\xi=\eta\delta$. 4′. $\alpha\beta$, $\gamma\delta$ in T imply the existence of ξ, η, ζ in S such that $\alpha\beta=\xi\eta$, $\gamma\delta=\eta\zeta$; $\alpha\beta$, $\alpha'\beta'$, $\alpha''\beta''$ in T imply the existence of ξ, η, ζ, τ in S such that $\alpha\beta=\xi\eta$, $\alpha'\beta'=\eta\zeta$, $\alpha''\beta''=\zeta\tau$. 5′. $\xi\eta=\xi'\eta'$, $\eta\zeta=\eta'\zeta'$ imply $\xi\zeta=\xi'\zeta'$. Basic for the proof of sufficiency of each set of postulates is the theorem: If $\alpha\beta\times\gamma\delta=\lambda\mu$ then $\alpha\beta=\alpha'\beta'$ implies $\alpha'\beta'\times\gamma\delta=\lambda\mu$, $\gamma\delta=\gamma'\delta'$ implies $\alpha\beta\times\gamma'\delta'=\lambda\mu$; $\lambda\mu=\lambda'\mu'$ implies $\alpha\beta\times\gamma\delta=\lambda'\mu'$. The independence of the first part of 4′ is discussed. It is assumed that the relation symbolized by "$=$" is reflexive, symmetric and transitive. Postulates 1, 2, 3, 4′, 5′ permit interpretation of dyads as rational numbers, excluding zero. (Received September 20, 1947.)

12. L. V. Toralballa: *On Newton's interpolation series in n variables.*

The finite difference analog of Taylor's expansion theorem in several variables is the theorem: If $f(x_1, x_2, \cdots, x_n)$ possesses partial derivatives of all kinds and orders at the origin, and moreover the limit as i approaches infinity of the square root of the absolute value of $\partial^i f/\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_n^{i_n}$ is zero for all sets $i_1, i_2, \cdots, i_n$ such that $i_1+i_2+\cdots+i_n=i$, then $f(x_1, x_2, \cdots, x_n)=f(0, 0, \cdots, 0)+(x_1\Delta_{x_1}+x_2\Delta_{x_2}+\cdots+x_n\Delta_{x_n})f(0, 0, \cdots, 0)+(1/2!)(x_1\Delta_{x_1}+x_2\Delta_{x_2}+\cdots+x_n\Delta_{x_n})^{(2)}f(0, 0, \cdots, 0)+\cdots$, where $(x_1\Delta_{x_1}+x_2\Delta_{x_2}+\cdots+x_n\Delta_{x_n})^{(i)}f(0, 0, \cdots, 0)=\sum(i!/\alpha_1!\alpha_2!\cdots\alpha_n!)\cdot x_1^{(\alpha_1)}x_2^{(\alpha_2)}\cdots x_n^{(\alpha_n)}\Delta_{x_1}^{\alpha_1}\Delta_{x_2}^{\alpha_2}\cdots\Delta_{x_n}^{\alpha_n}f(0, 0, \cdots, 0)$, where the summation extends over all the compositions of i, zeros being allowed. This is proved by induction. As a special case one has $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}=\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}(x_1^{(i_1)}/i_1!)(x_2^{(i_2)}/i_2!)\cdots(x_n^{(i_n)}/i_n!)\Delta_{x_1}^{i_1}\Delta_{x_2}^{i_2}\cdots\Delta_{x_n}^{i_n}0^{m_1}0^{m_2}\cdots0^{m_n}$. (Received September 22, 1947.)

13. G. C. Webber: *Non-existence of odd perfect numbers of the form $p^\alpha 3^{2\beta}p_1^{2\beta_1}p_2^{2\beta_2}p_3^{2\beta_3}$.*

Sylvester (C. R. Acad. Sci. Paris vol. 106) proved that an odd integer n, divisible by 3, cannot be perfect unless it contains at least five prime divisors. In this paper it is proved that the number of prime divisors of such n must be at least six. The principal tools used are, first, results concerning the form of the cyclotomic polynomial when that polynomial is divisible by an odd prime; second, certain inequalities obtained by consideration of $\sigma(n)<2n$ and $\sigma(n)>2n$. (Received September 11, 1947.)

14*t*. E. M. Wright: *Equal sums of like powers.*

Let $s\geqq 2$ and let $P(k, s)$ be the least value of j such that the equations $\sum_{i=1}^{j}a_{i1}^{h}=\sum_{i=1}^{j}a_{i2}^{h}=\cdots=\sum_{i=1}^{j}a_{is}^{h}$ $(1\leqq h\leqq k)$ have a nontrivial solution in integers. Prouhet (C. R. Acad. Sci. Paris vol. 33 (1851) p. 225) showed that $P(k, s)\leqq s^k$. The author proves that (i) $P(k, s)\leqq(k^2+k+2)/2$, (ii) if k is odd, $P(k, s)\leqq(k^2+3)/2$, (iii) $P(2, s)=3$, (iv) $P(3, s)=4$. (Received September 25, 1947.)

ANALYSIS

15. R. J. Duffin: *On a question of Hadamard concerning super-biharmonic functions.*

A function w satisfies the equation $\Delta^2 w(x, y) = p(x, y)$ in a region of the plane and vanishes together with its first derivatives on the boundary of the region. Consider then the assertion "$p \geqq 0$ implies $w \geqq 0$." In mechanical language the assertion can be translated to say that if a positive distribution of pressure p is applied to the surface of a thin plate which is clamped on its boundaries, then the displacement w of the plate is nowhere negative. It is shown that the assertion is not true for some regions. Of special interest is the region bounded by two infinite parallel lines. If a positive pressure is applied over a sufficiently limited portion of such a strip, then the resultant displacement at large distances is essentially a damped sine wave whose wavelength and decrement depend only on the breadth of the strip and not on the particular form of the distribution of pressure. (Received September 22, 1947.)

16. Paul Erdős: *On bounded polynomials.*

Let x_i^n $(1 \leqq i \leqq n)$ be a point group in $(-1, 1)$. It has *property* A if for every nth degree polynomial f satisfying $|f(x_i^m)| < 1$ for $m > (1+c_1)n$ we have $|f(x)| < c_2$ in $(-1, 1)$, with c_2 independent of f and n but dependent on c_1. Bernstein proved the roots of the Tchebycheff polynomials, Zygmund the roots of the Legendre polynomials, have property A. The following property is proved necessary and sufficient for property A: if the y_i^n are the projections of the x_i^n upon the upper half of the unit circle then for any sequence of positive numbers a_n approaching infinity and sequence of arcs of length a_n/n the number of y_i^n on the nth arc is greater than $a_n/\pi - o(a_n)$. (Received September 11, 1947.)

17*t*. Roger Godement: *Les fonctions de type positif et la théorie des groupes.*

Let G be a locally compact topological group. A function ϕ on G is said to be of positive type if $\sum_{i,j}\alpha_i\bar{\alpha}_j\phi(s_i^{-1}s_j) \geqq 0$ for all systems of points $(S_1, \cdots, S_n)$ of the group and of complex numbers $\alpha_1, \cdots, \alpha_n$. Continuous functions of positive type are shown to be closely related to representations of G by unitary operators in Hilbert space; by this method, the existence of irreducible unitary representations of G is established. These results duplicate to a large extent those of Gelfond and Raĭkow (*Irreducible unitary representations of arbitrary locally bicompact groups*, Rec. Math. (Mat. Sbornik) N.S. vol. 13 (1943)). The spectral theory of functions of positive type is generalized from the case of the group of real numbers to that of arbitrary locally compact groups, yielding a generalization to the latter case of Beurling's theorem. It is shown that a continuous ϕ of positive type is (in a unique way) the sum of an almost periodic function of positive type and a function ψ of positive type such that the mean value of $|\psi|^2$ is 0. Special properties of functions of positive type belonging to L^2 are studied. (Received October 23, 1947.)

18*t*. J. W. Lawson: *A sufficiency theorem for the Plateau problem.*

A hypersurface V_{n-1} is considered, embedded in the Euclidean space S_n, and the analogue of the Plateau problem set up in parametric form. For the multiple integral problem, a field is constructed using the sufficiency condition which is the main result of the paper. Hypersurfaces satisfying the Euler differential condition being termed minimal, the theorem is: If there exists one point in the space, through which no tangent hyperplane passes, then the minimal hypersurface is minimizing. (Received August 22, 1947.)

19*t*. Norman Levinson: *The asymptotic nature of solutions of linear differential equations.*

Let x denote a column vector. Let $dx/dt=(A+\Phi+R)x$ where (1) A is a constant matrix with simple characteristic roots μ_i; (2) $\Phi(t)$ is a complex matrix such that $\Phi\rightarrow 0$ as $t\rightarrow\infty$ and the elements of Φ are all of bounded variation over $(0, \infty)$; (3) the elements of the complex matrix, R, are all absolutely integrable $(0, \infty)$. Let the characteristic roots of the matrix $A+\Phi$ be denoted by $\lambda_i(t)$. If the real parts of μ_i are all distinct (or if certain less restrictive conditions are met) then there exist n independent vectors $x^{(k)}(t)$ each a solution of the system such that $x^{(k)}(t)\sim C^{(k)} \exp\left(\int_0^t\lambda_k(t)dt\right)$ as $t\rightarrow\infty$. Each $C^{(k)}$ is a characteristic vector of A associated with the root μ_k. (Received August 29, 1947.)

20*t*. G. G. Lorentz: *Tauberian theorems and Tauberian conditions.*

Let $\sigma_m=a_{m1}s_1+a_{m2}s_2+\cdots$ be a regular Silverman-Toeplitz sequence-to-sequence transformation A by which a sequence s_n is evaluable A to σ if $\lim \sigma_n=\sigma$. A sequence $n_1<n_2<\cdots$ of positive integers determines a gap condition: $u_n=0$ when $n\neq n_1, n_2, \cdots$. A sequence $c_1, c_2, \cdots$ of positive constants determines an order condition: $u_n=o(c_n)$. Relations among the sequences n_k and c_k are determined such that if $\sum u_n$ converges whenever $\sum u_n$ is evaluable A and satisfies the gap (or order) condition, then $\sum u_n$ converges whenever $\sum u_n$ is evaluable A and satisfies the order (or gap) condition. For the Cesàro methods and the Euler-Abel power series method, there is given a characterization of the sequences c_k for which the order condition $u_n=O(c_n)$ is a Tauberian condition which, with summability, implies convergence. (Received September 30, 1947.)

21*t*. M. E. Munroe: *Homomorphisms on Banach spaces.*

If G is a closed linear subspace of the Banach space E, then G^* (the conjugate space to G) is algebraically isomorphic to the factor space E^*/Γ where $\Gamma=\{f|f(x)=0$ for $x\in G\}$. The natural homomorphism $T(E^*)=G^*$ is continuous and open when E^* has its norm topology and G^* has its norm topology. The above results are already known. It is shown that T is continuous and open with respect to the weak and weak* topologies in E^* and G^*. T is not closed for any of the usual topologies. In fact, it does not even map all closed convex sets into closed sets. However, T preserves weak* closure of linear subspaces. Remarks on weak and weak* convergence are appended, including an example to show that in any infinite-dimensonal E^* there is a directed set weak* convergent to zero but containing no bounded subset having zero as a point of accumulation. The same thing is true of weak convergence. (Received September 11, 1947.)

22. J. C. Oxtoby: *On the ergodic theorem of Hurewicz.*

Let μ be a measure in a set S and assume that S is σ-finite. Let T be a 1:1 measurable transformation of S onto itself, and let $F(X)$ be a countably additive finite function of a measurable set. Write $F_n(X)=\sum_0^n F(T^iX)$, $\mu_n(X)=\sum_0^n\mu(T^iX)$, and let $\phi_n(x)$ be a point function such that $\int_X\phi_n d\mu_n=F_n'(X)$, where F_n' is the absolutely continuous part of F_n with respect to μ_n. Such a function exists by the Radon-Nikodym theorem.

It is shown (1) that in any case the limit $\phi(x)=\lim_{n\to\infty}\phi_n(x)$ exists except on a set N such that $\mu(T^iN)=0$ for $i\geq 0$, and (2) that if $\sum_0^\infty \mu(T^iX)=0$ or ∞ for each measurable set X, then $\mu(T^iN)=0$ for every integer i, $\phi(x)$ is integrable, and $\phi(x)=\phi(Tx)$ except on a set whose images all have measure zero. If in addition $F(X)=0$ for every invariant set of measure zero, then $\int_X\phi d\mu=F(X)$ for every invariant set of finite measure. Result (2) generalizes Hurewicz' theorem (Ann. of Math. (2) vol. 45 (1944) pp. 192–206) and directly implies the ergodic theorem of Halmos (Proc. Nat. Acad. Sci. U.S.A. vol. 32 (1946) pp. 156–161, Theorem 1). It is derivable from either of these theorems. The proof of (1) requires supplementary considerations. (Received September 10, 1947).

23. Everett Pitcher: *A proof of lower semicontinuity.*

A short direct proof is made of a general theorem of McShane (Duke Math. J. vol. 2 (1936) pp. 597–616, Theorem 3.1) on lower semicontinuity of simple integrals of the calculus of variations. (Received September 19, 1947.)

24. Arthur Sard: *The remainder in approximations by moving averages.*

Let $g(s, t)$ be a function which, for each real number t in a set T, is of bounded variation in s on each finite s interval. Given any function $x(s)$, put $y(t)=\int_{-\infty}^{\infty}x(s)d_sg(s, t)$ and $R[x]=x(t)-y(t)$, $t\in T$. Integrals on infinite ranges are to be understood either as Lebesgue-Stieltjes integrals or as improper Lebesgue-Stieltjes integrals. $R[x]$ exists if $y(t)$ and $x(t)$ exist and are finite for each $t\in T$. Assume that $R[x]$ exists and vanishes whenever $x(s)$ is a polynomial of degree $n-1\geq 0$. Put $k(s', t)=R[\psi_{s'}]$, $\psi_{s'}(s)=0$ if $s\leq s'$, $\psi_{s'}(s)=p(s, s')$ if $s>s'$, $p(s, s')=(s-s')^{n-1}/(n-1)!$. Let $x(s)$ be a function with absolutely continuous $(n-1)$th derivative on each finite s interval. Put $R^*[x]=\int_{-\infty}^{\infty}x^{(n)}(s')k(s', t)ds'$ and $I=\int_{-\infty}^{\infty}d_sg(s, t)\int_0^s p(s, s')x^{(n)}(s')ds'$. A necessary and sufficient condition that $R[x]$ and $R^*[x]$ exist and be equal is that I exist and that the order of integration in I be invertible, $t\in T$. A sufficient condition is that $\int_{-\infty}^{\infty}|x^{(n)}(s')|M(s', t)ds'$ be finite, $t\in T$, where $M(s', t)=\int_{-\infty}^{s'}|p(s, s')||d_sg(s, t)|$ if $s'\leq 0$, $M(s', t)=\int_{s'}^{\infty}|p(s, s')||d_sg(s, t)|$ if $s'>0$. The sufficient condition also implies that $R^*[x]$ exists as a Lebesgue-Stieltjes integral. (Received September 19, 1947.)

25. I. E. Segal: *Operator algebras associated with locally compact groups.* Preliminary report.

Certain uniformly closed self-adjoint algebras of bounded operators on Hilbert spaces, determined either by a locally compact group or by a locally compact group of transformations of a locally compact space, with an invariant measure, are studied, from a viewpoint suggested by quantum kinematics. Some results for arbitrary groups are obtained, and it is also shown that the self-adjoint elements of the algebra determined by the covering group of the inhomogeneous de Sitter group constitute a model for the bounded observable associated with an elementary particle, with the following properties: (1) the usual relativistic and non-relativistic models are limiting cases, (2) space-time is relativistically invariant and space is discrete, (3) the rest-mass and spin of the particle are automatically determined. (Received September 2, 1947.)

26*t*. J. E. Wilkins: *The isoperimetric problem of Bolza with finite side conditions.*

In view of recent improvements in the theory of the problem of Bolza, due primarily to McShane and Hestenes, it is now possible to give a much simpler treatment of the problem of Bolza with auxiliary finite conditions than that originally given by Bower (*The problem of Lagrange with finite side condition*, Contributions to the calculus of variations 1933–1937, University of Chicago Press). (Received September 15, 1947.)

27*t*. G. M. Wing: *Summability with a governor of integral order.*

The series $\sum_{j=0}^{\infty} a_j$ is said to be summable by means of a governor of order k (k a positive integer), or (G, k) summable, if $\lim_{n\to\infty} \sum_{j=0}^{n} s_j p_{n-j}(k)/p_n(k+1) = \sigma$ exists and $\lim_{n\to\infty} p_n(k)/p_n(k+1) = 0$, where $s_n = \sum_{j=0}^{n} a_j$, $p_n(0) = |a_n|$, and $p_n(k) = \sum_{j=0}^{n} p_j(k-1)$ $(k = 1, 2, \cdots)$. The method (G, k) reduces to the method (G) of G. Piranian (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 882–889) in the case $k = 1$, and is shown to preserve all of the characteristic properties of that method. It is proved that (G, k) includes the Cesàro method of order k, and several conditions are given under which the new method is stronger than the corresponding Cesàro mean. One of these yields a new proof of a theorem of Piranian on (G) summability of certain series derived from polynomials. An example is given to show that there exists no integer k' such that all Cesàro summable series are (G, k') summable. (Received August 29, 1947.)

APPLIED MATHEMATICS

28*t*. Stefan Bergman: *An inversion formula for the integral operator of the second kind in the theory of a compressible fluid.*

The stream function $\psi(\lambda, \theta)$ of a compressible fluid flow (see NACA, TN972) can be represented in the region $[|\lambda| < 3^{1/2}|\theta|]$ in the form $\psi = \operatorname{Im} \int_{C_1} \mathrm{E}(Z, \bar{Z}, t) \cdot f(2^{-1}Z(1-t^2))dt/(1-t^2)^{1/2}$. Here $Z = \lambda + i\theta$, $\bar{Z} = \lambda - i\theta$, $\mathrm{E} = A_1\mathrm{E}^{(1)} + [2^{-1}Z(1-t^2)]^{1/3} \cdot A_2\mathrm{E}^{(2)}$, where A_κ are suitably chosen (complex) constants satisfying the condition $\operatorname{Im}\{i^{-4/3}A_1\bar{A}_2\} \neq 0$, and $\mathrm{E}^{(\kappa)} = \sum_{n=0}^{\infty} q^{(n,\kappa)}/(-t^2Z)^{n-1/2+2\kappa/3}$, $q^{(n,\kappa)}$ satisfying the equations $(n+2/3)q_Z^{(n,\kappa)} + q_{Z\bar{Z}}^{(n+1,\kappa)} + Fq^{(n+1,\kappa)} = 0$, $n = 0, 1, 2, \cdots$; C_1 is a curve in the complex $t-1$ plane connecting $t = -1$ and $t = 1$. Let $\lim_{\lambda\to 0} \psi = \chi_1(\theta)$, $\lim_{\lambda\to 0}(\partial\psi/\lambda M) = \chi_2(\theta)$. The line $\lambda = 0$ (that is, $M = 1$) is the transonic line. Then f can be expressed in terms of the χ_{κ}, namely, $f(\zeta) = 3^{1/2}[2\pi \operatorname{Im} (D_1\bar{D}_2)]^{-1}\{(-2i\zeta)^{7/6}\bar{D}_2\int_0^1 \chi'(\sigma)\tau^{-1/3}d\tau + (-2i\zeta)^{1/6} \cdot \sum_{\kappa=1}^{2} \bar{D}_\kappa \int_0^1 [\chi_\kappa(\sigma) - \alpha_0^{(\kappa)}](1-\tau)^{-1}\tau^{-2/3}d\tau - 3^{-1}(-2i\zeta)^{-1/6} \sum_{\kappa=1}^{2} \bar{D}_\kappa \int_0^1 [\chi^*(\sigma) - 3\alpha_0^{(\kappa)}] \cdot (1-\tau)^{-4/3}\tau^{-2/3}d\tau + (-2i\zeta)^{1/6}[\bar{D}_0\alpha_0^{(\kappa)} - \sum_{\kappa=1}^{2}\bar{D}_\kappa\alpha_0^{(\kappa)}]\}$, where $\chi_\kappa^*(\sigma) = \int_0^\sigma \omega^{-2/3}\chi_\kappa(\omega)d\omega$, $\sigma = -2i\zeta(1-\tau)$, and the D_κ and $\alpha_0^{(\kappa)}$ are suitably chosen constants. (Received October 25, 1947.)

29. Lipman Bers: *Subsonic gas flows past a straight cascade.* Preliminary report.

The author deals with a subsonic gas flow past a straight cascade of profiles. The gas is assumed to satisfy Chaplygin's simplified density-speed relation. The problem of finding the flow is reduced to a mapping problem involving the domain exterior to the profiles of the cascade and a certain standard Riemann surface. The mapping problem is shown to be equivalent to a nonlinear integral equation, similar to the one occurring in the theory of a single airfoil (cf. Bers, NACA TN No. 1006, 1946). Under the assumption that the profiles are convex, it is shown that the integral equation possesses a solution. (Received October 23, 1947.)

30*t*. Eric Reissner: *Note on the membrane theory of shells of revolution.*

The solution of the equilibrium equations of the membrane theory of shells of revolution is expressed in terms of a stress function which is slightly more general than Neményi's stress function (Bygningstatiske Meddelelser (1936)). The essential difference between Neményi's stress function and Pucher's stress function (Beton und Eisen vol. 33 (1934)) is set in evidence. It appears that Neményi's function is somewhat more convenient for the applications than is Pucher's function. Finally, a formula is derived which expresses explicitly Pucher's function in terms of Neményi's function. (Received September 9, 1947.)

GEOMETRY

31*t*. Reinhold Baer: *Projectivities of finite projective planes.*

Consider a projectivity p of the finite projective plane P in which the theorem of Desargues may or may not be valid. Denote by m the order of the projectivity p, by $n+1$ the number of points on a line in P and by F the set of fixed elements of the projectivity p. The author obtains relations between the arithmetic properties of m and n and the geometric properties of the configuration F. A typical example of such a relation is the following proposition: If m is a power of a prime q, and if F is not a complete subplane of P, then q is a divisor of $(1+n+n^2)n(n^2-1)$. (Received August 12, 1947.)

32*t*. Salomon Bochner: *Curvature and Betti numbers.*

If on a compact space S_n with a positive definite Riemannian metric the Ricci (mean) curvature is positive, and if the space is conformally flat, or more generally if the deviation from conformal flatness is a small fraction of the Ricci curvature, then all Betti numbers vanish, $B_p=0$, $1\leq p\leq n-1$. On the other hand, for n even, it is easy to give a criterion under which $B_{2l}\geq 1$, $l=1, 2, 3, \cdots$. (Received September 17, 1947.)

33*t*. Edward Kasner: *Trajectories and catenaries.*

The author discusses analogies and differences between the theories of general trajectories and general catenaries in an arbitrary field of force. In each case, the focal locus is a circle. This circle becomes a straight line only when the field is constant or elastic. However, while the angular ratio is 1:1 for trajectories, it becomes 3:2 for catenaries. Again the fundamental ratio 1:3 for special (rest) trajectories becomes 1:2 for special catenaries in any field. Analogous results exist for brachistochrones, for velocity curves, and for systems S_k. The locus of the centers of all the focal circles for a given lineal element is determined. (Received September 26, 1947.)

LOGIC AND FOUNDATIONS

34*t*. Ira Rosenbaum: *Enumeration of first order classes, ordered q-ads, and q-adic relations in a universe of m objects* $(1, 2, \cdots, m)$.

Let $a^b \vdots (c-1)_m$ be the coefficient of a^b in the m-adic number-system expression of $c-1$. Define the nth ordered q-ad of an m-membered universe as follows: $q_n^m=(\{n, 1\}, \{n, 2\}, \cdots, \{n, q\})$ where $\{n, k\}=1+(m^{k-1} \vdots (n-1)_m)$ and $1\leq n\leq m^q$. Defining q-adic relations (predicates, propositional functions) extensionally as classes of ordered q-ads, $R_n^{q(m)}=(\{n, 1\}, \{n, 2\}, \cdots, \{n, m^q\})$, where now $\{n, k\}=1+(2^{k-1} \vdots (n-1)_2)$, and k denotes the ordered q-ad, q_k^m, and this q-ad is to be a member of $R_n^{q(m)}$ if and only if $\{n, k\}=1$. Here $1\leq n\leq 2^{m^q}$. A 1–1 relation is thus established between the numbers

1, 2, $\cdots$, m^q and the m^q ordered q-ads and between the numbers 1, 2, $\cdots$, 2^{m^q} and the 2^{m^q} q-adic relations definable in an m-membered universe. By methods like those of Bull. Amer. Math. Soc. Abstract 53-5-265, it is possible to determine the number n of any order q-ad whose elements and order are known, and the number n of any q-adic relation whose member q-ads are known. For $q=1$, the formula for $R_n^{q(m)}$ defines the nth class of individuals in an m-membered universe. $R_n^{q(m)}(q_k^m)$ is analytic or contradictory as $\{n, k\}$ is 1 or 2, that is $R_n^{q(m)}(q_k^m)=\{n, k\}$. (Received August 18, 1947.)

35*t*. Ira Rosenbaum: *On a method of determining the nth rational p-valued q-adic relation in a universe with* $\aleph_0$ *elements.*

Both ordered q-ads of a universe with $\aleph_0$ elements and the rational numbers are effectively enumerable, the enumeration of the latter proceeding in familiar fashion, the fraction k/t occurring in the $[(k+t-1)(k+t-2)/2+k]$th place, fractions equal to a preceding fraction being considered subsequently struck from the array. A denumerably infinite array of columns, each containing a denumerable infinity of entries, may be envisaged, the value $\{n, r\}$ in the rth row and nth column being the rth digit in the p-adic decimal representation of the nth rational number. The denumerably infinite set of values $\{n, r\}$ obtained for fixed n and variable r defines the nth rational p-valued q-adic relation, $\{n, r\}$ indicating the degree in which the rth ordered q-ad is a member of the nth rational relation. Since every rational number when converted into a decimal, in general, becomes a terminated, or a pure repeating, or a mixed repeating decimal, one can represent the table of the nth rational p-valued q-adic relation in abbreviated form. Thus the table of the 18th rational 10-valued q-adic relation is $\dot{1}4285\dot{7}$. (Received September 8, 1947.)

36*t*. Ira Rosenbaum: *On a method of determining the number n of any arbitrary rational p-valued q-adic relation defined by a table.*

Any denumerably infinite set of integers coinciding with the digits of a terminated, pure or mixed repeating p-adic decimal may be taken as defining a rational p-valued q-adic relation of a universe with a denumerable infinity of individuals. To determine the number n of the rational p-valued q-adic relation in question requires two major steps, (1) transformation of the decimal into a fraction, this being reduced to its lowest terms, and (2) determination of the order number n of the resulting rational number in the enumeration of rational numbers. The number n of the resulting rational, say k/t, will be not greater than $[(k+t-1)(k+t-2)/2+k]$ and its determination hence a finite process. The transformation of the decimal into a fraction utilizes only known mathematical methods, for example, when $p=10$, the transformation of a mixed repeating decimal into a fraction is accomplished by multiplying the integer formed by the non-repeating digits by that formed by the repeating digits and then subtracting the former integer, subsequently dividing the result by $(10^r-1)10^s$ where s is the number of non-repeating digits, r the number of repeating digits. Related methods suffice in other cases. Reduction to lowest terms is also, generally, readily accomplished. (Received September 8, 1947.)

37*t*. Ira Rosenbaum: *On a method of enumerating ordered q-ads in a universe with a denumerable infinity* ($\aleph_0$) *of elements.*

In a universe with a denumerable infinity ($\aleph_0$) of elements, the set of ordered q-ads, for every finite integer q, is denumerable, since the number of q-ads is then $\aleph_0^q$ and $\aleph_0^q=\aleph_0$. Each q-ad may be regarded as a point with integral coordinates in a space of q dimensions. In order to enumerate the ordered q-ads of a universe of the type in question one first writes down the integers in the single q-partition of q, thus obtaining the first ordered q-ad. One then determines the number of q-partitions of $q+1$, the actual q-partitions of $q+1$, an order among these q-partitions, the permutations of each partition, an order among these permutations, and then lists the 1st, 2nd, $\cdots$ permutations of the first q-partition of $q+1$, of the second q-partition of $q+1, \cdots$, of the last q-partition of $q+1$; and similarly for $q+n$, $n=2, 3, 4, \cdots$. In this way every ordered q-ad will be listed once and only once. One may assign an order number to each ordered q-ad as it is written down and hence speak of the nth ordered q-ad in a universe with $\aleph_0$ elements. The above process embraces that of Gödel (Monatshefte für Mathematik und Physik vol. 37 (1930)). (Received September 8, 1947.)

38t. Ira Rosenbaum: *On absolute junctions of p-valued q-adic relations in an m-membered universe.*

A set of m^q integers, W_i, $1 \leq W_i \leq p$, may be regarded as defining a q-adic p-valued relation of an m-membered universe; each of the m^q integers W_i indicates the degree in which the ith ordered q-ad q_i^m is a member of the relation; the number n of the relation is determinable from the condition $n-1=\sum_{j=0}^{c} W_j \cdot p^{c-1-j}$ where $c=m^q$. Conversely, given the number n of a q-adic p-valued relation, the integer in the kth row of its table is $\{n, k\}=1+(p^{k-1} \vdots (n-1)_p)$, $1 \leq k \leq m^q$. One has: $R_n = R_s \cdot = \cdot (k)(\{n, k\}=\{s, k\})$, $R_n \subset R_s \cdot = \cdot (k)(\{n, k\} \geq \{s, k\}) R_n \cap R_s \cdot = \cdot R_i$, $i=2x(k)(\{x, k\}=\max(\{n, k\}, \{s, k\}))$, $R_n \cup R_s \cdot = \cdot R_j$, $j=2x(k)(\{x, k\}=\min(\{n, k\}, \{s, k\}))$, $-R_n=R_{p^c-n+1}$. For $q=1$, the definitions reduce to those of p-valued class-junctives; for $p=2$, to those of the traditional (two-valued) theory of classes and relations. For $p=2$, $q=1$, classes defined by a table with only a single "1" in them are *unit* classes; for $q \geq 2$, similarly, we obtain unit-relations (that is, those with only a single q-ad as member); and similarly for $p \geq 3$. (Received September 8, 1947.)

39t. Ira Rosenbaum: *On converses, relative products and sums, powers and multiples of p-valued binary relations in an m-membered universe.*

Given two arbitrary p-valued binary relations R_n and R_t, letting $\{u, v\}=1+(p^{v-1} \vdots (u-1)_p)$, $x=i+[r/m]$ (where $[r/m]$ denotes the integral part of r/m) unless r/m is an integer s in which case $x=i+m(s-1)$, $y=\rho(r, m)+m(i-1)$ (where $\rho(r, m)$ denotes the remainder obtained on dividing r by m, unless r is exactly divisible by m, when $\rho(r, m)=m$) and adapting and generalizing a device of Schroeder's, one obtains (for the m^2 values of r $(=1, 2, \cdots, m^2)$ and hence of x and y) the m^2 values defining the relative product R_n/R_t and relative sum $R_n \oplus R_t$ by use of the formulae $\sum_{i=1}^{m}(\{n, x\} \cap \{t, y\})$ and $\prod_{i=1}^{m}(\{n, x\} \cup \{t, y\})$ respectively, $\sum$ and $\prod$ denoting p-valued logical sums and products. Using the m^2 values defining R_n/R_t and $R_n \oplus R_t$, determine the numbers of these tables (and relations) by previous methods. Letting $t=n$, one obtains the tables of the *square* R_n^2 and double $2R_n$, of R_n, from the formulae for relative product and sum respectively. Since $R_n^{k+1}=R_n^k/R_n$ and $(k+1)R_n=kR_n \oplus R_n$, powers and multiples of relations are definable generally. Letting $\rho(r, m)$ and $[r/m]$ have their previous significance, the converse, $Cnv'R_n$, of R_n has the table whose values, for the m^2 values of r, are given by $\{n, [r/m]+1+m(\rho(r, m)-1)\}$ and with these values the number of $Cnv'R_n$ is obtainable as previously. (Received September 5, 1947.)

40*t*. Ira Rosenbaum: *On p-valued q-adic relations in a universe with a denumerable infinity ($\aleph_0$) of elements.*

Although the set of ordered q-ads in a universe with $\aleph_0$ elements is denumerably infinite and, indeed, effectively so, the set of p-valued classes and p-valued q-adic relations is nondenumerably infinite, possessing the power of the continuum. This is easily seen since the number of p-valued q-adic relations in the universe in question is $p^{\aleph_0^q} = p^{\aleph_0} = 2^{\aleph_0}$. It follows that tables defining the p-valued q-adic relations cannot be completely enumerated. One may, however, consider a 1–1 correlation to exist between the real numbers of the interval $0 \leqq x \leqq 1$ and the set of p-valued q-adic relations. One may consider the relation correlated to a given real number, of the range indicated, to be defined by the successive digits in the p-adic decimal representation of the real number in question, the rth digit in the decimal indicating the degree in which the rth ordered q-ad is a member of the relation correlated with the real number. If the real number correlated with a p-valued q-adic relation is irrational, one may term the relation itself *irrational*; otherwise *rational*. The values in the tables defining irrational p-valued q-adic relations can be but incompletely determined, those of rational relations completely determined. (Received September 8, 1947).

41*t*. Ira Rosenbaum: *On the nth q-adic p-valued relation in an m-membered universe.*

Defining the kth ordered q-ad of an m-membered universe by the relation $q_k^m = (\{k, 1\}, \cdots, \{k, q\})$ where $\{k, i\} = 1 + (m^{q-i} \vdots (k-1)_m)$, generalize the definitions of classes and relations given in a preceding abstract. There any given individual (or ordered q-ad) either was, or was not, a member of a given class (or relation); so that an essentially two-valued theory of classes and relations resulted. The admission of more than these two possibilities (complete membership, complete non-membership) leads to many-valued theories of classes and relations as in the corresponding case of propositional logic. Let 1 denote complete membership, p complete non-membership, and intermediate integers intermediate degrees of membership. Then for the nth q-adic p-valued relation of an m-membered universe $R_n^{p,q(m)} = (\{n, 1\}, \{n, 2\}, \cdots, \{n, m^q\})$ where $\{n, k\} = 1 + (p^{k-1} \vdots (n-1)_p)$, k denotes the ordered q-ad q_k^m and this q-ad is a member of R_n in the degree indicated by $\{n, k\}$. For $q = 1$, the nth p-valued class of individuals in an m-membered universe is obtained. In all cases $R_n^{p,q(m)}(q_k^m) = \{n, k\}$. Given n^q integers i, $1 \leqq i \leqq p$, the number n of the q-adic p-valued relation defined by them is determinable as in Bull. Amer. Math. Soc. Abstract 53-5-265. (Received August 19, 1947.)

42*t*. Ira Rosenbaum: *The nth genus of rth order k-adic predicate in the functional calculi of higher order.*

The genus of an rth order k-adic predicate is here defined so as to depend solely on the pattern of orders of the arguments of the function. It is less complex, more general, than the concept of logical type. The nth genus of rth order k-adic predicate is either (1) undefined, or (2) is defined as that in which the order of the ith argument, for each i such that $1 \leqq i \leqq k$, is specified by $\{n, i\} = 1 + (r^{k-i} \vdots (n-1)_r)$. It is undefined or defined according as (1) for no value of i is $\{n, i\} = r-1$ or (2) for some i, $\{n, i\} = r-1$ and for no i is $\{n, i\} > r-1$. Given a set of k arguments of suitable orders the genus-number n of a function of those arguments is determinable. The number $G_{r,k}$ of genera of rth order k-adic functions is given by $G_{r,k} = r^k - (r-1)^k$ and by two recursive relations: $G_{r,k} = rG_{r,k-1} + (r-1)^{k-1}$ and $G_{r,k} = G_{r-1,k} + r^k - 2(r-1)^k$

$+(r-2)^k$. The number of genera of rth order functions of degree not greater than k is given by $\sum_{t=1}^{r} G_{r,t}=((r^{k+1}-1)/r-1)-(r-1)^{k+1}/(r-2)$. The number of genera of k-adic functions of orders not greater than r is given by $\sum_{t=1}^{r} G_{t,k}$ which is algebraically calculable with the help of the Euler-Maclaurin sum formula for $\sum_{m=1}^{n} m^k$. (Received August 18, 1947.)

STATISTICS AND PROBABILITY

43t. Mark Kac: *On the distribution of certain Wiener functionals.* Preliminary report.

Let $x(t)$, $x(0)=0$, $0\leq t<\infty$, be elements of the Wiener space and $V(x)$ a non-negative piecewise continuous function satisfying certain additional conditions. Let $\sigma(\alpha; t)$ be the probability (Wiener measure) that $\int_0^t V(x(\tau))d\tau<\alpha$. It is shown that the double real Laplace transform $\int_0^\infty\int_0^\infty \exp\,(-u\alpha-st)d\sigma_\alpha(\alpha;\,t)dt$ is equal to $\int_{-\infty}^{\infty}\psi(s,\,u;\,x)dx$, where $\psi(s,\,u;\,x)$ is the (unique) fundamental solution of the differential equation $\psi''/2-(s+uV(x))\psi=0$ subject to the conditions $\psi(x)\rightarrow 0$ as $x\rightarrow\pm\infty$, $|\psi'(x)|<M$, $\psi'(-0)-\psi'(+0)=2$. The special cases $V(x)=x^2$, $V(x)=|x|$, $V(x)=(1+\text{sign } x)/2$ treated elsewhere by a somewhat different method are used as illustrations of the present method. Generalizations to more-dimensional Wiener processes and to Markoffian processes other than that of Wiener are indicated. (Received August 29, 1947.)

TOPOLOGY

44. E. H. Spanier: *Borsuk's cohomotopy groups.*

Let X be a compact topological space with dim $X<2n-1$. Borsuk showed how the homotopy classes of continuous maps of X into the n-sphere S^n form a group $\pi^n(X)$, called the nth cohomotopy group of X. In a similar manner the relative Borsuk group $\pi^n(X,\,A)$ of a pair consisting of a compact space X and closed subset A is defined. The elements of this group are the classes of maps of X into S^n which map A into a point. A coboundary operator, which is a homomorphism from $\pi^n(A)$ to $\pi^{n+1}(X,\,A)$, is defined, as well as the homomorphism from $\pi^n(Y,\,B)$ to $\pi^n(X,\,A)$ induced by a continuous map from $(X,\,A)$ to $(Y,\,B)$. The cohomotopy groups with these auxiliary concepts satisfy the Eilenberg-Steenrod axioms for cohomology groups except that the cohomotopy groups are defined only for those integers n such that $n>(\text{dim } X+1)/2$. The Hopf theorems about the classes of maps of an n-dimensional space into S^n and the Steenrod theorems about the classes of maps of an $(n+1)$-dimensional space into S^n are proved in the framework of the cohomotopy groups. These results are summarized in a mixed exact sequence involving cohomotopy and cohomology groups (Received September 10, 1947).

45t. A. H. Stone: *On products of paracompact metric spaces.*

A Hausdorff space is *paracompact* (J. Dieudonné, *Une généralisation des espaces compacts*, J. Math. Pures Appl. vol. 23 (1944) pp. 65–76) if it has arbitrarily fine neighborhood-finite open coverings. It is proved that a metrisable space is paracompact if and only if it has a basis for its open sets consisting of a countable number of families of open sets, the sets in each family being pairwise disjoint. The following two theorems are among those deduced from this characterization. A necessary and sufficient condition for a product of nonempty paracompact metric spaces to be paracompact is that all but a countable number (at most) of the factors be compact. For the product of a given metric space X with *every* compact Hausdorff space to be normal, it is necessary that X be paracompact; sufficiency here was proved by Dieudonné. (Received September 26, 1947.)

46. Hing Tong (National Research Fellow): *Some characterizations of normal and perfectly normal spaces.* Preliminary report.

Let R be a topological space. The following theorems are proved: (1) A necessary and sufficient condition for R to be normal is that if ψ and ϕ are respectively upper and lower semicontinuous functions over R such that $\psi \leqq \phi$ (for every point in R) there is a function ρ continuous over R so that $\psi \leqq \rho \leqq \phi$ (for every point in R). (2) The following properties concerning R are equivalent: (α) R is perfectly normal. (β) for every closed set A in R there is a function ϕ continuous over R such that the set of zeros of ϕ is precisely A. (γ) Every upper semicontinuous function over R is the limit of a monotonically decreasing sequence of continuous functions over R. (Received September 20, 1947.)

T. R. Hollcroft,
Associate Secretary

THE NOVEMBER MEETING IN ST. LOUIS

The four hundred twenty-ninth meeting of the American Mathematical Society was held at Washington University, St. Louis, Missouri, on Friday and Saturday, November 28–29, 1947. All sessions were held in Brown Hall. The total attendance was about 150, including the following 115 members of the Society:

V. W. Adkisson, J. J. Andrews, W. L. Ayres, Reinhold Baer, R. W. Ball, M. A. Basoco, P. O. Bell, Herman Betz, H. D. Block, L. M. Blumenthal, D. G. Bourgin, H. R. Brahana, P. B. Burcham, R. H. Bruck, A. V. Bushkovitch, R. H. Cameron, J. E. Case, Abraham Charnes, Harold Chatland, H. J. Cohen, Byron Cosby, M. E. Daniells, M. M. Day, Hubert Delange, J. L. Doob, T. L. Downs, Otto Dunkel, John Dyer-Bennet, J. D. Elder, Benjamin Epstein, Paul Erdos, D. H. Erkiletian, G. M. Ewing, J. W. Gaddum, R. E. Gaskell, B. E. Gatewood, Michael Golomb, S. H. Gould, Cornelius Gouwens, L. M. Graves, R. E. Graves, Franklin Haimo, Marshall Hall, O. H. Hamilton, Charles Hatfield, F. F. Helton, G. A. Herr, D. L. Holl, L. A. Hostinsky, F. B. Jones, L. M. Kelly, Marie Kernaghan, L. A. Knowler, J. C. Koken, Max Kramer, M. Z. Krzywoblocki, Joseph Landin, J. S. Leech, Solomon Lefschetz, Walter Leighton, A. J. Lorenz, W. S. Loud, Szolem Mandelbrojt, C. W. Mathews, Kenneth May, S. W. McCuskey, W. C. McDaniel, J. V. McKelvey, R. J. Michel, Josephine Mitchell, E. E. Moise, F. V. E. Morfoot, K. L. Nielsen, Rufus Oldenburger, Isaac Opatowski, Gordon Pall, S. T. Parker, P. M. Pepper, B. J. Pettis, G. B. Price, M. O. Reade, Francis Regan, T. M. Renner, P. R. Rider, R. E. Roberson, W. H. Roever, Arthur Rosenthal, R. G. Sanger, A. C. Schaeffer, H. M. Schaerf, Robert Schatten, K. C. Schraut, M. G. Segner, M. E. Shanks, Marlow Sholander, Annette Sinclair, M. F. Smiley, G. W. Smith, C. E. Springer, R. H. Stark, M. H. Stone, E. B. Stouffer, H. P. Thielman, W. J. Thron, E. A. Trabant, H. L. Turritin, F. E. Ulrich, G. B. Van Schaack, W. A. Vezeau, Bernard Vinograde, G. L. Walker, S. E. Walkley, S. E. Warschawski, M. F. Willerding, L. C. Young.

At 10:00 A.M. Friday Professor Marshall Hall of the Ohio State University delivered an hour address entitled *Foundations of geometry—retrospect and prospect.* Professor Reinhold Baer presided. At 9:00 A.M. Saturday Professor Szolem Mandelbrojt of the College de France and the Rice Institute delivered an hour address entitled *Analytic continuation and infinitely differentiable functions—a general principle*, with Professor L. C. Young presiding. Two sessions for short research papers were held at 11:15 A.M. Friday, and additional such sessions at 2:00 P.M. Friday and at 10:15 A.M. Saturday. Presiding officers for these sessions were, respectively, Professors Gordon Pall, L. M. Blumenthal, M. H. Stone, and G. B. Price.

An informal gathering of midwestern applied mathematicians was arranged for lunch on Friday for the purpose of a preliminary discussion of the future needs and plans for applied mathematics in the Midwest. It was the sense of the group that no specific conclusions should be reached at that time. A committee of three midwestern applied mathematicians was elected by those present, with instructions to consider, in cooperation with the Committee on Applied Mathematics of the American Mathematical Society, what questions, if any, might warrant further discussion by a more representative group of midwestern applied mathematicians, and to initiate such a discussion if deemed desirable.

Those attending the meetings were the guests of Washington University at a very successful tea, held at 4:00 P.M. Friday in the Women's Building. At 6:30 P.M. Friday, a dinner was held in the dining room of McMillan Hall, with an attendance of 112. Professor J. E. Case was toastmaster. Assistant Dean Graham welcomed the Society on behalf of Washington University. Dean Stouffer expressed the appreciation of the Society for the hospitality of Washington University, and warmly commended Professor P. R. Rider and the other members of the Local Committee on Arrangements for their very efficient work.

Abstracts of all papers presented at the meeting are given below. Papers read by title are indicated by the letter "*t*." Paper number 48 was presented by Professor Chatland, number 54 by Mr. Block, number 58 by Professor Ulrich, number 72 by Mr. Kelly, and number 82 by Dr. Moise.

ALGEBRA AND THEORY OF NUMBERS

47*t*. A. A. Albert: *Power-associative rings*. II.

A power-associative algebra A over a field F is trace-admissible when there is a function $t(x, y)$ on A, A to F such that $t(x, y)=t(y, x)$, $t(xy, z)=t(x, yz)$, $t(x, y)=0$ if xy is nilpotent, $t(e, e)\neq 0$ if e is idempotent. When the radical is defined to be the maximal nilideal a trace-admissible algebra has the usual structure properties. An algebra A is called a standard algebra if the identities $z(xy)+y(zx)+(zy)x=z(yx)+(zx)y+(yz)x$, $w[x(yz)]+[(wy)x]z+[(wz)x]y=(wx)(yz)+(wy)(xz)+(wz)(xy)$ are satisfied. All associative and Jordan algebras are standard. All standard nilalgebras are solvable and all solvable standard algebras strongly nilpotent. If the field is nonmodular standard algebras are trace-admissible and the radical of A is the radical of the Jordan algebra $A^{(+)}$. For every algebra A there is an attached algebra $A(\lambda)$ defined by the product $x\cdot y=\lambda xy+(1-\lambda)yx$ for λ in F. Then $A^{(+)}=A(1/2)$. The author calls A over F quasiassociative if there exists an extension K such that $A_K=B(\lambda)$ for B associative and λ in K. Then either A is associative or $\lambda^2-\lambda$ is in F. He calls A J-simple if $A(1/2)$ is a simple Jordan algebra, and shows that all flexible J-simple algebras are either the exceptional Jordan algebra or quasiassociative. In particular, every simple standard algebra is either associative or a Jordan algebra. (Received October 13, 1947.)

48. Harold Chatland and H. B. Mann: *On extensions of domains of integrity*.

Let I, I' denote domains of integrity with unit element. The authors call I' an extension of I if $I' \supset I$ and if $p = qr$, p, q, in I, r in I' imply r in I. Two elements a, b of I are called coprime in I if $d|a$, $d|b$ implies $d|1$. The following theorems are proved. Theorem 1. If a, b in I are coprime in every extension of I then there exists elements x, y in I such that $ax+by=1$. Theorem 2. To every I there exists an extension I' such that every ideal in I is the intersection of a principal ideal of I' with I. Theorem 3. If any two elements in I have a greatest common divisor in I then $(a, d)=1$, $(b, d)=1$ implies $(ab, d)=1$. Received October 20, 1947.

49. Paul Erdös: *On additive number theoretic functions.*

Let $f(n)$ be an additive number theoretic function, $|f(p^\alpha)| < c$ for all p and α. Previously the author proved that the density of integers for which $f(n+1) > f(n)$ is 1/2. Now it is shown that the necessary and sufficient condition that for every k the density of integers with $f(n+k-1) > f(n+k-2) > \cdots > f(n)$ equals $1/k!$ is that $\sum_p (f(p))^2/p$ diverges. Also the necessary and sufficient condition that for every k the inequalities $f(n+k-1) > f(n+k-2) > \cdots > f(n)$ have a solution is that $\limsup f(n) = \infty$. The condition $|f(p^\alpha)| < c$ can be weakened, but $f(n) = \log n$ shows that $f(p^\alpha)$ must be restricted in some way. It is not yet clear what the necessary condition is. (Received October 18, 1947.)

50*t*. Franklin Haimo: *Preservation of divisibility under homomorphisms.*

To each element of an additive abelian group G, attach the collection of its heights with respect to the primes (Prüfer, Math. Zeit. vol. 20 (1924) pp. 165–187). Under a homomorphism, these heights will tend to increase. H, a subgroup of G, is said to be a *Liapin subgroup* of G if each element of the factor group G/H has the same set of heights as one of its antecedents in G. (Cf. Liapin, Rec. Math. (Mat. Sbornik) N.S. vol. 8 (1940) pp. 205–237). It is proved that if G is a torsion-free abelian group, every subgroup of which is Liapin, then G is the direct sum of copies of the additive group of rationals. Finite ordered subgroups are always Liapin in any containing group; and sequentially compact subgroups of division-closure groups are Liapin. In particular, closed subgroups of compact groups are Liapin; whence it follows that direct sums of copies of the integers can never be compact groups. (Received October 21, 1947.)

51. Gordon Pall: *The minimum of an indefinite, binary, quadratic form.*

The first two Markoff minima of an indefinite, real binary quadratic form are obtained simultaneously by a simple method. (Received October 24, 1947.)

52*t*. H. J. Ryser: *Rational vector spaces.*

Two finite-dimensional rational inner product spaces are said to be equivalent if their rational inner products determine the same class of positive definite matrices (Bull. Amer. Math. Soc. Abstracts 52-9-267, 53-1-21). Let V_n be an arbitrary rational inner product space of dimension n and let M_n denote the set of all polynomials of degree less than n with rational coefficients. There exists a rational inner product $\int_0^\infty f(x)g(x)d\psi(x)$ for M_n such that M_n is equivalent to V_n. The function $\psi(x)$ is non-decreasing with $\psi(0)=0$, has an infinite number of points of increase on $0 \leq x < \infty$. Every rational inner product space that is denumerable and infinite-dimensional has a basis whose elements are orthogonal and of length one. Thus there exists a set of polynomials $Q_1(x)$, $Q_2(x)$, $\cdots$ with rational coefficients such that every polynomial with rational coefficients is expressible as a finite sum $\sum_{i=1}^{n} a_iQ_i(x)$ where the a's are rational, and, moreover, $\int_{-1}^{1} Q_i(x)Q_j(x)dx=\delta_{ij}$, $i, j=1, 2, \cdots$. Results on algebraic number fields are extended. It is shown that every abelian field is a subfield of a cyclotomic I-field, and for every integer $n>2$, there exists an infinitude of abelian I-fields of degree n. (Received October 29, 1947.)

53. M. F. Smiley: *Application of a radical of Brown and McCoy to non-associative rings.*

The theory of the F_1-radical of Brown and McCoy (Amer. J. Math. vol. 69 (1947) pp. 46–58) is extended to arbitrary non-associative rings. After certain fundamental properties of non-associative rings are established, the proofs given in the associative case require no essential change. In particular, *a non-associative ring R has zero F_1-radical if and only if it is isomorphic to a subdirect sum of simple non-associative rings with unit elements and the sum is direct if the descending chain condition holds for the (two-sided) ideals of R.* For a non-associative algebra of finite order *which has a unit element,* this radical coincides with the radical defined by A. A. Albert (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 891–897). For a hypercomplex alternative ring (Max Zorn, Ann. of Math. (2) vol. 42 (1941) pp. 676–686), this radical reduces to that defined by Zorn and therefore also to that defined by N. Jacobson (Amer. J. Math. vol. 67 (1945) pp. 300–320, and the author's abstract, *The radical of an alternative ring,* Bull. Amer. Math. Soc. Abstract 53-11-373). (Received October 20, 1947.)

ANALYSIS

54. H. D. Block and H. P. Thielman: *Note on commutative polynomials.*

Two functions $f(x)$, $g(x)$ are said to be commutative with each other if $fg=f[g(x)]=gf=g[f(x)]$. An "entire set of commutative polynomials" is a set of polynomials which contains one of every positive degree, and which is such that any two polynomials of the set are commutative with each other. A linear transformation of a function $g(x)$ is defined as $\lambda g\lambda^{-1}$, where λ is a linear function and the operation is substitution. It is shown in this paper that the only entire sets of commutative polynomials are linear transformations of x^n, and of $\cos(n \arccos x)$. (Received October 23, 1947.)

55. D. G. Bourgin: *An approximate isometry.*

If T is a generalized ϵ isometry of the Banach space $C(Q)$ of bounded continuous functions onto $C(Q_1)$ where Q_2 and Q are completely regular spaces, then there is an isometry within $k\epsilon$ of T. This extends a theorem of Hyers and Ulam who require that T be a homeomorphism and that Q_1 and Q_2 be compacta. In particular, the theorem implies that if Q_1 and Q_2 are Hausdorff compact, then the existence of a generalized ϵ isometry on $C(Q_1)$ onto $C(Q_2)$ implies the topological equivalence of Q_1 and Q_2. (Received November 24, 1947.)

56*t*. D. G. Bourgin: *Some operators.*

Operators of Riesz type are developed for constant coefficient partial differential equations for other than the normal hyperbolic or elliptic cases. For instance, the parabolic equation is included. (Received October 24, 1947.)

57. R. E. Graves: *A generalization of the Riemann-Stieltjes integral appropriate to the Wiener space C.* Preliminary report.

Let $f(t)\in L_2$ on $[0, 1]$, let $\{\alpha_k(t)\}$ $(k=1, 2, \cdots)$ be a complete orthonormal set on $[0, 1]$, each of whose elements is of bounded variation, and let $x(t)$ be an element of the Wiener space C. Define $c_k=\int_0^1 f(t)\alpha_k(t)dt$. The generalized Stieltjes integral $(G)\int_0^1 f(t)dx(t)$ is defined to be $\lim_{n\to\infty}\int_0^1\sum_1^n c_k\alpha_k(t)dx(t)=\sum_1^\infty c_k\int_0^1\alpha_k(t)dx(t)$. The author proves that the limit on the right exists for all $x(\cdot)\in C$ except for a set of Wiener measure zero. He proves further that any two generalized Stieltjes integrals defined with different sets of $\alpha_k(t)$'s agree for all $x(\cdot)\in C$ except for a set of Wiener measure zero. The paper concludes with a proof that the generalized Stieltjes integral is a true generalization in the sense that if $f(t)$ is of B.V., so that the ordinary Stieltjes integral exists for all $x(\cdot)\in C$, then the generalized Stieltjes integral agrees with the ordinary Stieltjes integral for all $x(\cdot)\in C$ except for a set of Wiener measure zero. (Received October 20, 1947.)

58. Szolem Mandelbrojt and F. E. Ulrich: *Concerning regions of flatness for holomorphic functions and their derivatives.*

Let $C(\alpha, R)$ denote the circle $|z-\alpha|<R$. Suppose $f(z)$ is holomorphic and different from zero in each circle $C(\alpha, R)$, where α and R vary over a set S of values (α complex, R real and positive). For a given $0<\eta<1$, let $M(\alpha, R; \eta)=\text{Max }|f(z)|$ for $z\in C(\alpha, \eta R)$, $m(\alpha, R; \eta)=\text{Min }|f(z)|$ for $z\in C(\alpha, \eta R)$. Let $L(\alpha, R; \eta)=\text{Min }[\log M(\alpha, R; \eta)/\log m(\alpha, R; \eta), M(\alpha, R; \eta)/m(\alpha, R; \eta)]$. If to each $0<\eta<1$ there corresponds a finite positive number $A(\eta)$ such that $L(\alpha, R; \eta)<(A\eta)$ for all α and R belonging to S, the circles $C(\alpha, R)$ will constitute regions of flatness for $f(z)$. This definition is somewhat of the same character as that proposed by J. M. Whittaker. The authors have proved the following theorem: Suppose the circles $C(\alpha, R)$ are regions of flatness for the function $f(z)$ and $f'(z)\neq 0$ in any of these circles. If there exist three positive constants A, B, and d such that $R|[\log f(\alpha)]'|>A$, $\log|f(\alpha)|>BR$ and $R\geq d$, then the circles $C(\alpha, R)$ are regions of flatness for $f'(z)$. (Received October 22, 1947).

59*t*. Szolem Mandelbrojt and Norbert Wiener: *Infinitely differentiable functions on the half line.*

The authors prove a theorem which asserts that an odd, infinitely differentiable function on the half line is identically zero if an infinite sequence of odd derivatives are zero at the origin and if the maxima of the nth derivatives grow suitably (the function and its derivatives are supposed bounded on the half line). The statement is along the same lines as previous theorems by S. Mandelbrojt. The proof differs from those given by the latter for his theorems in that the authors employ an auxiliary function, holomorphic in the half plane with an argument corresponding to an order of differentiation, real or complex, with non-negative real part, and a value given by the value of the derivative of that order at the origin of the infinitely differentiable function. (Received October 22, 1947.)

60*t*. Morris Marden: *The number of zeros of a polynomial in a circle.*

Let $f(z)=\sum_0^n a_k z^k$ and $f^*(z)=z^n\bar f(1/z)$. Let the sequence $f_j(z)=\sum_0^{n-j} a_k^{(j)} z^k$ be defined by the recursion formula $f_{j+1}=\bar a_0^{(j)} f_j(z)-a_{n-j}^{(j)} f_j^*(z)$, $f_0(z)=f(z)$, $j=1, 2, \cdots, n$. As proved in the present note, if $\delta_j=f_j(0)\neq 0$ for $j=1, 2, \cdots, n$, then the number of zeros of $f(z)$ in the unit circle is equal to the number of negative P_j among the products $P_j=\delta_1\delta_2\cdots\delta_j$, $j=1, 2, \cdots, n$. Furthermore, if there is a $k<n$ such that $f_{k+1}(z)=0$ and $P_j\neq 0$ for $j=1, 2, \cdots, k$, then $f(z)$ has $n-k$ zeros on the unit circle and a number of zeros in the unit circle equal to the number of negative P_j for $j=1, 2, \cdots, k$. By use of the first theorem, the Schur-Cohn determinant criterion (Math. Zeit. vol. 14 (1922) pp. 110–148) for the number of zeros in the unit circle is given an elementary proof not involving the theory of Hermitian forms, as previous proofs. (Received October 24, 1947.)

61. P. M. Pepper: *Algebraic harmonic functions.*

Using Bergman's integral operator method the author gives sufficient conditions on the coefficients of the development $h=\sum_{\mu=0}^{+\infty}\sum_{\nu=-\mu}^{\mu}A_{\mu\nu}R^{\mu}P_{\mu}^{(\nu)}(\cos\theta)e^{i\nu\phi}$, in spherical harmonics, in order that h be an algebraic harmonic function. In terms of the $A_{\mu\nu}$, the expressions $a(\mu, \nu)=A_{\mu,\nu-\mu}2^{\mu}\nu!/\{(2\mu)!i^{\nu-\mu+1}\}$ for $0\leqq\nu\leqq 2\mu$ and $a(\mu, \nu)=0$ otherwise, are defined. For non-negative integers n and σ determinants $\Gamma_{n\sigma}(\mu_1, \cdots, \mu_\omega; \nu_1, \cdots, \nu_\omega)$ of order $\omega=(n+1)(\sigma+1)$ are formed in which the element of the kth row and the lth column is $a(\mu_k-q_l, \nu_k-r_l)$ where $q_l=[l/(\sigma+1)]$ and $r_l=l-(\sigma+1)q_l$, the brackets denoting the greatest integer less than or equal to the quantity within them. A sufficient condition for h to be algebraic is that there exist four non-negative integers m, n, ρ, σ such that the determinant relations $\Gamma_{n\sigma}(\mu_1, \cdots, \mu_\omega; \nu_1, \cdots, \nu_\omega)=0$ hold for each selection of 2ω integers $\mu_1, \cdots, \mu_\omega; \nu_1, \cdots, \nu_\omega$ for which, in each pair (μ_k, ν_k) either $\mu_k>m$ or $\nu_k>\rho$ (or both). A derivative process of generating harmonic functions is given. New types of singular lines of algebraic harmonic functions are given. (Received October 23, 1947).

62. B. J. Pettis: *Interiority and continuity of homomorphisms between metric groups.*

Necessary and sufficient conditions are given that a homomorphism ϕ between two metric groups X and Y be interior and that it be continuous. Defining ϕ to be permutative if transformations $[S_\alpha]$ and $[T_\beta]$ exist in X and Y respectively such that for each $x\in X$ and each α there is some β such that $\phi(S_\alpha(x))=T_\beta(\phi(x))$, where $[S_\alpha]$ and $[T_\beta]$ have certain topological properties, it follows that if ϕ is permutative and has a closed graph and X is complete then ϕ is interior. A dual definition leads to a dual theorem giving sufficient conditions that ϕ be continuous. From these two theorems there immediately follow, with somewhat weaker hypotheses, the well known interiority and continuity theorems of Schauder, Banach, Freudenthal, and Dunford for spaces of type (F) and for metric groups with separability assumptions. (Received October 21, 1947.)

63. G. B. Price: *Researches in the theory of functions of several real variables.* II. *Derivatives of functions.*

The Bertrand-Peano derivative of a function $F_0^{(n)}$ at $X_0^{(n)}$ is defined to be $\lim\Delta[F^{(n)}; X_0^{(n)}X_1^{(n)}\cdots X_n^{(n)}]/\Delta[X^{(n)}; X_0^{(n)}X_1^{(n)}\cdots X_n^{(n)}]$, the limit being taken as $X_1^{(n)}, \cdots, X_n^{(n)}$ tend to $X_0^{(n)}$ subject to suitable restrictions. The denominator of the fraction is an increment of the independent variable $X^{(n)}:(x_1, \cdots, x_n)$, and the numerator is the corresponding increment of the function $F^{(n)}:(f_1, \cdots, f_n)$. The definition is sufficiently broad to include ordinary partial derivatives and Jacobians in one treatment; it leads to significant advances in the theory of differentiation. This paper develops the theory of BP-derivatives under the following section headings:

sets of increments; *BP*-derivatives; *BP*-derivatives and factorable sets of increments; increments of *BP*-differentiable functions; Stolz differentiable functions; functions with derivatives of dimension 1; functions with derivatives of dimension greater than 1 only; continuity of *BP*-differentiable functions; derivatives of composite functions; directional derivatives; other limits for derivatives; functions with zero derivatives; linear spaces of differentiable functions; and history and the literature. (Received August 29, 1947).

64*t*. H. M. Schaerf: *On unique invariant measures.*

Let a measure space in the group G be the union of a sequence of measurable sets of finite measure. Under an assumption weaker than Weil's condition M there are proved (1) the *structure theorem*: If the measure is unique invariant then any measurable set A, whose measure is not greater than that of a measurable set B or equal to it, is almost congruent by finite or denumerable partition respectively with some measurable subset of B or with B; (2) a necessary and sufficient condition for the uniqueness of an invariant measure. The second result furnishes a simple proof for Weil's theorem on the uniqueness of an invariant measure under Weil's condition M and a new similar theorem. Extensions to measures invariant under a transitive group of transformations are indicated. (Received August 13, 1947.)

65*t*. W. J. Thron: *Twin convergence regions for continued fractions* $b_0+K(1/b_n)$. II.

Let two regions B_0 and B_1 in the complex plane be defined as follows: $re^{i\theta}\in B_0$ if and only if $r\geqq f(\theta)$, $re^{i\theta}\in B_1$ if and only if $r\geqq 4/f(\pi-\theta)$, where $f(\theta)>0$ and moreover $f(\theta)$ is such that the complements of the regions B_0 and B_1 are both strictly convex (that is, the curvature of the boundary is never equal to zero). Then B_0 and B_1 are "best twin convergence regions" for continued fractions of the form $b_0+K(1/b_n)$. This is an improvement over an earlier result of this nature proved by the author (Amer. J. Math. vol. 66 (1944) pp. 428–438). The proof depends on the fact that the regions B_0 and B_1 are of the desired type if and only if $f(\theta)=f_0\cdot\exp\left(\int_{\pi/2}^{\theta}\tan\alpha(\phi)\,d\phi\right)$, where $\alpha(\theta)$ has period 2π and satisfies $|\alpha(\theta)|<\pi/2$, $\alpha'(\theta)<1$. (Received October 23, 1947.)

66. L. C. Young: *Area and length.*

The current definitions of area and length have objections which have been repeatedly emphasized, particularly by T. Radó. Moreover, they are still restricted to parametric images of very special figures and they lack analogy with one another and with volume or hyper-volume. Intrinsic definitions, free from these drawbacks, are proposed. For the topological image of a plane set, the new area agrees with the Carathéodory-Hausdorff 2-dimensional measure μ. More generally, consider a "multiplicity-function" $M(x)$ where x denotes a point of the space in which the surface S is situated, and where $M(x)$ takes integer values, positive on S. The new area of S is then the sum $\sum_n\mu(E_n)$ where E_n is the set in which $M(x)\geqq n>0$. Its definition depends on the choice of $M(x)$ and in fact two such multiplicities are introduced: a crude one which leads to the "Banach" area, and a more satisfactory one (the "Morrey" multiplicity) which leads to the "intrinsic area." Various theorems connect these, and corresponding definitions of length, with classical definitions: thus the intrinsic area is not less than the classical double integral and agrees with the latter in the case of a Lipschitzian representation. (Received October 18, 1947.)

APPLIED MATHEMATICS

67*t*. D. G. Bourgin: *A boundary problem for the damped wave equation.*

This work extends the class of problems for which explicit solutions can be obtained. The data are given in implicit form on two nonparallel time-like straight lines and a space-like third side. The method in part involves co-images and Laplace transforms. The paper will appear in the Quarterly of Applied Mathematics. (Received October 24, 1947.)

68. Abraham Charnes: *Wing-body interaction in linear supersonic flow.*

The problem treated is to determine the steady uniform irrotational supersonic flow about a wing symmetrically affixed to a body. With the usual linearizing assumptions, if ϕ_1 is the velocity potential for flow about the wing in the absence of the body, ϕ_2 is sought such that $\phi_1+\phi_2$ is the potential for the flow about the wing plus the body. Bodies considered are circular and polygonal cylinders. In the former case ϕ_2 is obtained as a Fourier series involving inverse Laplace transforms of ratios of modified Bessel functions and their derivatives. With certain restrictions on ϕ_1, which the known ϕ_1 satisfy, convergence of ϕ_2 and appropriate derivatives is rigorously established. (Received October 24, 1947.)

69. M. Z. Krzywoblocki: *On the two-dimensional steady turbulent flow of a compressible fluid far behind a solid symmetrical body.* II. *Vorticity transfer theory.*

In the theory of incompressible turbulent flow it is the opinion that the velocity distributions obtained by the momentum transfer theory or the vorticity transfer theory are the same. Whereas the temperature distributions based on the vorticity transfer theory fit the results obtained from tests better than the distributions based on the momentum transfer theory. The question arises whether these statements are true for a compressible turbulent flow. To answer this question, the author solved the entire system of equations based on both theories. The first part contained the set of equations based on the momentum transfer theory (Bull. Amer. Math. Soc. Abstract 53-11-409). In the present paper the second part was solved. The following conclusions were drawn: (a) the velocity distributions based on both theories are the same only to the approximation of the first order. Beginning from the approximation of the second order there are increasing discrepancies between both velocity patterns, (b) the temperature distribution is different in both cases beginning from the approximation of the first order, (c) the density and pressure distributions are different in both cases beginning from the first approximation different from zero, that is, approximation of the second order. (Received September 15, 1947).

70*t*. H. E. Salzer: *Coefficients for expressing the first thirty powers in terms of the Hermite polynomials.*

Exact expressions are given for x^n, $n=0, 1, 2, \cdots, 30$, in terms of the Hermite polynomials $H_n(x)=(-1)^n e(x^2)(d^n/dx^n)e(-x^2)$, where $e(x)=e^x$. The computation followed a formula in a paper by E. Feldheim and an overall functional check was performed to guarantee the accuracy of the final manuscript. This table is useful for approximating a polynomial of high degree by a polynomial of much lower degree, which is best in the least square sense where the interval is $[-\infty, \infty]$ with weight factor $e(-x^2)$, by virtue of the well known property that if $q_r(x)$ denotes the partial sum of degree r in the expansion of a function $f(x)$ in terms of Hermite polynomials, and $r_r(x)$ is any polynomial of degree $\leq r$, distinct from $q_r(x)$, then $\int_{-\infty}^{\infty} e(-x^2) [f(x)-q_r(x)]^2 dx < \int_{-\infty}^{\infty} e(-x^2)[f(x)-r_r(x)]^2 dx$. (Received October 22, 1947.)

Geometry

71. R. W. Ball: *Dualities of finite projective planes*. Preliminary report.

A duality δ of a finite projective plane is a one-to-one correspondence of a point P (line k) to a line P^δ (point k^δ) such that P is on k if and only if k^δ is on P^δ. If N is the number of absolute points (that is, points P such that P is on P^δ), in a finite plane, $N \equiv 1 \bmod p$ for all primes p that divide n (where there are $n+1$ points on every line). Hence every duality of a finite plane possesses at least one absolute point. If $2o(\delta)$ is the order of δ and if the progression $1+x2o(\delta)$ contains at least one prime p with $(n/p) = -1$, then $N = n+1$. And if $N \neq n+1$ and n not a square, then $o(\delta)$ is not 1, 2, or a power of a prime of the type $4k+3$. (Received October 21, 1947.)

72. L. M. Blumenthal and L. M. Kelly: *New metric-theoretic properties of elliptic space.*

In two recent papers elliptic spaces of finite and infinite dimensions were characterized metrically among the class of semimetric spaces (Trans. Amer. Math. Soc. vol. 59 (1946) pp. 381–400) and criteria for the superposability of congruent subsets were obtained (to appear in Trans. Amer. Math Soc.). The present paper continues the development of metric-theoretic properties of elliptic space and is principally concerned with the remaining fundamental problem of determining a congruence order of n-dimensional elliptic space with respect to sub-classes of semimetric spaces. Several such theorems are proved. The principal result obtained is that the elliptic plane has congruence order 9 with respect to the class of all separable semimetric spaces. The intrinsic difficulty of these problems is due to the presence in every elliptic space (of dimension exceeding 1) of congruent but not superposable subsets. (Received October 12, 1947.)

73. M. O. Reade: *On the boundary of minimal surfaces*. Preliminary report.

Classic results due to Carathéodory, Fatau, Seidel, et al. (see Seidel, Math. Ann. vol. 104 (1931) pp. 182–243) are used to obtain information concerning the boundary of minimal surfaces. A typical result is the following. Let $\mathfrak{X} = \mathfrak{X}(u, v)$, $u^2+v^2<1$, define a regular minimal surface $\mathfrak{M}$, and let $\zeta = \zeta(u, v) = \zeta(z)$ be the normal vector to $\mathfrak{M}$. If $\zeta_3(u, v) \leq 0$, $u^2+v^2<1$, then $\lim \zeta(z)$, $z \to e^{i\theta}$, exists for almost all θ; here z approaches $e^{i\theta}$ along a path not tangent to the unit circle $|z| = 1$. (Received October 24, 1947.)

74. Marlow Sholander: *Contributions to the theory of convex curves.*

Results representative of those contained in this paper are stated below. The midpoints of certain chords of a closed convex curve C are used to define a curve M which has the following property: If P is a point inside C and not on M, if the index of P with respect to M is m, and if the number of pairs of points on C which have P as midpoint is n, then $2m+1=n$. The set of all points which divide the members of a family of parallel chords of C in a fixed ratio form a curve which is called a cross curve of C. Cross curves derived from the same family of chords are said to be parallel. For all C which contain at most one linear segment it is shown that (i) two nonparallel cross curves have one and only one point of intersection, (ii) two points inside C lie on one and only one cross curve. (Received October 27, 1947.)

75. E. A. Trabant: *Some special Riemann spaces.*

The following two theorems are proved. Theorem 1: A three-dimensional Riemann space with element of arc-length $ds^2=g_{ij}dx^i dx^j$, $i, j=1, 2, 3$, and the components of the fundamental metric having the form $g_{11}=g_{22}=g_{33}=Af(x^1)$, $g_{23}=g_{32}=Bf(x^1)$, $g_{12}=g_{21}=g_{13}=g_{31}=0$ and $|A|\neq|B|$ may possess constant Riemannian curvature of value b if and only if $B=0$ and will possess a zero curvature invariant if and only if $f(x^1)=((Cx^1+D)/k)^k$, where $(3A^2-2B^2)k=8(A^2-B^2)$ and C, D are arbitrary constants. Theorem 2: In a n-dimensional Riemann space with element of arc-length $ds^2=g_{ij}dx^i dx^j$ where $g_{ij}=\delta^i_j f(x)$, $\delta^i_j=1$ if $i=j$, $\delta^i_j=0$ if $i\neq j$ and $i, j=1, 2, \cdots, n$, a necessary and sufficient condition in order that the space possess constant Riemannian curvature of value b, be an Einstein space, and may be mapped conformally on a flat space is that $f(x)=K^{-1}(x+C)^{-2}$ where $K=-b$, C is an arbitrary constant and $x=x^1, x^2, \cdots$, or x^n. (Received October 22, 1947.)

STATISTICS AND PROBABILITY

76. Benjamin Epstein: *The role of the Mellin transform in the study of the distribution of the product and quotient of independent random variables.*

It is well known that the Fourier transform is a powerful analytical tool in studying the distribution of the sums of independent random variables. In this paper it is pointed out that the Mellin transform is a natural analytical tool to use in studying the distribution of products and quotients of independent random variables. Formulae are given for determining the probability density functions of the product $\xi\eta$ and the quotient ξ/η where ξ and η are independent positive random variables with probability density functions $f(x)$ and $g(y)$, in terms of the Mellin transforms $F(s)=\int_0^\infty x^{s-1}f(x)dx$ and $G(s)=\int_0^\infty y^{s-1}g(y)dy$. An extension of the transform technique to random variables which are not everywhere positive is given. A number of examples including Student's t-distribution and Snedecor's F-distribution are worked out by the techniques of this paper. (Received September 15, 1947).

77t. Dorothy J. Morrow: *On the distribution of the sums of the characteristic roots of a determinantal equation.*

Consider p variates $x_1, \cdots, x_p$ distributed with a multivariate normal distribution. Let $X=\|x_{i\alpha}\|$ and $X^*=\|x^*_{i\beta}\|$ $(i=1, \cdots, p; \alpha=1, \cdots, n_1; \beta=1, \cdots, n_2)$ be observation matrices of two samples of $x_1, \cdots, x_p$ with n_1 and n_2 degrees of freedom and covariance matrices $S=\|XX'\|$ and $S^*=\|X^*X^{*\prime}\|$ respectively. Then the determinantal equation $|S^{-1}S^*-\lambda|=0$ has p roots, $\lambda_1, \cdots, \lambda_p$. Define T_0^2/n_1 to be the sum of the roots. Under the condition that n_1/n_2 is large, the cumulative distribution function of T_0^2 is expressed in series form. The first four terms of the series are found exactly by a method which extends to the general term, the series is shown to be alternating and convergent, and an upper bound for the remainder after k terms is given. The statistic, T^2_σ provides a test of the significance of the difference between covariance matrices for two samples and therefore has many practical applications. It is an extension of Hotelling's generalized T^2 test. (Received October 20, 1947.)

78*t*. D. N. Nanda: *Certain results on the roots of a determinantal equation.*

If $x=\|x_{ij}\|$ and $x^*=\|x^*_{ij}\|$ are two p-variate sample matrices with n_1 and n_2 degrees of freedom and $S=\|xx'\|/n_1$ and $S^*=\|x^*x^{*\prime}\|/n_2$ are the covariance matrices which under the null hypothesis are independent estimates of the same population covariance matrix, then the joint distribution of the roots of the determinantal equation $|A-\theta(A+B)|=0$ where $A=n_1S$ and $B=n_2S^*$ has been obtained by Hsu in 1939. This also gives the joint distribution of the squares of canonical correlations between two sets with p variates in one set and q in the other. The limiting distribution of any root specified by its rank order has been obtained in the form of linear combinations and products of incomplete gamma functions for the equations of 2, 3, 4, and 5 roots. The distribution of the sum of the roots of an equation of 2, 3, or 4 roots has also been obtained under either of the conditions that $n_1=p\pm 1$ or that the numbers of variates in two sets differ by one. The distribution is expressed as a simple function of algebraic functions and incomplete beta functions. (Received October 20, 1947.)

TOPOLOGY

79*t*. H. W. Becker: *On the enumeration of bridge circuits.* Preliminary report.

The pure active bridge circuits a_n^* (n-branch, without free *SPB* sub-circuits) are relatively few, and derivable from a_{n-k}^* by recurrence, $k=1, 2, 3$. The pure passive bridge circuits b_n^* are found in a_{n+1}^*, by deleting an active branch in all possible ways. The b_n^* are classified according to the node, mesh, symmetry, or direct path partitions, to the number of feeders (terminal branches), true bridgers (having negative terms in the transfer conductance), breaks, or makes (products in the hindrance sum, or its inverse), and to the multiplicity (of the general term in the conductance denominator), dimensionality (of surface applicable without crossovers), fracto number (of *SP* circuits in the equivalent fracto-series circuit), or yet other criteria. Then the number of bridge circuits without external (but with possibly internal) free *SPB* branches is $b'_n=\sum cCpP$, summed over all terms of possible $c=$ census of sub-b^* of given symmetry partition, $C=$ given topological binomial coefficient, $P=$ topological product of the *SPB* circuits inserted, and $p=$ their permutations among the branches chosen; the bridge circuits with external *SPB* branches, $b''_n=\sum b'_{n-k}\cdot s'_{k+1}$, s'_k a generalized *SP* number. Finally, the total number of n-branch bridge circuits $b_n=b'_n+b''_n$, which is computed through $n=12$, where they outnumber the *SP* circuits about 3 to 1. (Received May 10, 1947.)

80*t*. R. H. Bing: *Some characterizations of arcs and simple closed curves.*

Each compact continuum contains two points neither of which cuts between two open subsets of the continuum. A compact continuum is a simple closed curve if each pair of its points cuts between open subsets of it. (Received October 24, 1947.)

81. F. B. Jones: *A note on homogeneous plane continua.*

That a homogeneous compact plane continuum exists which is *not* a simple closed curve has been shown by R. H. Bing. However, it is shown in this paper that if a homogeneous compact plane continuum is aposyndetic (or semi-locally-connected), it *is* a simple closed curve. It follows that a homogeneous compact plane continuum which contains no weak cut point is a simple closed curve. (Received October 27, 1947.)

82. E. E. Moise and G. S. Young: *On imbedding continuous curves in 2-manifolds.*

Claytor (Ann. of Math. vol. 38 (1937) pp. 631–646) has proved that a compact locally connected continuum can be imbedded in a 2-sphere if it contains no one of four continua, two of which are the primitive skew curves, and two of which are curves which will be called *C*-curves. The authors have proved a theorem which seems to cast some light on the role of the *C*-curves in Claytor's theorem. Their result is somewhat more general than the following: If a compact locally connected continuum does not contain a *C*-curve and has no local cut points, then it can be imbedded in a 2-manifold. (Received October 24, 1947.)

83*t*. Hing Tong (National Research Fellow): *On some problems of Čech.*

Čech has proposed a few problems in his paper *On bicompact spaces* (Ann. of Math. (1937)). Two of them have been solved by Hewitt (Bull. Amer. Math. Soc. Abstract 52-5-206) and Pospíšil (Ann. of Math. (1937)). The purpose of this note is to discuss the remaining ones. Specifically, answers to the questions on p. 836, line 13, p. 837, footnote, p. 843, lines 27–28, p. 843, footnote, p. 844, lines 14–15, p. 844, lines 16–18 are no, can be weakened but not entirely removed, yes, can be weakened but not entirely removed, no, no respectively. (Received October 24, 1947.)

84*t*. G. S. Young: *The factorization of manifolds.* Preliminary report.

It is proved that no connected open subset of a compact combinatorial n-manifold (without boundary) is homeomorphic to the product of two spaces A and B, such that A is compact and can be continuously retracted onto a nowhere dense subset of itself. From this it follows that if A is a finite complex, it is a manifold. These results are partial extensions of work of Borsuk (Fund. Math. vol. 33 (1945) pp. 273–298) and some of the methods used are due to him. (Received October 24, 1947.)

R. H. Bruck,
Associate Secretary

THE NOVEMBER MEETING IN PASADENA

The four hundred thirtieth meeting of the American Mathematical Society was held at the California Institute of Technology, Pasadena, California, on Saturday, November 29, 1947. The attendance was approximately 115, including the following 82 members of the Society:

O. W. Albert, H. L. Alder, R. F. Arens, S. P. Avann, L. H. Bechtolsheim, E. F. Beckenbach, H. W. Becker, M. M. Beenken, Clifford Bell, E. T. Bell, H. F. Bohnenblust, Herbert Busemann, Ralph Byrne, W. D. Cairns, F. H. Campbell, L. H. Chin, F. M. Clarke, L. M. Coffin, F. E. Cothran, E. L. Crow, D. R. Curtiss, P. H. Daus, Harold Davenport, R. W. Davies, R. P. Dilworth, L. K. Durst, Arthur Erdélyi, G. C. Evans, G. E. Forsythe, W. H. Glenn, E. C. Goldsworthy, J. W. Green, Leonard Greenstone, H. J. Hamilton, W. L. Hart, Olaf Helmer, J. G. Herriot, M. R. Hestenes, P. G. Hoel, Alfred Horn, D. H. Hyers, Rufus Isaacs, C. G. Jaeger, P. B. Johnson, P. J. Kelly, Cornelius Lanczos, D. H. Lehmer, Harold Luxenberg, S. L. McDonald, G. F. McEwen, Rhoda Manning, A. V. Martin, A. D. Michal, E. D. Miller, T. E. Oberbeck, J. W. Odle, L. J. Paige, R. S. Phillips, W. T. Puckett, W. C. Randels, H. P. Robertson, J. B. Robinson, R. M. Robinson, C. H. Savit, Henry Scheffé, G. E. F. Sherwood, Ernst Snapper, I. S. Sokolnikoff, R. H. Sorgenfrey, D. V. Steed, A. C. Sugar, J. D. Swift, Alfred Tarski, E. B. Tolstead, C. A. Truesdell, S. E. Urner, F. A. Valentine, Morgan Ward, L. E. Wear, P. A. White, R. M. Winger.

The morning session was devoted to research papers and to the hour address, *Electronic aids to the theory of numbers*, by Professor D. H. Lehmer of the University of California. Professor Morgan Ward presided. Additional research papers were presented in the afternoon in two sections, analysis and algebra, at which Professors E. F. Beckenbach and E. T. Bell, respectively, presided.

Following the meetings, tea was served at the Athenaeum through the hospitality of the Department of Mathematics of the California Institute of Technology.

Abstracts of papers read at the meeting follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Dr. Karlin was introduced by Professor Morgan Ward. Paper number 87 was read by Dr. Horn, paper 88 by Professor Tarski, and paper 90 by Miss Chin.

ALGEBRA AND THEORY OF NUMBERS

85. H. L. Alder: *The nonexistence of certain identities in the theory of partitions and compositions.*

Let $q_{d,m}(n)$ be the number of partitions of n into parts differing by at least d, each part being greater than or equal to m. Identities involving $q_{1,1}(n)$ (Euler's identity) and $q_{2,1}(n)$ and $q_{2,2}(n)$ (Rogers-Ramanujan identities) are known. It is shown in this paper that, aside from a simple generalization of Euler's identity, no other such identities can exist. The following two theorems, both of which were proved for the case $m=1$ by D. H. Lehmer (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 538–544), are proved: The number $q_{d,m}(n)$ of partitions of n is not equal to the number of partitions of n into parts taken from any set of integers whatsoever unless $d=1$, or $d=2$ and $m=1, 2$. The number $q_{d,m}(n)$ of partitions of n is not equal to the number of partitions of n into distinct parts taken from any set of integers whatsoever unless $d=1$. The nonexistence of certain generalizations of an identity of I. Schur involving partitions into parts differing by 3 or more is also proved. Furthermore it is shown that the results above can be extended to compositions. (Received October 13, 1947.)

86. S. P. Avann: *Ternary distributive semi-lattices.*

A ternary distributive semi-lattice $\mathfrak{J}$ is a set of elements closed with respect to a ternary operation satisfying (T1) $(a, a, b)=a$, (T2) (a, b, c) symmetric in all three letters, (T3) $((a, b, c), d, e)=((a, d, e), b, (c, d, e))$. Representing the elements of $\mathfrak{J}$ by points and joining elements b and c by a "link" of length 1 whenever $(b, x, c)=b$ or c for all x, one obtains the graph: $(\mathfrak{J})$ $\mathcal{G}$ of $\mathfrak{J}$. Ternary distance $\rho[x, a]+\rho[x, b]+\rho[x, c]$ in $\mathcal{G}(\mathfrak{J})$ has a unique minimum for $x=(a, b, c)$. Conversely, if to each unordered triple a, b, c of elements of a connected graph $\mathcal{G}$ there corresponds an element $[a, b, c]$ having unique minimal ternary distance, then $\mathcal{G}$ is a ternary distributive semi-lattice with $(a, b, c)=[a, b, c]$. For each $a\in\mathfrak{J}$ a partially ordered set $\rho(a, \mathfrak{J})$ may be defined whose ascending chains are the minimal chains joining a to the other elements of $\mathfrak{J}$. Each $\rho(a, \mathfrak{J})$ is an ideal of a distributive lattice of rank equal to the number n, independent of a, of join irreducibles of $\rho(a, \mathfrak{J})$. If a has a complement a' satisfying (T4) $(a, x, a')=x$ for all x, then $\rho(a, \mathfrak{J})$ is a distributive lattice. (Received November 20, 1947.)

87. Alfred Horn and Alfred Tarski: *Measures in Boolean algebras.*

A non-negative functional m on a Boolean algebra B is called a ***measure on B*** if $m(x+y)=m(x)+m(y)$ for any disjoint $x, y\in B$, and $m(1)=1$; it is called ***strictly positive*** if $m(x)=0\rightarrow x=0$, and ***countably additive*** if $m(\sum_{n<\omega}x_n)=\sum_{n<\omega}m(x_n)$ for any disjoint $x_n\in B$. The definition of measure, in one of its equivalent forms, is applied to an arbitrary set $S\subseteq B$; every measure m on S is shown to be extendible to a measure m' on B (m' being two-valued if m is two-valued). Furthermore, various conditions for the existence and non-existence of strictly positive or countably additive measures are established. For example, if B has a strictly positive measure, every uncountable $S\subseteq B$ includes an uncountable subset T such that $x, y\in T\rightarrow x\cdot y\neq 0$. Also, if a countable set $D\subseteq B$ is "dense" in B (that is, if every $x\in B$ is a sum of elements in D), then (i) B is isomorphic with a field of sets of integers; (ii) B has a strictly positive measure which is not countably additive; (iii) B has a countably additive measure only if it is not atomless; (iv) B has a strictly positive countably additive measure only if it is atomistic. (Received October 22, 1947.)

88. Bjarni Jónsson and Alfred Tarski: *Boolean algebras with operators.*

A function F mapping a Boolean algebra B onto a set $C\subseteq B$ is called an *operator* if $F(x+y)=F(x)+F(y)$ for $x, y\in B$; F is ***complete*** if $F(\sum x_i)=\sum F(x_i)$ for $x_i\in B$ (whenever $\sum x_i\neq 0$ exists), and ***normal*** if $F(0)=0$. A function of many arguments is called an operator if it is an operator in each argument; similarly for complete or normal operators. Closure algebras, relation algebras, projective algebras are Boolean algebras with operators (see McKinsey-Tarski, Ann. of Math. vol. 45, and the following two abstracts). *Extension Theorem*: (i) Every algebra B with operators $F_0, F_1, \cdots$, can be extended to a complete atomistic algebra B' with complete operators $F_0', F_1', \cdots$; (ii) every identity in B involving Boolean addition, multiplication, and operators F_i becomes an identity in B' when F_i are replaced by F_i'. Given an n-termed relation (a set of n-termed sequences) R and sets $X_1, \cdots, X_{n-1}$, let $R^*(X_1, \cdots, X_{n-1})$ be the set of all y's with $(x_1, \cdots, x_{n-1}, y) \in R$ for some $x_i \in X_i$. *Representation Theorem*: Every algebra with normal operators is isomorphic to a field of sets with operators of the form R^*. Example: every closure algebra is isomorphic to a field of sets with the closure operator R^* where R is a reflexive and transitive binary relation. (Received October 21, 1947.)

89*t*. Bjarni Jónsson and Alfred Tarski: *Representation problems for relation algebras.*

A *relation algebra* (RA) A is a Boolean algebra with a binary associative operator $|$ and a unary operator $\smile$ such that: $|$ has the unit element e; $a^{\smile\smile}=a$; $(a|b)^{\smile}=b^{\smile}|a^{\smile}$; $a|\overline{a|b}\leq\bar{b}$. (See Tarski, Journal of Symbolic Logic vol. 6, and Abstract 88.) Examples: I. *Proper relation algebras* (PRA's)—families A' of subrelations of a binary relation U closed under set addition $R+S$, complementation $U-R$, relative multiplication $R|S$, conversion $R^{\smile}$; the set of all (a, a) with $a\in U$ is in A'. II (by McKinsey). *Frobenius algebras* (FA's)—families A'' of subsets of a group G closed under set addition and complementation, complex multiplication and inversion; the smallest subgroup of G is in A''. Results: 1. A is extendible to a complete atomistic RA. 2. A is isomorphic with an algebra where $R+S$, $R|S$, $R^{\smile}$ (but not complementation) have the meaning in I. 3. An atomistic A where $a^{\smile}|a\leq e$ (or $a^{\smile}|a=e$) for every atom a is isomorphic with a PRA (or FA). 4 (by McKinsey-Tarski). A is simple if and only if $a\neq 0\rightarrow 1|a|1=1$. 5. A PRA which is simple is isomorphic with a PRA A' where U is a Cartesian set-square; and conversely. Problems: Are all RA's isomorphic with PRA's? Are all RA's with $a|b=0\rightarrow(a=0\vee b=0)$ isomorphic with FA's? (Received October 21, 1947.)

90. Louise H. Chin and Alfred Tarski: *Remarks on projective algebras.*

By Everett-Ulam, Amer. J. Math. vol. 68, projective algebras (PA's) are Boolean algebras with additional operations $O_1(a)=a_x$, $O_2(a)=a_y$ satisfying certain postulates (some rather involved). Example: proper PA's—non-empty families of subsets of a Cartesian product $X\times Y$ closed under set addition, $A+B$; complementation, $(X\times Y)-A$; projections on X-axis, A_x, and Y-axis, A_y. By definition, in any PA: $a^x=$ (the largest b with $b_x=a_x$); similarly a^y; $p=1_{xy}$. A PA can now be characterized as a Boolean algebra with operations a^x, a^y, and a distinguished element p satisfying mutually independent postulates: $0^x=0$; $0^y=0$; $a\leq a^x$; $a\leq a^y$; $(a^x\cdot b)^x=a^x\cdot b^x$; $(a^y\cdot b)^y=a^y\cdot b^y$; $a\neq 0\rightarrow a^{xy}=1$; p is atom; $p=p^xp^y$. One redefines a_x as $a^x\cdot p^y$; similarly a_y. The two postulate systems for PA's prove equivalent. By Abstract 89 above, a^x, a^y being operators, every PA is extendible to a complete atomistic PA (result obtained differently by McKinsey, Bull. Amer. Math. Soc. Abstract 52-7-227); consequently, by Everett-Ulam, it is isomorphic with a proper PA. Both results extend to related algebras, for example, Boolean algebras with operations a^x, a^y satisfying all postulates above not involving p; PA's supplemented by an operator a^c with

$a^{**}=a$, $a^{*\smile}=a^{\smile *}$. (In a proper PA, A^* is obtained from A by interchanging coordinates in each point.) PA's are closely related to RA's of Abstract 89. (Received October 21, 1947.)

91*t*. C. A. Rogers: *The signatures of the errors of simultaneous Diophantine approximations.*

Let $\theta_1, \cdots, \theta_n$ be irrational numbers, such that $1, \theta_1, \cdots, \theta_n$ are linearly independent. It is well known that there are an infinity of simultaneous Diophantine approximations $u_1/u, \cdots, u_n/u$, such that $|\theta_i - u_i/u| < u^{-1-1/n}$, for $i=1, \cdots, n$. The author investigates what precision can be attained by simultaneous approximations, when restrictions are imposed on the signs of the differences $\theta_i - u_i/u$. It is shown that for any $\epsilon > 0$, there are always simultaneous approximations, with *each* difference having a prescribed sign, which satisfy $|\theta_i - u_i/u| < \epsilon u^{-1}$, for $i=1, \cdots, n$. Further there are always an infinity of simultaneous approximations with *one* of the differences having a prescribed sign, which satisfy $|\theta_i - u_i/u| < 2^{1-1/n}u^{-1-1/n}$, for $i=1, \cdots, n$. But if more than one of the n signs are prescribed this extra degree of accuracy is not always attainable. In fact, when two or more of the signs are prescribed, for any continuous function $f(u)$ tending to zero as u tends to infinity, there are irrationals $\theta_1, \cdots, \theta_n$, such that the inequalities $|\theta_i - u_i/u| < u^{-1}f(u)$, $i=1, \cdots, n$, have only a finite number of solutions subject to the prescribed conditions on the signs. The results are obtained mostly as corollaries of a more general investigation of the "asymptotic directions" of n linear forms in $n+1$ integral variables, which will appear in the Proc. London Math. Soc. (Received October 16, 1947.)

92. Ernst Snapper: *The center of the endomorphism ring of a group for which the decomposition theorem holds.*

Let $\mathfrak{B}$ be an abelian group with a commutative ring A as operator domain. Let $\mathfrak{B}$ be a direct sum of cyclic operator subgroups, $\mathfrak{B}=(v_1) \dotplus \cdots \dotplus (v_k)$, where $\mathfrak{a}_i$ is the annihilating ideal of v_i. Suppose that $\mathfrak{a}_1 \subseteq \mathfrak{a}_i$ $(i=1, \cdots, k)$, that is, that $\mathfrak{a}_1$ is the anhilating ideal of $\mathfrak{B}$. It is proved that then the center C of the operator endomorphism ring of $\mathfrak{B}$ consists of the multiplications of elements of $\mathfrak{B}$ with elements of A. Consequently, C is isomorphic with the factor ring $A/\mathfrak{a}_1$. The conditions of this theorem are satisfied, for instance, if A is a principal ideal ring and $\mathfrak{B}$ has a finite number of generators with respect to A. For a finite abelian group $\mathfrak{B}$ without operators, the theorem states that the center C of the endomorphism ring consists of the endomorphisms $v \rightarrow nv$, where n is a fixed whole number and v ranges over $\mathfrak{B}$. Consequently, C is then isomorphic with the factor ring $I/(p)$, where I is the ring of whole numbers and p is the smallest non-negative integer such that $pv=0$ for all $v \in \mathfrak{B}$. (Received October 18, 1947.)

93. Morgan Ward: *Divisibility and factorability properties of general numerical functions.*

Let $W=\phi(n)$ be a function on the positive integers to a fixed residuated distributive lattice $\mathfrak{S}$ with null element O and unit element I in which quotients are unique. ϕ has property D if $\phi(0)=O$, $\phi(1)=I$ and $\phi(n) \supseteq \phi(m)$ whenever n divides m. ϕ has property F if $\phi(nm)=\phi(n)\phi(m)$ whenever n, m are coprime. Among other results it is shown that a necessary and sufficient condition that a function ϕ have both properties D and F is that there exist a second function ψ such that $\psi(nm)=\psi(n) \cup \psi(m)$ for n, m coprime and $\phi(n)=\prod \psi(p^r)$. Here the product refers to multiplication in the lattice, and is extended over the powers of primes dividing n. $\psi(n)$ is uniquely determined by ϕ. (Received October 24, 1947.)

ANALYSIS

94. J. G. Herriot: *Inequalities for the capacity of a lens.*

A lens is a solid determined by the intersection of two spheres. It is the purpose of this paper to compare the electrostatic capacity C of the lens with certain more accessible geometrical quantities. These quantities are the surface radius $\bar{S}$ which is the radius of the sphere having the same surface area as the lens and the "outer radius" $\bar{r}$ of the meridian section of the lens. It is shown that $(4/\pi) \log 2 \leqq C/\bar{r} \leqq 4/\pi$, the extreme values corresponding to two equal tangent spheres and the circular disk respectively. Moreover, $C/\bar{r}$ is greater or less than one according as the dielectric angle of the lens is greater or less than π. It is shown that for two tangent spheres (limiting case of lens) $2^{1/2} \log 2 \leqq C/\bar{S} < 1$, the equality holding only for two equal spheres. Since $2^{1/2} \log 2 = .980$, it is seen that $\bar{S}$ is a close approximation to C in this case. For the symmetric lens $2^{3/2}/\pi \leqq C/\bar{S} \leqq 1$, the extreme values corresponding to the circular disk and sphere respectively. It is conjected that $C \leqq \bar{S}$ for all lenses. (Received October 15, 1947.)

95. M. R. Hestenes: *Quadratic forms in the calculus of variations.*

The second variation $J(\eta)$ of an integral $I(y)$ in the calculus of variations is a quadratic form of the variation η. Moreover these variations with a suitable norm form a Hilbert space, provided the class of variations is taken sufficiently large. In the present paper is studied the properties of quadratic forms in Hilbert space that have significant applications in the calculus of variations together with these applications. This method enables one to give a unified treatment of the second variation for the various simple integral and multiple integral problems that arise in the calculus of variations. (Received October 22, 1947.)

96. Samuel Karlin: *Bases in Banach spaces.*

This paper investigates four types of bases in separable Banach spaces. A basis in a Banach space E is a sequence of elements x_n such that each element x has a unique representation in $x_n(\|x-\sum_1^n a_n x_n\| \to 0)$. An absolute basis x_n is a basis such that whenever $\sum a_n x_n$ converges, then the series converges by every arrangement. It is shown that whenever E possesses an absolute basis and the conjugate space E^* is separable, then E^* is weakly complete. In addition, if E has an absolute basis and E^* is weakly complete, then E^* has an absolute basis. As a consequence, it can be deduced that the space of continuous functions has no absolute basis. A basis is said to be absolutely convergent if $\sum_{n=1}^{\infty} \|f_n(x)x_n\|$ converges for every x in E. A space with an absolutely convergent basis is necessarily finite-dimensional. If M is a proper subset of E possessing a basis x_n and a biorthogonal set f_n then a necessary and sufficient condition that M possess a projection operation on it is that there exist an extension g_n of f_n to E^* such that $\sum g_n(x)x$ converges for every x in E. This result is used to give several criteria that a space E be a conjugate space. Other related results are obtained. (Received October 5, 1947.)

97. Cornelius Lanczos: *Differentiation of a Fourier series.*

The error of a truncated Fourier series of n terms can be estimated in the following form: exp inx times a relatively slowly changing function of x. Differentiation of the first factor brings the large n in front and decreases the convergence of the differentiated series. The operation $[f(x+\pi/n)-f(x-\pi/n)]/(2\pi/n)$, on the other hand, bypasses the first factor (except multiplication by -1) and differentiates the second factor only. As n grows to infinity, the derivative $f'(x)$ is obtained but now without loss in convergence. By this method a function such as x^{-m}, which is not absolutely integrable but the mth derivative of an absolutely integrable function, becomes expanded into a trigonometric series. The method amounts to a multiplication of the standard Fourier coefficients a_k and b_k by the "attenuation factors," $\gamma_k=(\sin \pi k/n)/(\pi k/n)$, or generally the mth power of these factors. This operator has strong smoothing properties in damping out the oscillations of the square wave series and the delta function series and other analogous series. The method is equally applicable to the Fourier integral. (Received October 24, 1947.)

98. E. B. Tolsted: *Limiting values of subharmonic functions.*

Littlewood showed that under certain restrictions a subharmonic function defined in the interior of the unit circle has a radial limit at almost all points of the unit circumference (J. E. Littlewood, *On functions subharmonic in a circle* II, Proc. London Math. Soc. (2) vol. 28 (1928) pp. 383–394.) Priwaloff published a proof that such a subharmonic function has a non-tangential limit at almost all points of the unit circumference (I. Priwaloff, *Sur un problème limite des fonctions sousharmoniques*, Rec. Math. (Mat. Sbornik) Moscow vol. 41 (1934) pp. 3–10). In 1942 Tamarkin discovered a mistake in Priwaloff's theorem, and in 1943 Zygmund constructed a counterexample to the theorem. In this paper counterexamples to the Priwaloff theorem including Zygmund's example are presented; and generalizations of Littlewood's theorem are proved in which the subharmonic function is shown to have a limit for restricted types of approach to the unit circumference. (Received October 1, 1947.)

APPLIED MATHEMATICS

99. H. W. Becker: *Bridge circuit duality decomposition.*

Let C and C' be any pair of dual bridge circuits. Let C_∞ and C_0 be the fracto components of C with branch b opened and shorted, respectively; and let C_∞' and C_0' be the fracto components of C' with b', dual of b, likewise taken as key branch. Then C_∞ and C_0' are dual circuits, and so are C_0 and C_∞'. More generally, let S and S' be any complete sets of $S-P$ decompositions of C and C', using dual sets B and B' of key branches; then S and S' are dual sets of circuits. (Received October 18, 1947.)

100t. S. A. Schaef: *Notes on nonlinear heat conduction.*

Two types of analytic solutions are obtained for the nonlinear heat conduction equation $\partial(K(T)\partial T/\partial x)/\partial x=S(T)\partial T/\partial t$. One of these is a perturbation series, successive terms of which are obtained by the Laplace transform. The other is a set of exact solutions, each containing three arbitrary constants, for various special forms assumed for $K(T)/S(T)$. (Received September 7, 1947.)

GEOMETRY

101. R. M. Winger: *The parametric treatment of cyclic-harmonic curves.*

Moritz, in a series of papers culminating in an extensive monograph (University of Washington Publications in Mathematics (1923)), and Hilton (Ann. of Math. vol. 24 (1922–1923) p. 209) have discussed the curves with the polar equation $\rho = a \cos p\theta/q + k$, where p, q are relatively prime integers and a, k are real. In this paper the author obtains very concise parametric equations of these curves, proving ipso facto that they are rational. He is thus able to apply the technique for dealing with rational curves. The group properties of the curves are considered and several errors of Moritz are corrected. (Received October 20, 1947.)

LOGIC AND FOUNDATIONS

102. R. M. Robinson: *Recursion and double recursion.*

A simple proof is given of the fact that the function G_nx defined by the double recursion $G_0x = x+1$, $G_{n+1}0 = G_n1$, $G_{n+1}(x+1) = G_nG_{n+1}x$ *majorizes* all primitive recursive functions of one variable in the following sense: If Fx is a primitive recursive function, then there is a number n such that $Fx < G_nx$. Also, two primitive recursive functions Ax and $B(x, y)$ are found, such that the function G_nx defined by the double recursion $G_0x = Ax$, $G_{n+1}0 = 0$, $G_{n+1}(x+1) = G_nB(x, G_{n+1}x)$ *generates* all primitive recursive functions of one variable in the following sense: There exists a primitive recursive function $H(n, x)$, which is indeed a polynomial in n and x, such that if Fx is a primitive recursive function, then there is a number n such that $Fx = G_nH(n, x)$. Results of a similar character were found by Rózsa Péter, *Konstruktion nichtrekursiver Funktionen*, Math. Ann. vol. 111 (1935) pp. 42–60. The proofs given here depend on Theorem 3 in §7 of the author's paper, *Primitive recursive functions*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 925–942. (Received October 18, 1947.)

103t. Ira Rosenbaum: *On the numbers correlated with p-valued classes of referents and relata with respect to $R_n^{p,2(m)}$.*

The class, $sg'R_n'k$, of referents of the individual k with respect to the nth p-valued dyadic relation R_n, is readily determined as the xth of the p^m p-valued classes of an m-membered universe, where $x = 1 + (\sum_{j=0}^{m-1} W_{km-j} \cdot p^{km-j-1})$ and $W_{km-j} = (p^{km-j-1} : (n-1)_p)$. Similarly, the number of the p-valued class, $gs'R_n'k$, of relata of k with respect to R_n is determined as $y = 1 + (\sum_{j=0}^{m-1} W_{mj+k} \cdot p^{mj+k-1})$ with $W_{mj+k} = (p^{mj+k-1} : (n-1)_p)$. Since the number assigned to a sum of classes is determinable by previous methods when the numbers of the summands are known, the number correlated with the class, $R_n''C_s$, of referents of members of C_s is easily obtained, since this class is the logical sum of the classes of referents with respect to R_n of the members of C_s. Similarly determination of the numbers correlated with the classes of relata re R_n of the members of C_s makes possible determination of the number assigned to the class sum of these classes, that is, to the class, $Cnv'R_n''C_s$, of relata re R_n of members of C_s. When $s = 1$, $C_s =$ the universal class, and $R_n''C_s = D'R_n$ (the domain of R_n) while $Cnv'R_n''C_s =$ Ɑ$'R_n$ (the converse domain of R_n). (Received October 23, 1947.)

STATISTICS AND PROBABILITY

104. G. F. McEwen: *Evaluation of an integral occurring in a fluctuation problem of cosmic-ray showers.*

A. Nordsieck, W. E. Lamb, Jr., and G. E. Uhlenbeck (Physica Vol. 7 (1940)) obtained a solution of the statistical problem of the fluctuation in the number of particles as a function of the thickness, x, of matter penetrated when the effect of ionization is considered. They found for the number of particles, $\bar{N}(s, x) = \exp[-x]$

$\cdot\{1+\int_0 I_1(\eta)\exp[\eta^2/8x]d\eta\}$, where I_1 is the modified Bessel function of the first kind. Various approximation methods, including that of steepest descent, were used to obtain numerical estimates of the definite integral. It can be integrated by repeated applications of a reduction formula involving integration by parts in two ways. Final results are the two converging series: $\exp[-b^2x_1^2]\int_0^{x\leq x_1} I_1(x)\exp[-b^2x^2]dx=\{f_0(x)+(2b^2x)f_1(x)+(2b^2x)^2f_2(x)+\cdots\}\exp[-b^2(x-x_1)^2]-\exp[-b^2x_1^2]$, and $\exp[-b^2x_1^2]\int_0^{x\geq x_1} I_1(x)\exp[-b^2x^2]dx=[\exp(x_1/2)-1]([1+f_0(x_1)]/2-\{f_1(x)/(2b^2x)+f_2(x)/(2b^2x)^2+\cdots\}\exp[-b^2(x-x_1)^2]$, where the tabulated function $f_i(x)=\exp(-x)I_1(x)\leq 1$, and $2b^2x_1=1$, and $[1+f_0(x)]/2=\sum_{i=0}^{\infty}f_i(x)$. (Received October 20, 1947.)

TOPOLOGY

105*t*. R. F. Arens: *Representation of certain nearly commutative Banach algebras.*

It is proved that a certain type of Banach *-algebra in which $xx^*=x^*x$ is always in the center is equivalent to the ring of all continuous quaternion valued functions (vanishing at infinity) on a locally compact Hausdorff space which satisfy a certain condition of covariance under the ordinary orthogonal group. (Received October 25, 1947.)

J. W. GREEN
Associate Secretary

FIRST SUMMER SEMINAR OF THE CANADIAN MATHEMATICAL CONGRESS

This seminar in algebra and the theory of numbers was held at Victoria College in the University of Toronto from August 15 to September 13, 1947. The program consisted of three lecture series on research topics, four expository courses of graduate level, and sub-seminars with lectures on individual researches. About a hundred and thirty mathematicians took part in the seminar.

The lecture series were the following: *Théorie des idéaux algébrique*, by Professor Paul Dubreil; *Algebraic cohomology*, by Professor Saunders MacLane; *Topics from number theory*, by Professor L. J. Mordell.

The expository courses were: *Introductory topology*, by Professor C. H. Dowker; *Elementary theory of numbers*, by Professors R. D. James and Gordon Pall; *Introduction to modern algebra*, by Professor D. C. Murdoch; *Les fondements généraux des mathématiques*, by Professor Adrien Pouliot.

In the sub-seminars in algebra and number theory, conducted respectively by Professors Richard Brauer and Gordon Pall, the following papers were presented:

Ernst Snapper: *Linear equations in Noetherian rings.*

H. S. M. Coxeter: *On quadratic residues, and anallagmatic tesselations.*

S. A. Jennings: *Group algebras of certain infinite groups.*

Christine Williams: *Normal chains in groups.*

R. D. James: *Viggo Brun's method.*

L. K. Hua: *Exponential sums.*

D. C. Murdoch: *A theorem on nilpotent groups.*

G. de B. Robinson: *Nakayama's prime factor theorem.*

Karl Menger: *Generalization of a theorem of Minkowski.*

Edward Rosenthall: *The sum of cubes.*

Richard Brauer: *Class-number of algebraic number fields.*

R. M. Thrall: *Quasi-Frobenius algebras.*

C. J. Nesbitt: *A remark on representation of algebras.*

J. S. Frame: *Double coset traces.*

L. K. Hua: *Geometry of matrices.*

R. H. Bruck: *Applications of loop theory.*

Grace E. Bates: *Loop homomorphisms.*

Irving Reiner: *Meyer's theorem on primes in a binary quadratic form.*

L. J. Mordell: *The minimum of an indefinite binary quadratic form.*

Peter Scherk: *Densities.*

Hans Tornehave: *Generalisation of Kronecker's theorem.*

Leslie Peck: *On a theorem of Hajos.*

W. L. G. Williams: *Algebraic invariants and a certain congruence.*

R. D. James: *Linnik's proof of the three prime theorem.*

Peter Scherk: *An analogue to algebraic geometry.*

Harold Davenport: *On a theorem of Minkowski.*

Irving Kaplansky: *Complete algebras.*

H. S. Vandiver: *On the solutions of certain general types of equations in a finite field.*

A. T. Brauer: *The distribution of quadratic and nth power residues.*

Marie-Louise Dubreil-Jacotin: *On the imbedding problem.*

Bjarni Jónsson: *Boolean algebras.*

It was decided to hold in the summer of 1949, at the time of the regular quadrennial meeting of the Congress, another seminar devoted to topics from applied mathematics and cognate fields of pure mathematics.

During the course of the month arrangements were made for the publication by the Congress of a mathematical journal. This is to be an international quarterly journal of mathematical research, the first issue appearing January 1949. The editorial board is: H. S. M. Coxeter (Editor-in-Chief), Abel Gauthier, Leopold Infeld, R. D. James, R. L. Jeffery, and G. de B. Robinson (Managing Editor).

R. ERIC O'CONNOR, S.J.
Secretary, Canadian Mathematical Congress

HARRY BATEMAN
1882–1946

Harry Bateman was born in Manchester, England, May 29, 1882. He was the third and youngest child of Samuel and Marnie Elizabeth (Bond) Bateman. His father, who was born in Congleton, Cheshire, was a druggist and commercial traveler. His mother was born in New York City in 1853 (her father, who came from Lancaster, having been a planter in the West Indies and America). He lived from 1884 to 1890 in Oldham, Lancashire, and his early education was received at home since, as he records, his mother did not wish him to acquire the Lancashire accent. He recounts two incidents of these early years in a manner which conveys some impression of the quiet, dry humor which was characteristic of him in later life. In order not to spoil this impression we use his own words: "One day a Mr. Pullinger, to whom my father had been apprenticed, was visiting us. As a result of some questions he had put to me he recommended me to study mathematics. I was quite impressed but my memory played me a trick when a lady asked me a few days later what I was going to study. My reply was that I was going to study acrobatics. She then asked me where I was going to perform and I was at a loss for an answer. Since I have learned recently that Dr. Thomas Young was an expert tight rope walker and harlequin my mistake does not seem so bad after all." The second story of his early days runs as follows: "Perhaps my love for the exact sciences dates from the day when I went with my sister to the home of one of her girl friends. The father of this girl was very stout and when I met him I gazed at him in astonishment and finally spoke thus: 'Mr. Booth, the next time I come to see you I am going to bring with me mother's inch tape and measure you. I think your waist line is about two yards.' 'No, Harry,' replied the good humoured Mr. Booth, 'it's nearly three'." From 1891 to 1900 he attended Board School and Grammar School in Manchester. He held Manchester City Council and Langworthy Scholarships at the Grammar School where he specialized in mathematics and ended by winning a Derby Scholarship and sizarship at Trinity College, Cambridge. When he was at Board School, and not yet twelve years old, a teacher named Arthur Gronowsky offered a prize of one shilling to the boy who was first able to demonstrate the first twelve propositions in the first book of Euclid. A shilling seemed a lot in those days to young Bateman, and he set to work to win the prize. They had good teachers then in Manchester (and, doubtless, still have) and this small piece of bread cast on the waters by one of them certainly brought returns.

At Trinity College Bateman won a major scholarship in 1902 and took his B.A. in 1903, being bracketed Senior Wrangler with P. E. Marrack. He was Smith's Prizeman and won a fellowship in 1905 and took his M.A. in 1906. He worked very hard during his last year (1904–1905) at Cambridge for in addition to his dissertations for the Smith's Prize and fellowship, he marked papers for the Briggs Correspondence School and coached for the Mathematical Tripos. After winning his fellowship he studied for a year on the continent, visiting Göttingen and Paris. In 1906 he was appointed Lecturer in Mathematics at Liverpool University. Before taking up this appointment he visited Professor Carey who was head of the department of mathematics at Liverpool, and he recounts that Carey's boys "besieged me with questions." At night one of them said to his father, "You told us that Mr. Batemen was a Senior Angler but he doesn't seem to know anything about fishing."

After one year at Liverpool Bateman was appointed Reader in Mathematical Physics at Manchester University. In 1910 he was appointed Lecturer in Mathematics at Bryn Mawr College where Charlotte Angas Scott, another English mathematician, was head of the department of mathematics. He spent only two years at Bryn Mawr and we can only surmise that he was not particularly successful as a teacher of young ladies or that he did not find the work particularly congenial. In 1912 he received an appointment as Johnston Scholar at Johns Hopkins University where Frank Morley, an old Cambridge mathematician, was head of the department of mathematics. The Johnston Scholarship in those days at Hopkins was a research scholarship, and the holder could give a seminar if he felt like doing so and if there were any students who felt like taking the course. Bateman had married in the summer of 1912 Ethel Horner Dodd and a son was born in 1914. The death of this son in 1917 was a blow which left its mark but which was borne with a courage which revealed the inner strength of the man. The Batemans later adopted a daughter Joan, who prepared the list of publications at the end of this memoir.

Bateman spent five years at Hopkins, holding the Johnston Scholarship for three years and being Lecturer in Mathematics for the two year period 1915–1917. In order to add to the small income from his scholarship and, later, his lectureship, he taught at the Bureau of Standards and at Mount Saint Agnes College and reviewed papers for the Weather Bureau; he also spent the summer of 1915 in Washington working for the Department of Terrestrial Magnetism.

In 1917 he was appointed Professor of Mathematics, Theoretical Physics and Aeronautics at Throop College (now the California Institute of Technology), Pasadena, California. He held this position until his sudden death from coronary thrombosis on January 21, 1946, while on the train to New York to receive an award from the Institute of Aeronautical Sciences.

Bateman's distinction as a mathematical physicist was widely recognized. He was a Fellow of the Royal Society (1928), a member of the National Academy of Sciences (1930) and of the American Philosophical Society (1924). He was vice president of the American Mathematical Society (1935) and Gibbs' Lecturer of the Society (1943). The last years of his life were devoted mainly to work connected with the war; he was a member of the War Preparedness Committee of the American Mathematical Society and chief consultant in aeronautics for the American Mathematical Society and the Mathematical Association of America.

The writer of this memoir was first brought into contact with Harry Bateman under the following circumstances. In 1914 I was awarded a Traveling Studentship in Mathematical Physics by the National University of Ireland and was looking about for some place to study. My professor, A. W. Conway, told me that there was a young man, Bateman, at Hopkins and that he thought that I could not do better than study with him. I followed this advice and, looking back over a third of a century, I judge the advice to have been sound. Bateman, a frail slight man of 32, was lecturing on The Absolute Calculus and Electrodynamics (remember that this was 1914 and that four years or more had to elapse before most of us in this country heard of Einstein's General Theory of Relativity). As I recall the situation, six students started the course and by March I was, if my memory is correct, the only student. I do not think that this diminution of the size of his class bothered the lecturer very much, and I have sometimes thought that if the vicissitudes of student life had prevented my attendance, the lecture would have been none-the-less delivered. By common standards he was not (in those early days) a good lecturer. He was too detached, too objective and perhaps too scornful of histrionic effects, and we were too untrained to profit as much as we should have from the instruction he gave us. As time went on the scene changed and he must have changed with it for I have heard enthusiastic reports of his lectures from students who took courses under him in the late twenties and thirties at the California Institute of Technology. As I think back over my two years of association (1914–1916) with him I remember well a feeling of amazement, mingled with discouragement, which came over me when I discovered the thoroughness of the man. He already possessed a large, carefully indexed card-catalogue on each card of which was written in his minute, but beautifully clear, handwriting an abstract of a paper which he had read. I am told that in later years this card-catalogue crowded him out of his office and almost out of his home. No wonder, then, that his books and papers bristle with references which are a veritable mine of useful source material. His memory was phenomenal. No matter what stubborn integral or intractable differential equation you showed him, a moment's thought and a reference to the card catalogue never failed to produce something useful. General theories did not seem to have for him the same attraction as the special instance; the only exception to this was his devotion, which marked him as a true disciple of Hamilton, to the variational principle. As a master of the special instance I have not met his equal, nor one who approached him, and I do not think that we shall see his like again.

Bateman's best work centered around the development of the properties of special functions and the solution of the equations of Mathematical Physics. His first book, ***Mathematical analysis of electrical and optical wave motion***, is unique and characteristic of the man. Into less than 160 small pages is crowded a wealth of information which would take an expert years to digest. Some of the material in this book may be found in expanded form in his monumental work, ***Partial differential equations of mathematical physics***, published by the Cambridge University Press in 1932 and republished with minor additions by the Dover Press in 1944. This book has already taken its place beside Lamb's *Hydrodynamics* and Love's *Elasticity* as one of the classics which are part of the equipment of every worker in applied mathematics. Amongst the many results involving the functions of special importance in mathematical physics which are due to Bateman we select for special mention the formula

$$\int_0^x J_\mu(t)J_\nu(x-t)\,\frac{dt}{t}=\frac{J_{\mu+\nu}(x)}{t},\qquad R(\mu)>0,\ R(\nu)>-1$$

($J(t)$ being the familiar Bessel function of the first kind). This result is treated fully in Watson's book on Bessel functions and has been discussed by L. J. Mordell (J. London Math. Soc. vol. 5, pp. 203–208).

In the early years of the present century, when Bateman was a student at Cambridge, the theory of integral equations dominated the mathematical scene. Fredholm's epoch making paper (Acta Math., vol. 27) appeared in 1903 and Hilbert was actively developing the subject at Göttingen when Bateman visited there in 1906. It was natural, then, that this new field of mathematical research should have occupied the attention of the young analyst. Papers 15, 19, 21, 22, 25, 26, 27, 28 and 42 of the bibliography below contain substantial contributions to the subject and the Report (1) to the British Association for the Advancement of Science is a valuable account of the theory as it stood in 1910. In the paper (39) Bateman applied (independently of Herglotz, who had the same idea) integral equation theory to the propagation of earthquake waves through the interior of the earth. He showed how to determine, from a knowledge of the time taken by an earthquake wave to reach various points on the surface of the earth, the velocity of propagation of the wave at various points in the interior of the earth. The full importance of this result has not yet, in our opinion, been sufficiently exploited. The velocity of propagation tells us the ratio of the appropriate elastic constant to the density. If we know this for both types of waves, longitudinal and transverse, we can determine how the density varies with the pressure. Knowing this we can set up differential equations whose solutions tell us the density and pressure throughout the interior of the earth as functions of the distance from the center. It should then be possible, from a knowledge of the variation of compressibility with temperature, to estimate the variation of temperature throughout the earth's interior.

The field in which Bateman stood preeminent was that of electrodynamics. In 1908 Hargreaves published in volume 21 of the Transactions of the Cambridge Philosophical Society a paper in which he showed that Maxwell's differential equations were merely the expression in differential or *local* form of relations between integrals over two-dimensional and three-dimensional spreads in four-dimensional space-time. Every electrical engineer knows that the relation div $B=0$ (B = magnetic induction) is merely the differential, or local, form of statement of the fact that, in magnetostatics, the flux of magnetic induction through any closed surface is zero. This integral or *global* statement is much closer to the physics of the matter than the local statement div $B=0$. For non-static phenomena Maxwell's equations

$$-\frac{dB}{dt}=c\,\text{curl}\,E; \qquad \text{div}\,B=0$$

are merely the local form of the global statement that the flux of the magnetic induction-electric intensity tensor across any closed two-dimensional spread in four-dimensional space is zero. In the paper

(50) Bateman exploited fully this idea and showed that the group of transformations under which Maxwell's electrodynamic equations are invariant is the group of conformal transformations of four-dimensional space-time. The fundamental significance of this paper from the point of view of relativity theory was not generally recognized but Klein, in his *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, vol. 2, 1927, directs attention to this significant and pioneer work of Bateman.

This brings us to the end of the first stage of Bateman's career. In 1912 he was thirty years old, had published some 64 papers and had been two years in America. He was preparing to accept Morley's offer of a Johnston Scholarship at Hopkins (roughly equivalent to a National Research Council fellowship of today). As we look back on the situation we cannot escape the inevitable Why? Here was a man of international reputation, pleasant (if self-effacing) personality, and he had to spend the next five years in a position designed for a young unmarried Ph.D. of promise or for an established scholar on leave-of-absence or sabbatical leave. When we think of the "odd-jobs" he had to do to eke out a subsistence, the reading of papers for the Weather Bureau, the hot Washington summer at the Bureau of Standards, the teaching at Mount Saint Agnes and then recall that during this period he wrote his book on electrical and optical wave-motion, we can only subscribe to the old Latin tag: *Per aspera ad astra.*

The influence of Morley upon Bateman is shown by the publication of several papers on geometrical topics (66), (67), (68), (72). His main work, however, during the Hopkins period (1912–1917) dealt with electromagnetic theory. He was particularly interested in the lines of electric force due to a moving electron and (following J. J. Thomson) in the connection between these and the structure of the aether. During this period he wrote his book on differential equations. While this book has not generally been found well adapted to beginning courses in the subject, it well repays study, particularly by those who are more interested in the applications of differential equations to mathematical physics than in the applications to differential geometry and the theory of functions.

Shortly after his appointment in 1917 to the chair of Mathematics, Theoretical Physics and Aeronautics at Throop College (now the California Institute of Technology) Bateman published his first paper (94) on hydrodynamics, a subject which was to engage a large share of his attention during the last years of his life. However, most of his papers published during the decade 1918–1928 deal with electromagnetic theory. During this period his monograph on electromagnetism appeared as Bulletin No. 4 of the National Research Council (1922). In addition to his papers on electromagnetism he published a paper (97) containing an interesting application of integral equation theory to mathematical economics, two papers (106) and (119) on the numerical solution of integral equations, several papers (100), (104), (114), (116) on potential theory, papers (98), (133) and (134) on elasticity, and a paper (128) on geometry.

In 1925 Bateman was appointed a member of a committee on hydrodynamics of the National Research Council and was assigned the problem of writing those sections of a report on hydrodynamics which dealt with viscous fluids and compressible flow. The report appeared in 1932 as Bulletin No. 84 of the National Research Council. It is a large report of some 634 pages and of these Bateman's part ran to over 500 pages and is a veritable mine of information and of references to all papers of significance prior to 1932. The section on compressible fluids has been widely used and the report has been for several years out of print. Paper (136) is an important contribution to the theory of two-dimensional compressible fluid flow. The work done by Bateman on this subject during the last five years of his life is not yet available to the public due to secrecy imposed by war conditions.

Bateman was always interested in the problem of numerical computation. In collaboration with A. A. Bennett and W. E. Milne he wrote a report on the numerical integration of differential equations which was published in 1933 as Bulletin No. 92 of the National Research Council. In 1944 he published, in collaboration with R. C. Archibald, a *Guide to tables of Bessel functions* which appeared in the first volume of the journal: Mathematical Tables and other Aids to Computation.

His Gibbs Lecture, *The control of an elastic fluid*, appeared in the Bulletin of the American Mathematical Society, vol. 51 (1945) and in the same year articles by him on dynamics and elasticity appeared in the Encyclopaedia Britannica. His last paper, appropriately entitled *Some integral equations of potential theory*, appeared after his death in the Journal of Applied Physics vol. 17 (1946). We learn from a note by Professor E. T. Bell in the Quarterly of Applied Mathematics vol. 4 (1946) that he was engaged, in the time he could snatch from his war work, on "what he regarded as his most useful contributions to mathematical scholarship: an exhaustive work on definite integrals, and a critical census of all the special functions that have been considered in mathematics." It is pleasant to know that the California

Institute of Technology has invited Professor Erdélyi to prepare these works for publication.

This notice fails to convey an adequate impression of the personality of the great scientist and scholar who was lost to the world in the passing of Harry Bateman. Modest, reserved, cultured and self-effacing he was, in its true sense, a gentle man. He was very English and he must have long looked back over his shoulder, before becoming, in 1927, a citizen of the new exuberant country in which he spent more than half his life, to the quiet England of Victoria and Edward VII. He was an expert chess player and participated, when eighteen years of age, in a chess tournament between England and America. We profit from his labors and are the better for his having lived amongst us.

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F. D. MURNAGHAN

BOOK REVIEWS

Advanced calculus. By D. V. Widder. New York, Prentice-Hall, 1947. 16+432 pp. $5.00.

This text gives careful treatments of a restricted selection of topics culminating in the theory of real Laplace transforms.

Content: Limits and derivatives of functions of one variable are sketched in four pages. The treatment of partial derivatives emphasizes the theory of functional relations. The differential geometry of curves and surfaces is developed after a short chapter on vectors. A thorough treatment is given to maxima and minima. Single integrals are developed in the form of Stieltjes integrals, with Riemann integrals merely mentioned as a special case. But multiple integrals are given a relatively conventional treatment including some of the usual elementary applications. The theory of line and surface integrals is carried as far as the basic integral relations. Indeterminate forms are handled rather conventionally. The chapters on infinite series and improper integrals are closely parallel. They begin with a relatively detailed treatment of the elementary convergence tests; then follow the theory and applications of uniform convergence; the discussion closes with an introduction to the process of summation. Brief treatments are given to the gamma and beta functions, and to Stirling's formula. Various topics are included in the theory of Fourier series and integrals. The last two chapters develop the theory of real Laplace transforms and their applications to the solution of differential and difference equations. Such subjects as complex variables, matrices, variations, numerical methods, and statistics are omitted entirely; differential equations are only touched on in connection with line integrals and Laplace transforms.

Style: The formulation of most of the definitions and theorems is illustrated by the first numbered definition in the book:

DEFINITION 1. $f(x) \in C$ *at* $x = a \leftrightarrow \lim_{x \to a} f(x) = f(a)$.

Precision and rigor constitute an outstanding feature of this text. The extent to which these attributes are carried is illustrated by the following definition (page 30):

DEFINITION 8. ***The directional derivative*** *of* $f(x, y)$ ***in the direction*** ξ_α *at* (a, b) *is*

$$\left.\frac{\partial f}{\partial \xi_\alpha}\right|_{(a,b)} = \lim_{\Delta s \to 0} \frac{f(a + \Delta s \cos \alpha, b + \Delta s \sin \alpha) - f(a, b)}{\Delta s}.$$

Great care is taken to show the relation between hypothesis and conclusion. Effective use is made of "counter examples." There are many exercises.

The amount of space devoted to specific applications is indicated by the following numbers of pages: mercator maps, 1; least squares, 1; mass, center of gravity, moment of inertia, and force of attraction, 7; work, 1; vibrating strings, 8.

Criticisms: Only a few misprints were noted, such as: p. 13, Case II, $s=\varphi(r, s, t)$ should read $x=\varphi(r, s, t)$; p. 101, Theorem 1, C' should be C^1; p. 263, line 1, S_n should be σ_n. On page 5, and in the Index of Symbols, "not" is denoted by $|$, whereas in the text, "not" is denoted by /.

The text contains only 40 figures, many of which are rudimentary.

Italic letters are used for both scalar and vector variables. (While this is logically sound, it is a questionable psychological hazard.)

The author's English tends at times to be cryptically terse. For example, Exercise 8 on page 328 reads: Prove the rest of the orthogonality and normality relations.

On page 50 the author states: By a *vector* we mean a directed line segment. Farther down this page the author writes: DEFINITION 1. *A vector r is a triple of numbers* (r_1, r_2, r_3). A similar difficulty occurs in connection with homogeneity (p. 14) and ∇ (p. 65).

Conclusion: Students who can adapt themselves to Professor Widder's style will surely find this text to be elegant and cogent, and an admirable introduction to the finesse of mathematical methods.

C. C. TORRANCE

Antennae: an introduction to their theory. By J. Aharoni. New York, Oxford University Press, 1947. 8+265 pp. $8.50.

Antenna theory is a promising field of activity for mathematicians of varying degrees of expertness and "purity." Most of its boundary value problems and all of its integral equations are both difficult and intriguing. Antenna theory offers an opportunity for fundamental contributions both to the theory of partial differential equations and to the theory of integral equations of the first kind. Unfortunately it is not easy for the mathematician to become acquainted with antenna problems, for most books on the subject he would find unreadable. The present volume by J. Aharoni may remedy this situation. Although *Antennae: an introduction to their theory* is not written primarily for mathematicians as such or for the purpose of stimulating their interest in antenna problems, it is a book which is likely to be intelligible to them. Neither special engineering background nor interest in the nonmathematical features of antenna theory are required of the reader.

The book contains a clear and comprehensive account of up-to-date mathematical developments in this field and is divided into three chapters according to the methods of mathematical analysis. The first chapter, entitled Antennae and boundary-value problems, begins with Maxwell's partial differential equations, boundary conditions, and a general statement of the problem. This chapter contains an account of those solutions which may be obtained by the method of separation of independent variables. Thus it includes plane, cylindrical, and spherical waves in free space; spherical waves along coaxial cones; free and forced oscillations on spherical and spheroidal conductors.

The second and longest chapter is devoted to integral equations for antennas. Certain solutions of Maxwell's equations make it possible to convert boundary-value problems into integral equations. The low frequency circuit theory follows very readily from these equations and is treated first. It is followed by an approximation method of solving a more general integral equation and a discussion of the numerical results obtained by this method. The remainder of the chapter is of greater interest to engineers than to mathematicians. The third and last chapter presents a brief but exceptionally clear exposition of the "wave guide theory" of antennas. Mathematically this and the first chapter are closely related.

The book should appeal both to the applied mathematician who is interested in practical results and to the "purer" mathematician who might be interested in more fundamental mathematical problems that arise in the field of antenna theory.

S. A. SCHELKUNOFF

NOTES

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A new monthly scientific journal, Applied Mechanics Review, devoted to criticism and review of current world literature in applied mechanics, will be published by the American Society of Mechanical Engineers. The first issue is scheduled to appear in January, 1948.

Associate Professor J. J. Wheeler of the University of Kansas has retired with the title Emeritus.

Adjunct Professor W. E. F. Appuhn of the Polytechnic Institute of Brooklyn has been appointed to a professorship at St. Francis College, Brooklyn, New York.

Professor May M. Beenken of State Teachers College, Oshkosh, Wisconsin, has been appointed to an associate professorship at Immaculate Heart College, Los Angeles, California.

Assistant Professor B. W. Brewer of Agricultural and Mechanical College of Texas has been appointed to an assistant professorship at Oregon State College.

Assistant Professor W. R. Callahan of Northeastern University has been appointed to an assistant professorship of applied mechanics at New York University.

Dr. W. B. Caton of the University of Maine has been appointed to an assistant professorship at Washington State College.

Dr. J. D. Elder of the University of Michigan has been appointed to an associate professorship at St. Louis University.

Dr. G. E. Forsythe of Boeing Aircraft Company has been appointed to an assistant professorship of meteorology at the University of California at Los Angeles.

Dr. W. C. G. Fraser of Dartmouth College has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Dr. H. E. Goheen of the Office of Research and Inventions, Navy Department, has been appointed to an assistant professorship at the University of Delaware.

Assistant Professor A. A. Grau of the University of Kentucky has been appointed to an associate professorship at the University of Alabama.

Assistant Professor W. W. Gutzman of the United States Naval Postgraduate School has been appointed to a professorship at the University of South Dakota.

Associate Professor P. E. Lewis of Oklahoma Agricultural and Mechanical College has been appointed to an assistant professorship at North Carolina State College.

Mr. H. C. McKenzie of the University of Wisconsin has been appointed to an assistant professorship at Western State College, Gunnison, Colorado.

Professor H. F. MacNeish of Brooklyn College has been appointed to a visiting professorship at the University of Miami.

Mr. B. L. Miller of Swarthmore College has accepted a position as physicist with the Bartol Research Foundation, Franklin Institute.

Mr. B. N. Moyls of Harvard University has been appointed lecturer at the University of British Columbia.

Dr. E. N. Nilson of Pratt and Whitney Aircraft has accepted a position as analytical engineer at United Aircraft Corporation, East Hartford, Connecticut.

Professor F. S. Nowlan of the University of British Columbia will be visiting lecturer at the University of Illinois in Chicago for the current academic year.

Professor Rufus Oldenburger of the Illinois Institute of Technology is on leave of absence.

Dr. Irving Reiner of Cornell University is at the Institute for Advanced Study for the current academic year.

Mr. L. D. Rodabaugh of the Bureau of the Census has been appointed to an associate professorship at Southern Illinois University, Carbondale, Illinois.

Mr. E. J. Specht of the University of Minnesota has been appointed to a professorship at Emmanuel Missionary College, Berrien Springs, Michigan.

Assistant Professor E. C. Stopher of State Teachers College, Brockport, New York, has been appointed to an assistant professorship at Miami University, Oxford, Ohio.

Dr. A. H. Van Tuyl of Stanford University has accepted a position as mathematician with the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland.

Associate Professor Alexander Weinstein of Carnegie Institute of Technology has accepted appointments as senior associate at the

Naval Ordnance Laboratory and professor at the University of Maryland.

The following promotions are announced:

Claire F. Adler, New York University, to an associate professorship.

E. D. Eaves, University of Tennessee, to a professorship.

Cornelius Gouwens, Iowa State College of Agriculture and Mechanic Arts, to a professorship.

Coleman Herpel, Pennsylvania State College, Altoona, Pennsylvania, to an associate professorship.

I. M. Hostetter, Oregon State College, to an associate professorship.

W. J. Kirkham, Oregon State College, to an associate professorship.

A. G. Montgomery, College of St. Thomas, to an assistant professorship.

J. M. H. Olmsted, University of Minnesota, to an associate professorship.

R. R. Reynolds, Oklahoma Agricultural and Mechanical College, to an assistant professorship.

Peter Scherk, University of Saskatchewan, to an associate professorship.

W. S. Snyder, University of Tennessee, to an associate professorship.

The following appointments to instructorships are announced: University of Alabama: Miss Irma Berkowitz; University of Buffalo: Miss Jean L. Blaney, Mr. B. J. Clark, Mr. F. P. Kowalewski, Miss Jane L. Noller, Mr. D. D. Strebe; Bethany College: Miss Catherine E. Moser; University of Chicago: Mr. I. R. Hershner; Hunter College: Mr. M. E. White; Johns Hopkins University: Mr. George Shapiro, Mr. D. R. Waterman; University of Nebraska: Dr. Edwin Halfar, Dr. W. G. Leavitt; Oregon State College: Mr. J. F. Price, Mrs. Hazal Schroeder, Mrs. Evelyn Thurman, Mr. R. J. Wise; University of Tennessee: Mr. Ralph Donnell, Mr. Clyde Miller, Mr. Jack Moshman, Miss Ella M. Sowder, Miss Frances Street; University of Texas: Mr. W. L. Shepherd.

Professor Emeritus William Gillespie of Princeton University died on September 13, 1947, at the age of seventy-six years. He had been a member of the Society for forty-three years.

Professor J. C. Fitterer of Colorado School of Mines died on March 12, 1947. He had been a member of the Society for thirty-six years.

Dr. H. A. Ruger died in July 1947. He had been a member of the Society for twenty-five years.

APSIDAL ANGLES FOR SYMMETRICAL DYNAMICAL SYSTEMS WITH TWO DEGREES OF FREEDOM

J. L. SYNGE

The bob of a spherical pendulum (or particle on a smooth sphere under gravity) oscillates between two levels, and the change of azimuth in passing from the lower to the higher level, or vice versa, is the same for all such passages during a single motion. Using the language of orbit theory, we shall refer to this change of azimuth as the *apsidal angle*. It is a function of the two constants of the motion, total energy and angular momentum, but the remarkable fact is that the apsidal angle α always satisfies the inequalities[1]

$$\pi/2 < \alpha < \pi. \tag{1}$$

The purpose of the present note is not to add to the theory of the spherical pendulum, but rather to place the problem in a more general setting. Consider a particle of unit mass which moves on a surface of revolution Σ under the action of a conservative field of force for which the potential energy is independent of azimuth.

Let A be the axis of Σ. Then Σ is determined completely by the section C of Σ by a half-plane terminated by A. Since the particle cannot pass from a portion of Σ to a disconnected portion of Σ, we may suppose C to be a single connected curve.

Several cases present themselves:

(i) C is an open curve with both ends on A (Σ homeomorphic to a sphere).

(ii) C is an open curve with one end on A and the other end at infinity (Σ homeomorphic to a paraboloid of revolution).

(iii) C is an open curve with both ends at infinity (Σ homeomorphic to a cylinder).

(iv) C is a closed curve which does not meet A (Σ homeomorphic to a torus).

In any one of these cases, we assign coordinates R, ϕ, where R is arc length measured along C and ϕ is the azimuthal angle. The range of ϕ is taken to be $(-\infty, \infty)$, an increase of 2π in ϕ leading us back to the same point. The range of R depends on the type of C. In case (i), the range of R is finite; in case (ii), the range may be taken as

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[1] For references to earlier work, see P. Appell, *Traité de mécanique rationnelle*, vol. 1 (1926) p. 518; see also A. Weinstein, Amer. Math. Monthly vol. 49 (1942) pp. 521–523, W. Kohn, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 107–131.

$(0, \infty)$; in case (iii), the range is $(-\infty, \infty)$; in case (iv), the range is taken to be $(-\infty, \infty)$, an increase in R equal to the length of C leading us back to the same point.

The line element of Σ may be written

$$d\sigma^2 = dR^2 + B(R)d\phi^2, \tag{2}$$

and the potential energy is $V(R)$. At any point common to Σ and the axis A we have $B(R) = 0$. In case (iv) B and V are periodic functions of R, the period being the length of C. We shall assume B and V to be of class C_2 in all domains where the motion of the particle is considered.

Although the dynamical system described above consists of a particle on a surface of revolution, it is clear that any results obtained will be available also for a conservative dynamical system with two degrees of freedom and one ignorable coordinate, the Lagrangian function being

$$L = 2^{-1}[A(R)\dot{R}^2 + B(R)\dot{\phi}^2] - V(R). \tag{3}$$

We shall, however, continue to refer to the system as if it consisted of a particle on a surface of revolution.

According to Jacobi's principle of stationary action, the path of the particle, if moving with total energy E, is a geodesic in a two-space with metric ds where

$$ds^2 = (E - V)d\sigma^2. \tag{4}$$

Let us introduce instead of R a new coordinate r, defined by

$$r = \int^R [E - V(u)]^{1/2}du, \tag{5}$$

the lower limit of integration being arbitrary. Then (4) becomes, by use of (2) and (5),

$$ds^2 = dr^2 + G(r)d\phi^2, \tag{6}$$

where

$$G(r) = [E - V(R)]B(R). \tag{7}$$

The form (6) may itself be regarded as the line element of a surface of revolution S. *Accordingly the study of particle paths on a surface of revolution Σ reduces to the study of geodesics on another surface of revolution S.*[2]

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[2] In general, S cannot be imbedded in a Euclidean 3-space. Since we shall be concerned solely with the intrinsic properties of S, that fact is not important for the purposes of the present paper.

The geodesics for (6) satisfy the Lagrangian equations

$$\frac{d}{ds}\frac{\partial w}{\partial r'}-\frac{\partial w}{\partial r}=0,\qquad \frac{d}{ds}\frac{\partial w}{\partial \phi'}-\frac{\partial w}{\partial \phi}=0, \tag{8}$$

where the prime means d/ds and

$$w=2^{-1}[r'^2+G(r)\phi'^2]. \tag{9}$$

Since ϕ is ignorable, we have the first integral

$$G(r)\phi'=k, \tag{10}$$

a constant analogous to the constant of angular momentum. Equation (10) replaces the second of (8); instead of the first of (8), we may use the first integral

$$r'^2+G(r)\phi'^2=1. \tag{11}$$

Elimination of s between (10) and (11) gives

$$(dr/d\phi)^2=G(G-k^2)/k^2. \tag{12}$$

The two constants, E and k, are involved in the theory, but they play very different roles. The constant E is implicit in the line element (6), through G, but the line element does not involve k. Since

$$E=T+V, \tag{13}$$

where T is the kinetic energy, and T cannot be negative, we see that E is greater than or equal to V at any point in the path of the particle. This places on E the condition

$$E\geqq \text{minimum of } V \text{ on } \Sigma. \tag{14}$$

The constant k must satisfy a more exacting inequality. Since $r'^2\geqq 0$ and $G(r)\geqq 0$ by (7) and the positive definite character of (2),

$$k^2\leqq G(r) \tag{15}$$

follows from (10) and (11), for all values of r on the particle path; this may also be seen directly from (12).

The *apsides* of a particle path correspond to maximum and minimum values of R, and the *apsidal angle* α is the increment in the azimuthal angle ϕ in passing from minimum to maximum or from maximum to minimum.[3] Since the transformation (5) is monotone, and ϕ is not transformed in the passage from Σ to S, it follows that *the apsidal distances* (R_1, R_2) *for the particle path on* Σ *correspond by* (5) *to the apsidal distances* (r_1, r_2) *for the corresponding geodesic on* S, *and the apsidal angles are equal.*

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[3] The surface of revolution is a particular case of the radial manifold, and radial apsides coincide with potential apsides; cf. J. L. Synge, Trans. Amer. Math. Soc. vol. 34 (1932) pp. 481–522.

To find the apsidal distances, we put $dr/d\phi=0$ in (12). We have then to consider the two equations $G=0$ and $G-h^2=0$. By (10), $G=0$ implies $h=0$, and so $G=0$ implies $G-h^2=0$. Consequently we have to consider only the equation

$$G = h^2. \tag{16}$$

However, the case $h=0$ is singular, and we shall exclude it from further consideration. By (10), it implies $\phi'=0$ for values of r not making $G=0$, and so the motion is along a meridian curve (simple pendulum motion for a spherical pendulum).

If (16) has no roots, there are no apsides. To study the existence of apsides, let us regard E as assigned, so that the function $G(r)$ is determined. Oscillation between apsidal distances r_1, r_2 $(r_1<r_2)$ will occur if, and only if, there exists a pair of roots of (16):

$$G(r_1) = h^2, \qquad G(r_2) = h^2 \qquad (r_1 < r_2), \tag{17}$$

with the inequality (15) satisfied for all intermediate values of r:

$$G(r) \geqq h^2 \qquad (r_1 \leqq r \leqq r_2). \tag{18}$$

The scheme for finding apsidal distances is then clear. The constant E being assigned, we consider the graph of $G(r)$ versus r. Unless the graph shows at least one relative maximum, a pair of apsidal distances cannot occur. Let us suppose that $r=r_0$ gives a relative maximum, so that the derivative vanishes:

$$(dG/dr)_{r=r_0} = 0. \tag{19}$$

We then draw a parallel to the r-axis below this maximum, cutting the curve of $G(r)$ at r_1 and at r_2, with $r_1<r_0<r_2$, and assign to h the value given by (17). Then there exists a geodesic with apsidal distances r_1, r_2, and, by (12), the apsidal angle is given by

$$\alpha = \int_{r=r_1}^{r=r_2} d\phi = h\int_{r_1}^{r_2} [G(G-h^2)]^{-1/2}dr. \tag{20}$$

It is interesting to note that if we had chosen $h^2=G(r_0)$ we would have obtained a circular geodesic (corresponding to the conical pendulum). This is easily seen from equations (8)–(12). In fact, *any geodesic on* S *possessing two apsidal distances weaves in and out across a circular geodesic.*

Let us now consider bounds for the apsidal angle α, assuming that in the range (r_1, r_2) there is only one value r_0 satisfying (19). Then in each of the ranges (r_1, r_0), (r_0, r_2) G is a monotone function of r. We may write (20) in the form

$$
\begin{aligned}
\alpha &= h\int_{r=r_1}^{r=r_0} [G(G-h^2)]^{-1/2}\,|dr/dG|\,dG \\
&\quad + h\int_{r=r_2}^{r=r_0} [G(G-h^2)]^{-1/2}\,|dr/dG|\,dG \qquad (21)\\
&= h\int_{h^2}^{G_0} [G(G-h^2)]^{-1/2}[\,|dr/dG|_1 + |dr/dG|_2]dG,
\end{aligned}
$$

where $G_0 = G(r_0)$, and the subscripts 1 and 2 refer respectively to the ranges (r_1, r_0) and (r_0, r_2).

Consider now the ratio

$$
\frac{|dG/dr|}{(G_0-G)^{1/2}} \tag{22}
$$

in the range (r_1, r_2). As r approaches r_0, this ratio has the limit

$$
(-2d^2G/dr^2)^{1/2},
$$

evaluated at $r = r_0$. Accordingly the ratio (22) is bounded above and below, and we may write

$$
m \leqq (G_0-G)^{-1/2}|dG/dr| \leqq M \qquad (r_1 \leqq r \leqq r_2), \tag{23}
$$

where m and M are in general functions of E and h. Equivalently,

$$
M^{-1}(G_0-G)^{-1/2} \leqq |dr/dG| \leqq m^{-1}(G_0-G)^{-1/2} \qquad (r_1 \leqq r \leqq r_2). \tag{24}
$$

We might have $m = 0$, in which case the upper bound is infinite. With the inequalities (24), equation (21) gives

$$
M^{-1}I \leqq \alpha \leqq m^{-1}I, \tag{25}
$$

where

$$
I = 2h\int_{h^2}^{G_0} [G(G_0-G)(G-h^2)]^{-1/2}dG. \tag{26}
$$

Changing the variable of integration by the transformation

$$
G = h^2\cos^2\theta + G_0\sin^2\theta, \tag{27}
$$

we get

$$I = 4\int_0^{\pi/2} [1 + (G_0/h^2 - 1)\sin^2\theta]^{-1/2} d\theta, \tag{28}$$

and hence, since $G_0 > h^2$,

$$2\pi h G_0^{-1/2} < I < 2\pi. \tag{29}$$

Substitution in (25) gives the following result: *For motion on a surface of revolution* Σ *(or, equivalently, for a geodesic on a surface of revolution* S *with line element* (6)), *the apsidal angle* α *satisfies the inequalities*

$$2\pi M^{-1} h G_0^{-1/2} < \alpha < 2\pi m^{-1}, \tag{30}$$

where h *is the constant of the first integral* (10), G_0 *is* $G(r_0)$ *where* r_0 *satisfies* (19), *and* M, m *are respectively upper and lower bounds of the ratio* (22) *in the range of oscillation* (r_1, r_2), *determined by* $G = h^2$.

It is interesting to see what happens when the geodesic under consideration lies close to the circular geodesic $r = r_0$. Then h^2 approaches G_0 and the two bounds M and m approach the common value $(-2d^2G/dr^2)^{-1/2}$ evaluated at r_0; we get in the limit

$$\alpha = 2\pi(-2d^2G/dr^2)^{-1/2}, \tag{31}$$

evaluated at r_0. This value is easily checked by differentiating (12), dividing by $dr/d\phi$, and linearizing the resulting equation.

Having transformed the problem of the apsidal angle for motion on a surface of revolution to that for geodesics on a surface of revolution, we can bring to bear on it well known results in differential geometry. Consider, on the surface of revolution S, the triangle AOB the sides of which are geodesics. Let O be the pole of S, that is, the point where it is met by the axis. Then OA and OB are meridians. Further, suppose that A and B are adjacent apsides on the geodesic AB, so that the angles ABO and BAO are right angles and AOB is the apsidal angle α. Then α is the excess of the angle sum over π, and so by the well known formula

$$\alpha = \iint K dS, \tag{32}$$

where K is the Gaussian curvature of S for the metric (6) and the integration extends over the triangle AOB, dS being the element of area.

The formula (32) brings an interesting fact to light. Consider the small oscillations of a spherical pendulum about the position of equilibrium. It is known that the particle path is approximately a central ellipse ($\alpha=\pi/2$). Consider the triangle AOB as described above. To that triangle on S there corresponds on the sphere Σ a triangle which we may also call AOB; the sides AO and OB are great circles joining O, the lowest point of the sphere Σ, to adjacent apsides A and B; AB is the particle path between apsides.

Now, for small oscillations, the area of the triangle AOB on Σ is small. The area of AOB on S is still smaller, since the factor $E-V$ involved in passing from $d\sigma^2$ to ds^2 is small for small kinetic energy. How then can the integral in (32) have a finite value, approximately $\pi/2$? The answer is to be found in K. For the metric (6), the circle $V=E$ is singular, and the Gaussian curvature of S tends to infinity as we approach this circle. If we take the zero of potential energy at the lowest point of the sphere, that is, at the point O, then small oscillations correspond to small values of E, and the radius of the circle $V=E$ is small. But inside this circle, considered in S, the Gaussian curvature is great, in such a way that its greatness balances the smallness of the area of the circle.

These considerations point to the desirability of the further investigation of the geometry of dynamical systems when the Jacobi metric (6) is used; this remark applies not merely to the simple symmetric case discussed, above, but to a general dynamical system. The locus $V=E$ has an interesting geometry. It is a null-domain, in the sense that the length of any curve drawn in it is zero. It forms a barrier across which a geodesic cannot pass; in fact, geodesics approaching it are bent back sharply. Furthermore, on it the curvature becomes infinite.

It might seem that the reduction of the question of dynamical apsidal angles to the question of geodesic apsidal angles might yield some simple general bounds. So we ask: For the whole class of surfaces of revolution, do there exist upper and lower bounds for apsidal angles? We shall show that the answer is in the negative, by constructing models in which the apsidal angles pass any given bound. It is simplest to use limiting models with sharp edges; passage to the limit from smooth surfaces will not destroy the inequalities.

Consider a long circular cylinder with flat ends, the pole O being the center of one end E. Consider a geodesic which consists in part of a chord of E, not a diameter. The midpoint of this chord is an apse; in fact, it is a point of minimum distance from O. Pursuing the geodesic in one direction, it passes over the edge of E, and winds round and round the curved sides of the cylinder. The azimuthal angle increases by 2π in each such revolution, and the next apse cannot occur until the geodesic reaches the other end of the cylinder.

Obviously, by making the cylinder sufficiently long, we can make the apsidal angle as great as we please.

To get a small apsidal angle, we take as our surface the two faces of a flat circular disk. The pole O is the center of one face E. Consider a geodesic consisting in part of a chord of E which subtends at O a small angle β. The midpoint of the chord is an apse. Follow this geodesic in one direction over the edge of the disk and on to the other face E'. The chord on E' and the chord on E will make equal angles with the radius drawn from O to the point where the geodesic passes over the edge. The midpoint of the chord on E' will also be an apse. The apsidal angle is $\alpha = \beta$, and can be made as small as we please by making β sufficiently small.

Consequently, for the whole class of surfaces of revolution, there is neither an upper nor a lower bound for the geodesic apsidal angle.

CARNEGIE INSTITUTE OF TECHNOLOGY

NONLINEAR NETWORKS. IIb

R. J. DUFFIN

This note is concerned with the quasi-linear properties of an n-dimensional transformation

$$\begin{aligned} y_1 &= S_1(x_1, \cdots, x_n), \\ &\cdots\cdots\cdots\cdots, \\ y_n &= S_n(x_1, \cdots, x_n). \end{aligned} \tag{1}$$

More precisely it is shown by imposing certain conditions A on the functions S, that the transformation has the property of possessing a unique inverse. In this property, then, the transformation is analogous to a nonsingular linear transformation. The actual conditions A imposed on the functions S_i are so chosen that the transformation (1) shall be a generalization of the equations which define the steady flow of current in electrical networks made up of quasi-linear conductors. It is reasonable to believe, however, that the methods developed here can, with suitable modification, be used to study the quasi-linear properties of types of transformations which have nothing to do with electrical networks.

At least three different methods of attack are available: The first method is to set up a form (analogous to a positive definite quadratic form) such that the equations (1) are the conditions that this form take on its minimum value. This insures the existence of a solution. A second positive definite form involving the differences of two transformations proves the uniqueness. It is well known that for linear transformations this method has the disadvantage of being applicable only for self-adjoint transformations. A similar disadvantage appears in the nonlinear case. This method was exploited in two previous notes: *Nonlinear networks.* I, and *Nonlinear networks.* IIa. (No appeal is made in this note to results obtained in the previous notes.)

The second method, which is the one employed in this note, is to impose conditions A_1 such that the Brouwer fixed point theorem is available. This insures the existence of the inverse transformation. Corresponding to the differences of two transformations, we associate a linear transformation somewhat analogous to a differential transformation. Conditions A_2 are then imposed, which insure that the associated linear transformation satisfies A_1. Hence, the linear trans-

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formation has an inverse. But a linear transformation with an inverse has a unique inverse, so transformation (1) has a unique inverse.

The third method is that of induction on the dimension. For example, suppose that all transformations of dimension $n-1$ have a unique inverse. With y_n fixed, solve the last equation for x_n in terms of $x_1, \cdots, x_n-1$ and substitute in the previous equations. If conditions A are so chosen that they are dominant under this inbreeding, it follows that the resulting $(n-1)$-dimensional transformation is of the same form and, hence, has a unique inverse by the inductive hypothesis. It is to be noted that if the conditions A are to be dominant, they must be neither too strong nor too weak. For example, the so-called Maxwell junction equations discussed in the previous note would not be invariant in form under this inbreeding. The physical significance of this fact is that a network with one or more concealed junctions may not in general be described in terms of Maxwell's equations applied to the unconcealed junctions. However, for networks of linear conductors it is possible to show by virtue of the distributive law that Maxwell's equations are applicable to the unconcealed junctions; engineers speak of this as a star-mesh transformation. Hence, in engineering terminology the inductive method may be spoken of as the star-mesh method. Theorem 6 below states that the star-mesh transformation preserves conditions A. An independent proof may be constructed for Theorem 6 so the inductive method could be applied. While the inductive method is direct, it is too synthetic to be illuminating, so it has not been used.

As a concrete example of the type of transformations to be treated here, consider the system

$$y_1 = (x_1 - x_2)^3 + (x_1 - x_3)^3,\ y_2 = (x_2 - x_3)^3 + (2x_2 - x_1)^3,$$
$$y_3 = (3x_3 - 2x_1 - x_2)^3 + (3x_3 - 2x_2 - x_1)^3.$$

Inspection of this system shows that the conditions of Theorem 1 are satisfied. Hence, choosing any real numbers for y_1, y_2, y_3, there is a unique real solution x_1, x_2, x_3. It is assumed hereafter that all functions and constants are real-valued.

1. **Connected transformations.** Clearly, it is always possible to express transformation (1) in the form

$$\begin{aligned} y_1 &= P_1(x_1, x_1 - x_2, \cdots, x_1 - x_n), \\ y_2 &= P_2(x_2 - x_1, x_2, \cdots, x_2 - x_n), \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ y_n &= P_n(x_n - x_1, x_n - x_2, \cdots, x_n). \end{aligned} \tag{2}$$

The set of functions $P_i(t_1, t_2, \cdots, t_n)$, $i=1, 2, \cdots, n$, will be called an n-dimensional *connected foundation* if for $i, j=1, 2, \cdots, n$ and all values of the variables:

(a) *P_i is a continuous function of the n variables.*

(b) *P_{ij} is either an increasing function unbounded at $\pm\infty$, or is constant.*

(c) *There is a sequence of integers $a, b, \cdots, g, h$ (dependent on i) such that in the chain $P_{ia}, P_{ab}, P_{bc}, \cdots, P_{gh}, P_{hh}$, each function is unbounded at $\pm\infty$.*

Here P_{ij} indicates P_i considered as a function of t_j, other variables being held fixed at arbitrary values. If the set of functions P_i forms a connected foundation, then transformation (2) will be called a *connected transformation*. These are the conditions A referred to above.

THEOREM 1. *For any assigned values of $y_1, \cdots, y_n$, a connected transformation has a unique solution $x_1, \cdots, x_n$.*

PROOF. The theorem will be an obvious consequence of the following two lemmas. It is clear, moreover, that the lemmas are actually more general than Theorem 1. However, under the conditions of the lemmas, Theorem 4 is not generally true, and in this respect transformation (2) would behave less like a linear transformation.

LEMMA 1. *Transformation (2) has at least one solution if*: (a) *P_i is a continuous function.* (b_1) *P_{ij} is nondecreasing.* (c_1) *For each i there is a chain in which the functions are unbounded at $+\infty$ and for each i there is a chain in which the functions are unbounded at $-\infty$.*

PROOF. Let y be a constant vector and let ϵ be a positive number. The relations $v_1=\epsilon x_1+P_1(x_1, x_1-x_2, \cdots, x_1-x_n)-y_1$, and so on, define a continuous vector field v when the vector x ranges in and on a cube with corners at $(\pm k, \pm k, \cdots, \pm k)$, $k>0$. From condition (b_1) it follows for $x_1=k$ that $v_1 \geqq \epsilon k+P_1(0, 0, \cdots, 0)-y_1$. Likewise for $x_1=-k$ it follows that $v_1 \leqq -\epsilon k+P_1(0, 0, \cdots, 0)-y_1$. Clearly, then, for $k>\max |P_i(0, 0, \cdots, 0)-y_i|\epsilon^{-1}$ it follows that on the surface of the cube the vector v is pointing outside the cube. Brouwer's fixed point theorem states that if a continuous vector points outside a cube on the surface, then there is at least one point inside the cube where the vector vanishes.[1] At this point

$$y_i = P_i(x_i - x_1, \cdots, x_i - x_n) + \epsilon x_i.$$

It will now be shown that the quantities x_i are bounded independent of ϵ. For some integer i, $x_i \geqq x_j$, $j=1, 2, \cdots, n$. Consider a chain sequence $i, a, \cdots, g, h$, for which the functions P_{ia}, $P_{ab}, \cdots, P_{hh}$ are unbounded at $+\infty$, and first suppose $x_i, x_a, \cdots, x_h$ are all positive. Then since x_i is positive

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[1] S. Eilenberg suggested to the writer the use of Brouwer's theorem instead of a more special procedure.

$$\begin{aligned} y_i &\geqq P_i(x_i - x_1, \cdots, x_i - x_a, \cdots, x_1 - x_n) \\ &\geqq P(0, \cdots, x_i - x_a, \cdots, 0). \end{aligned}$$

Thus there is a positive constant c_i independent of ϵ such that $x_i - x_a \leqq c_i$. Hence, $x_j - x_a \leqq c_i$ for any j and $x_a - x_j \geqq -c_i$. Therefore

$$\begin{aligned} y_a &\geqq P_a(x_a - x_1, \cdots, x_a - x_b, \cdots, x_a - x_n) \\ &\geqq P_a(-c_i, \cdots, x_a - x_b, \cdots, -c_i). \end{aligned}$$

Thus there is a positive constant b_a such that $x_a - x_b \leqq b_a$ and $x_j - x_b \leqq c_i + b_a = c_a$. This process is continued until finally we have

$$\begin{aligned} y_h &\geqq P_h(x_h - x_1, \cdots, x_h, \cdots, x_h - x_n) \\ &\geqq P_h(-c_g, \cdots, x_h, \cdots, -c_g), \end{aligned}$$

and there is a positive constant c_h such that $x_h \leqq c_h$.

$$\begin{array}{lccccccccc} x_i = & (x_i - x_a) & + & (x_a - x_b) & + \cdots + & (x_g - x_h) & + & x_h, \\ x_i \leqq & c_i & + & c_a & + \cdots + & c_g & + & c_h. \end{array}$$

The same inequality holds if some member of the sequence $x_i, \cdots, x_h$ is not positive. Suppose x_e is the first member of this sequence which is not positive, then

$$\begin{array}{lccccccc} x_i = & (x_i - x_a) & + & (x_a - x_b) & + \cdots + & (x_d - x_e) & + & x_e, \\ x_i \leqq & c_i & + & c_a & + \cdots + & c_d. & & \end{array}$$

The constants c depend only on the vector y and the growth of the functions P_i. There are such constants for each integer i so the components of the vector x have a finite upper bound.

A symmetrical argument, using chains unbounded at $-\infty$, shows that the components of x have a finite lower bound.

As ϵ approaches zero, it follows that the vector x has at least one limit point, and since P_i is continuous this proves Lemma 1.

LEMMA 2. *Transformation* (2) *may not have more than one solution if*: (b_1) P_{ij} *is nondecreasing.* (c_2) *For each* i *there is a chain in which each of the functions is an increasing function.*

PROOF. If the functions P_i are homogeneous linear functions, then transformation (2) becomes

$$
(3)\quad
\begin{aligned}
y_1 &= p_{11}x_1 + p_{12}(x_1 - x_2) + \cdots + p_{1n}(x_1 - x_n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
y_n &= p_{n1}(x_1 - x_n) + p_{n2}(x_n - x_2) + \cdots + p_{nn}x_n.
\end{aligned}
$$

Conditions (b) and (b_1) are equivalent and state that the constants p_{ij} are non-negative. Conditions (c), (c_1), and (c_2) are equivalent and state that for each i there is a sequence of integers $a, b, \cdots, h$ such that no member of the chain $p_{ia}, p_{ab}, \cdots, p_{gh}, p_{hh}$ vanishes.

Let x and x' be two vectors with transforms y and y'. Let $\delta x = x' - x$ and $\delta y = y' - y$. Let $t_1 = x_1$, $t_2 = x_1 - x_2, \cdots, t_n = x_1 - x_n$ and let $t_1' = x_1'$, $t_2' = x_1' - x_2', \cdots, t_n' = x_1 - x_n'$. Then $\delta y_1 = P_1(t_1', \cdots, t_n') - P_1(t_1, \cdots, t_n) = [P_1(t_1', t_2', \cdots, t_n') - P_1(t_1, t_2', \cdots, t_n')] + [P_1(t_1, t_2', \cdots, t_n') - P_1(t_1, t_2, \cdots, t_n')] + \cdots + [P_1(t_1, \cdots, t_{n-1}, t_n') - P_1(t_1, t_2, \cdots, t_n)]$. Let $p_{1j} = [P_1(t_1, \cdots, t_j', t_{j+1}', \cdots, t_n') - P(t_1, \cdots, t_j, t_{j+1}', \cdots, t_n')]\,(t_j' - t_j)^{-1}$ if $t_j' - t_j \neq 0$ and $P_{ij} = 1$ if $t_j' - t_j = 0$.

Thus $\delta y_1 = p_{11}\delta x_1 + p_{12}(\delta x_1 - \delta x_2) + \cdots + p_{1n}(\delta x_1 - \delta x_n)$. By similar definitions there are corresponding expressions for $\delta y_2, \cdots, \delta y_n$. The constants p_{ij} define a linear connected transformation; hence, by Lemma 1 this transformation has an inverse. But a linear transformation with an inverse has a unique inverse, so if $\delta y = 0$, then $\delta x = 0$. This completes the proof of Lemma 1.

Let us designate the inverse of transformation (2) by

$$
\begin{aligned}
x_1 &= R_1(y_1, \cdots, y_n),\\
&\cdots\cdots\cdots\cdots,\\
x_n &= R_n(y_1, \cdots, y_n).
\end{aligned}
$$

THEOREM 2. *The inverse of a connected transformation is a continuous transformation.*

PROOF. The proof of Lemma 1 with $\epsilon = 0$ shows that R is a bounded transformation. Thus, if $y' \to y$, then for some sub-sequence $x' = Ry'$ approaches a limit, say x. Now $y' = Px'$, so $y = Px$, since P is continuous. This uniquely defines x, so all sub-sequences have the same limit.

THEOREM 3. *In an n-dimensional connected transformation ($n > 1$) delete the nth equation and let x_n = constant in the rest. Then there remains an $(n-1)$-dimensional connected transformation.*

PROOF. The foundation for the new system is, if $x_n = c$,

$$
P_i'(t_1, \cdots, t_{n-1}) = P_i(t_1, \cdots, t_{n-1}, t_1 - c), \qquad i = 1, \cdots, n-1,
$$

so clearly conditions (a) and (b) are satisfied. The chain sequences which do not contain the integer n are left unchanged. A chain sequence of the form $i, a, \cdots, d, n, \cdots, k$ is replaced by the sequence $i, a, \cdots, d$. It follows that P'_{id} is unbounded because P_{dn} is unbounded. Thus (c) is satisfied.

THEOREM 4. *If transformation* (2) *satisfies* (a) *and* (b) *but not* (c), *then it does not have a unique solution. For some integer k, x_k may be given an arbitrary value.*

PROOF. The integers $1, \cdots, n$ are divided into sets C and D. We put the integer i in C if and only if there exists a sequence of integers $a, b, \cdots, k$ such that each of the functions $P_{ia}, P_{ab}, \cdots, P_{gk}, P_{kk}$ is unbounded. It is to be noted that if this situation obtains each of $a, b, \cdots, k$ is also in C. Moreover, if i is in C and if P_j actually contains x_i, then j is in C because P_{ji} is not constant and the chain $P_{ji}, P_{ia}, \cdots, P_{kk}$ is made up of unbounded functions. An integer k is in D if and only if it is not in C; and in view of the preceding remarks, P_{kk} is a constant.

The variables x_i and the equations $y_i = P_i$ are also divided into two sets depending on whether i belongs to C or D. Considering the D variables as constants, the C equation forms a connected transformation in the C variables. To prove this, write down the foundation P' as in the proof of Theorem 3, and it is clear that conditions (a) and (b) are satisfied. The chain condition (c) is satisfied by construction. Hence, no matter what values are assigned to the variables D, the C equations can be satisfied. The C variables do not occur in the D equations. Moreover, the D variables occur only as paired differences in the D equations because P_{kk} is a constant if k is in D. Hence, adding the same constant to each of the D variables gives a family of solutions if there is one solution.

THEOREM 5. *If transformation* (2) *satisfies* (a) *and* (b) *but not* (c), *then it does not have a solution x for all y.*

PROOF. The theorem is clearly true for a one-dimensional transformation, so we proceed by induction. According to Theorem 4, we may set $x_k = 0$. Delete the kth equation. What is left is an $(n-1)$-dimensional transformation of the form (2) and (a) and (b) are satisfied. If (c) is also satisfied, x_i is uniquely determined; hence y_k is uniquely determined by the other y_i. If (c) is not satisfied, these equations are singular by the inductive hypothesis.

THEOREM 6. *In the inverse of an n-dimensional connected transforma-*

tion $(n>1)$, *delete the nth equation and let* $y_n=$ *constant in the rest; then there remains the inverse of an* $(n-1)$-*dimensional connected transformation.*

PROOF. Thus $c=P_n(x_n-x_1, \cdots, x_n)$. Let $x_n-x_1=z_1$, then $c=P_n(z_1, z_1+(x_1-x_2), \cdots, z_1+x_1)$. By conditions (a), (b), and (c) it follows that P_n is a continuous increasing and unbounded function of z_1, so we may write $-z_1=H_1(x_1, x_1-x_2, \cdots, x_1-x_{n-1})$. From conditions (a) and (b), it follows that $H_1(t_1, t_2, \cdots, t_{n-1})$ is a continuous function and that H_{1j} is either a constant or is an increasing function unbounded at $\pm\infty$. Substituting z_1 in the first equation gives $y_1=P_1(x_1, x_1-x_2, \cdots, H_1(x_1, x_1-x_2, \cdots, x_1-x_{n-1}))$. The first function of the new foundation thus is

$$P_1'(t_1, \cdots, t_{n-1}) = P_1(t_1, \cdots, t_{n-1}, H_1(t_1, \cdots, t_{n-1})).$$

Clearly P_1' satisfies (a) and (b). The same result follows, by symmetry, for $P_2', \cdots, P_{n-1}'$. But the transformation P' has a unique inverse; hence, by Theorem 4 or 5 it follows that (c) is satisfied.

THEOREM 7. *If* $x_i=R_i(y_1, \cdots, y_n)$, $i=1, \cdots, n$, *is the inverse of a connected transformation and* R_{ij} *indicates* R_i *as a function of* y_j *then* R_{ij} *and* $R_{ii}-R_{ji}$ *are either constant or increasing functions, unbounded at* $\pm\infty$; *moreover,* R_{ii} *is not constant.*

PROOF. By repeated application of Theorem 6, we note that $R_1(y_1, c_2, \cdots, c_n)$ is the inverse of a one-dimensional connected foundation for any choice of constants $c_2, \cdots, c_n$. Hence R_{11} is an increasing function, unbounded at $\pm\infty$. By symmetry the same is true of R_{ii}.

Likewise, $x_1=R_1(y_1, y_2, c_3, \cdots, c_n)$ and $x_2=R_2(y_1, y_2, c_3, \cdots, c_n)$ is the inverse of a connected transformation of the form $y_1=P_1(x_1, x_1-x_2)$ and $y_2=P_2(x_2-x_1, x_2)$. Suppose that y_1 takes and increase $\delta y_1>0$ and that $\delta y_2=0$. By the result just proved $\delta x_1>0$ and $\delta x_1\to+\infty$ if $\delta y_1\to+\infty$. Consider the second equation. If P_{21} is constant and P_{22} is not constant, $\delta x_2=0$ so $\delta(x_2-x_1)=-\delta x_1<0$. If P_{22} is constant and P_{21} is not constant $\delta(x_2-x_1)=0$, so $\delta x_2=\delta x_1>0$. If neither P_{21} nor P_{22} is constant either $\delta x_2\leqq 0$ and $\delta(x_2-x_1)\geqq 0$ or $\delta x_2>0$ and $\delta(x_2-x_1)<0$. The first possibility is incompatible with $\delta x_1>0$. In the latter case if $x_1\to+\infty$ then x_2 can not remain bounded or y_2 would not be constant. By the same reasoning (x_2-x_1) is not bounded. A similar argument applies for $\delta x_1<0$, so this proves the theorem for R_{21} and $R_{11}-R_{21}$. By symmetry the theorem must hold for arbitrary indices.

2. **Linear connected transformations.** To put the connected trans-

formation (3) into conventional matrix form, define a *connected matrix* $\|s_{ij}\|$ as

$$s_{ij} = -p_{ij}, \qquad i \neq j,$$

and

$$s_{ii} = p_{i1} + p_{i2} + \cdots + p_{in}.$$

Then equations (3) become $y_i = \sum_{j=1}^{n} s_{ij}x_j$, $i = 1, \cdots, n$.

Sylvester [3][2] was the first to investigate linear transformations of this type, and he called the determinant $|s_{ij}|$ a *unisignant* determinant because of the nature of the following theorem.

THE SYLVESTER-BORCHARDT THEOREM. *The determinant* $|s_{ij}|$ *of an* n*-dimensional linear connected transformation is the sum of the products of the coefficients* p_{ij} *taken* n *at a time in such wise that the coefficients appearing in each product separately satisfy the chain condition.*

Later Maxwell [2] found that equations of the form (3) with $p_{ij} = p_{ji}$ define the steady flow of current in an electrical network of $n+1$ junctions, one of whose junction is held at zero potential. The other junctions have potentials $x_1, \cdots, x_n$, and the currents entering these junctions from outside are $y_1, \cdots, y_n$. The conductivity of the wire connecting the ith and jth junction is p_{ij}, $i \neq j$, while p_{kk} is the conductivity of the wire connecting the kth junction to the junction held at zero potential.

J. J. Thompson in an appendix to Chapter IV of the third edition of Maxwell's treatise stated a neat formula for the solution of the network equations. Equations of the form (3) with $p_{ij} = p_{ji}$ also appear in Maxwell's treatise in connection with the coefficients of capacity of neighboring conductors. Stieltjes [4] proved several theorems concerning these coefficients of capacity. Unfortunately these and later writers on electrical theory seem unaware of Sylvester's more profound treatment.

Let $\|r_{ij}\|$ be the inverse of the connected matrix $\|s_{ij}\|$. We shall now state some properties of these matrices which follow directly from the preceding theorems.

THEOREM 8. *The matrix* $\|s'_{ij}\|$ *corresponding to* $p'_{ij} = p_{ij} + p_{ji}$ *is positive definite.*

PROOF. If we let $p'_{ij} = p_{ij} + p_{ji}$, then $\|s'_{ij}\|$ is clearly a connected matrix and so is nonsingular. The theorem is then obvious from the identity

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[2] Numbers in brackets refer to the references cited at the end of the paper.

$$2\sum_{1}^{n}\sum_{1}^{n} s'_{ij}x_ix_j = \sum_{1}^{n}\sum_{1}^{n} p'_{ij}(x_i - x_j)^2 + 2\sum_{1}^{n} p'_{ii}x_i^2.$$

THEOREM 9. *If $n>1$ then $\|s_{nn}s_{ij}-s_{in}s_{nj}\|$, $i, j=1, \cdots, n-1$, is an $(n-1)$-dimensional connected matrix.*

PROOF. According to Theorem 6, $y_i=\sum_1^n s_{ij}x_j$, $i=1, \cdots, n-1$, where x_n is defined by $0=\sum_1^n s_{nj}x_j$, is an $(n-1)$-dimensional connected transformation. Then $x_n=-\sum_1^{n-1}s_{nj}x_j/s_{nn}$ and $y_i=\sum_1^{n-1}(s_{nn}s_{ij}-s_{in}s_{nj})x_j/s_{nn}$. But since $s_{nn}>0$, the matrix may be multiplied by this constant without destroying properties (a), (b), and (c).

THEOREM 10. *If $n>1$ then $\|r_{nn}r_{ij}-r_{in}r_{nj}\|$, $i, j=1, \cdots, n-1$, is the inverse of an $(n-1)$-dimensional connected matrix.*

PROOF. The proof parallels that of Theorem 9 but using Theorem 3 instead of Theorem 6. That $r_{nn}>0$ is clear from Theorem 7.

THEOREM 11. *For $i, j=1, \cdots, n$:* (a) $r_{ij}\geqq 0$. (b) $r_{ii}>0$. (c) $r_{ii}\geqq r_{ji}$. (d) $r_{ij}r_{kk}\geqq r_{ik}r_{kj}$. (e) $r_{ii}r_{kk}>r_{ik}r_{ki}$, $i\neq k$.

PROOF. Theorem 7 gives (a), (b), and (c). Then (d) and (e) follow from Theorem 10.

In the case $n=2$, the inequalities of Theorem 11 are sufficient to define all inverse connected matrices. This suggests the problem of giving a direct definition of the inverse of a connected transformation; however, the writer has been unable to accomplish this.

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STEINER'S FORMULAE ON A GENERAL S^{n+1}

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1. Introduction. Steiner's famous formulae on parallel curves and surfaces have attracted considerable interest recently, several mathematicians having developed various extensions of these theorems [3, 4, 6].[1] As stated by Steiner [5] these formulae have the following form:

THEOREM 1. *Let C be a convex curve in the plane of length L and area F, and let C_ρ be the curve parallel to C at a distance ρ from it (measured outward) with length L_ρ and area F_ρ; then*

$$L_\rho = L + 2\pi\rho, \qquad F_\rho = F + \rho L + \pi\rho^2.$$

THEOREM 2. *Let Σ be a convex surface in ordinary space of surface area S, enclosed volume V, and integrated mean curvature M; and let Σ_ρ be the surface parallel to Σ at a distance ρ from it (measured outward) with surface S_ρ and volume V_ρ; then:*

$$S_\rho = S + 2M\rho + 4\pi\rho^2, \qquad V_\rho = V + S\rho + M\rho^2 + 4\pi\rho^3/3.$$

We shall prove the following generalization of these results:

THEOREM 3. *Let S^{n+1} be a Riemann space of constant curvature, K, differentiable of class C^3 and complete in the sense of Hopf and Rinow. Let V^n be a hypersurface of S^{n+1} which is closed and bounding in S^{n+1} and of class C^3, and whose principal curvatures with respect to an outward normal are all negative. The area of V^n will be called A and its volume Vol. Its various mean curvatures (to be defined in §3) will be called M_i $(i=0, \cdots, n)$. Let V^n_ρ be a surface parallel to V^n at a distance measured along outward drawn geodesics where:*

$$\text{for} \quad K > 0: 0 \leqq \rho \leqq \pi/2K^{1/2}; \quad \text{and for} \quad K < 0: \rho \geqq 0.$$

Further let the area and volume of V^n_ρ be respectively A_ρ and Vol_ρ. *Then for $K>0$:*

$$A_\rho = \sum_{i=0}^{n} M_i (K^{-1/2} \sin [\rho K^{1/2}])^{n-i} (\cos [\rho K^{1/2}])^i,$$

$$\mathrm{Vol}_\rho = \mathrm{Vol} + \sum_{i=0}^{n} M_i \int_0^\rho (K^{-1/2} \sin [x^0 K^{1/2}])^{n-i} (\cos [x^0 K^{1/2}])^i dx^0;$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

And for $K<0$:

$$A_\rho = \sum_{i=0}^{n} M_i([-K]^{-1/2} \sinh [\rho(-K)^{1/2}])^{n-i}(\cosh [\rho(-K)^{1/2}])^i,$$

$$\mathrm{Vol}_\rho = \mathrm{Vol}$$

$$+ \sum_{i=0}^{n} M_i \int_0^\rho ([-K]^{-1/2} \sinh [x^0(-K)^{1/2}])^{n-i}(\cosh [x^0(-K)^{1/2}])^i dx^0.$$

Further simplifications of these formulae for special cases can be made by the use of the Gauss-Bonnet formula as developed by the author and A. Weil [1]. These results appear in §§3 and 5.

The methods used in deriving these results are similar to those of Vidal Abascal [6] who developed the special case of $n=1$ in a recent paper. Herglotz [4] has studied the same problem on spheres and on hyperbolic subspaces of pseudo-Euclidean space and has derived the above formulae for these restricted cases. Related formulae were developed by H. Weyl [7] in his study of the volume of tubes lying on spheres. Reference should also be made to the general study of parallel curves on a general two-dimensional surface of positive (non-constant) curvature by Fiala [2]. No such study is available for n-dimensional manifolds.

2. Calculations. In S^{n+1} we consider the geodesic parallel coordinate system in which the first fundamental form has the expression:

$$(1) \qquad ds^2 = (dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, \cdots, n).$$

This is so chosen that:

(a) $x^0=0$ is the hypersurface V^n;

(b) the curves $x^\alpha=$const. $(\alpha=1, \cdots, n)$ are geodesics normal to V^n, on which arc length is measured by x^0 positively outward from V^n;

(c) the positive orientation of S^{n+1} is given by the ordering $[x^0, x^1, \cdots, x^n]$ and that of V^n by $[x^1, \cdots, x^n]$.

Further let the values of $g_{\alpha\beta}$ for $x^0=0$ be denoted by $\gamma_{\alpha\beta}$, so that $\gamma_{\alpha\beta}$ are the components of the metric tensor V^n. When more machinery is available, we shall discuss the domain of S^{n+1} within which this is a proper coordinate system; but for the moment we assume that we are operating within such a domain.

If we assume that the normals to V^n in S^{n+1} are directed outward, we may recall the standard formula:

$$(2) \qquad \left(\frac{\partial g_{\alpha\beta}}{\partial x^0}\right)_{x^0=0} = -\Omega_{\alpha\beta}/2,$$

where $\Omega_{\alpha\beta}$ are the coefficients of the second fundamental form of V^n relative to S^{n+1}. From (2) it follows that:

$$(3) \qquad \left(\frac{\partial (g_{\alpha\beta})^{1/2}}{\partial x^0}\right)_{x^0=0} = -\frac{\Omega_{\alpha\beta}}{(\gamma_{\alpha\beta})^{1/2}}\cdot$$

We now wish to calculate $g^{1/2}$ (where $g=\det |g_{\alpha\beta}|$) in terms of $\Omega_{\alpha\beta}$, $\gamma_{\alpha\beta}$, and x^0; for the integration of this quantity gives the desired formulae. To do this, we first consider a fixed point P on V^n and the geodesic $G(P)$ through P normal to V^n (its arc length is x^0). By a linear transformation (constant coefficients) of x^α $(\alpha=1, \cdots, n)$ we can find a new coordinate system on V^n, $\bar x^\alpha = L^\alpha(x)$, in which the components of the first and second fundamental forms of V^n at P reduce to sums of squares, namely $\delta_{\alpha\beta}$ and $\bar\Omega_{\alpha\beta}$ (where $\bar\Omega_{\alpha\beta}=0$, $\alpha\neq\beta$).

Now in S^{n+1} apply the coordinate transformation:

$$\bar x^0 = x^0,$$
$$\bar x^\alpha = L^\alpha(x^\beta), \qquad \alpha, \beta = 1, \cdots, n.$$

The new line element is:

$$(4) \qquad ds^2 = (dx^0)^2 + \bar g_{\alpha\beta} d\bar x^\alpha d\bar x^\beta$$

where $\bar g_{\alpha\beta} = \delta_{\alpha\beta}$ at P, and $\partial \bar g_{\alpha\beta}/\partial x^0 = 0$, $\alpha\neq\beta$ at P. We now wish to prove the lemma:

LEMMA. *In the $\bar x$ coordinate system, $\bar g_{\alpha\beta}=0$ $(\alpha\neq\beta)$ for all points on $G(P)$.*

From (4) it follows that anywhere in S^{n+1}

$$(5) \qquad \bar R_{\alpha 0 0 \beta} = \frac{1}{2}\left(\frac{\partial^2 \bar g_{\alpha\beta}}{\partial x^0 \partial x^0}\right) - \frac{1}{4}\left(\bar g^{\gamma\delta}\frac{\partial \bar g_{\gamma\alpha}}{\partial x^0}\frac{\partial \bar g_{\beta\delta}}{\partial x^0}\right).$$

But since S^{n+1} is of constant curvature:

$$(6) \qquad \bar R_{\alpha 0 0 \beta} = -K\bar g_{\alpha\beta};$$

so

$$(7) \qquad \frac{1}{2}\left(\frac{\partial^2 \bar g_{\alpha\beta}}{\partial x^0 \partial x^0}\right) - \frac{1}{4}\bar g^{\gamma\delta}\left(\frac{\partial \bar g_{\gamma\alpha}}{\partial x^0}\frac{\partial \bar g_{\beta\delta}}{\partial x^0}\right) + K\bar g_{\alpha\beta} = 0.$$

For $\alpha\neq\beta$, equations (7) may be considered to be differential equations in $\bar g_{\alpha\beta}$ $(\alpha\neq\beta)$ whose coefficients involve constants and $\bar g_{\alpha\alpha}$ and

$\partial\bar{g}_{\alpha\alpha}/\partial x^0$. Taking these last functions as known, we consider the unique solution of (7) along $G(P)$ with initial values at P as given above, namely:

$$(8) \qquad (\bar{g}_{\alpha\beta})_P = 0; \qquad \left(\frac{\partial \bar{g}_{\alpha\beta}}{\partial x^0}\right)_P = 0 \qquad \text{for } \alpha \neq \beta.$$

Now $\bar{g}_{\alpha\beta} = 0$ $(\alpha \neq \beta)$ is a solution of (7) satisfying the initial conditions; hence it is the only solution, and the lemma is proved.

Returning now to (7) with $\alpha = \beta$, we find that as a result of the lemma the following equations hold along $G(P)$:

$$(9) \qquad \frac{\partial^2(\bar{g}_{\alpha\alpha})^{1/2}}{\partial x^0 \partial x^0} + K(\bar{g}_{\alpha\alpha})^{1/2} = 0.$$

Integrating (9) and taking (3) into account we find that:

$$(10) \qquad (\bar{g}_{\alpha\alpha})^{1/2} = -\bar{\Omega}_{\alpha\alpha}(K^{-1/2}\sin[x^0K^{1/2}]) + \cos[x^0K^{1/2}].$$

Hence along $G(P)$:

$$(11) \qquad \bar{g}^{1/2} = \prod_{\alpha=1}^{n}\left\{-\bar{\Omega}_{\alpha\alpha}(K^{-1/2}\sin[x^0K^{1/2}]) + \cos[x^0K^{1/2}]\right\}.$$

Returning to the original coordinate system we find that (11) transforms into:

$$(12) \qquad g^{1/2} = \gamma^{-1/2}\det\left| -\Omega_{\alpha\beta}(K^{-1/2}\sin[x^0K^{1/2}]) + \gamma_{\alpha\beta}\cos[x^0K^{1/2}]\right|$$

where $\gamma = \det|\gamma_{\alpha\beta}|$. Now equation (12) is valid on every geodesic $G(P)$ and hence holds throughout the entire region under consideration.

It is now possible to discuss the domain of validity of formula (12). First we must require that $g^{1/2} > 0$; this limits x^0 to be less than the minimum distance to the first conjugate point on any geodesic $G(P)$. An effective way to do this (but not the most general) is to make the assumptions stated in Theorem 3. Even with this restriction it is still possible that two distinct geodesics $G(P_1)$ and $G(P_2)$ will intersect on S^{n+1}, and hence cause a singularity in the coordinate system. This awkward complication may be avoided by supposing that we are dealing not with S^{n+1} but with a covering surface of it which puts these geodesics on separate sheets. We make this assumption, and thus in dealing with the volumes discussed in this paper we agree to count overlapping volumes with the necessary (finite) multiplicity. This means that our "parallel surfaces" are not necessarily true parallels in the sense of being the locus of points at a constant minimum distance from V^n, but they are more properly called "geodesic parallel surfaces."

3. **Results.** We are now in a position to prove Theorem 3. For

$$A_\rho = \int_{V^n} (g^{1/2})_{x^0=\rho} dx^1 \cdots dx^n \tag{13}$$

and

$$\mathrm{Vol}_\rho = \mathrm{V\dot{o}l} + \int_{V^n} \left\{ \int_0^\rho g^{1/2} dx^0 \right\} dx^1 \cdots dx^n \tag{14}$$

where $g^{1/2}$ is given by (12).

To simplify these results consider the expression:

$$\left| -\Omega_{\alpha\beta} + \lambda\gamma_{\alpha\beta} \right| = \theta_0 + \theta_1\lambda + \theta_2\lambda^2 + \cdots + \theta_n\lambda^n \tag{15}$$

where the θ_i are thus defined. Further let

$$M_i = \int_{V^n} \theta_i \gamma^{-1/2} dx^1 \cdots dx^n. \tag{16}$$

We call M_i the ith mean curvature of V^n in S^{n+1}. Since $\theta_n = \gamma$, $M_n = A$. For $K>0$ the expansion of (13) and (14) using the notation of (16) gives the formulae of Theorem 3. These formulae are indeed valid for $K<0$, but in this case they appear to involve complex numbers. By introducing the hyperbolic functions we find that in fact the entire expression is real, and so explicit formulae for this case are given in Theorem 3.

Further simplification can be obtained in certain cases by the use of the Gauss-Bonnet formula. To prepare for this we introduce the following notation:

$$\phi_i(\rho) = \int_0^\rho (K^{-1/2} \sin [x^0 K^{1/2}])^{n-i} (\cos [x^0 K^{1/2}])^i dx^0; \qquad K > 0,$$

$$\phi_i(\rho) = \int_0^\rho ([-K]^{-1/2} \sinh [x^0(-K)^{1/2}])^{n-i} (\cos [x^0(-K)^{1/2}])^i dx^0; \tag{17}$$

$$K < 0,$$

$$C_i = K^{(n+1)/2} \phi_i \left(\frac{\pi}{2K^{1/2}} \right) = \frac{\omega^{n+1}}{\omega^i \omega^{n-i}} K^{i/2} \tag{18}$$

where ω^j is the surface area of a j-dimensional unit sphere (its surface is of j dimensions) and $\omega^0 = 2$. In this notation the Gauss-Bonnet formula can be written:

$$(19)\quad \text{For } n \text{ even: } M_0C_0 + M_2C_2 + \cdots + M_nC_n = -\omega^{n+1}\chi'/2 = \omega^{n+1}\chi/4;$$

$$(20)\quad \text{For } n \text{ odd: } M_0C_0 + M_2C_2 + \cdots + M_{n-1}C_{n-1} + K^{(n+1)/2}\,\text{Vol} = \omega^{n+1}\chi'/2;$$

where χ' is the inner characteristic of the volume bounded by V^n and χ is the characteristic of V^n itself. These relations may be used to eliminate M_0 from the formulae of Theorem 3. We give two examples.

Example 1. When $n=1$, (20) gives: $M_0=2\pi-FK$, where we have written F instead of Vol to represent the area inclosed by the given curve. This leads to the results of Vidal Abascal which generalize the formulae of Theorem 1. Using the notations of Theorem 1, the calculations are as follows:

$$\begin{aligned} F_\rho &= F + \phi_0 M_0 + \phi_1 M_1 \\ (21)\qquad &= F + K^{-1}[1-\cos(\rho K^{1/2})][2\pi - FK] + K^{-1/2}\sin[\rho K^{1/2}]L \\ &= F\cos[\rho K^{1/2}] + K^{-1/2}\sin[\rho K^{1/2}]L + 2\pi K^{-1}(1-\cos[\rho K^{1/2}]). \end{aligned}$$

The derivative of this expression with respect to ρ gives the corresponding formula for L_ρ.

Example 2. When $n=2$, (19) gives:

$$(22)\qquad M_0 + AK = -4\pi x' = 2\pi\chi.$$

This leads to the formulae:

$$(23)\qquad \begin{aligned} A_\rho = A &+ M_1K^{-1/2}\sin(\rho K^{1/2})\cos(\rho K^{1/2}) \\ &+ 2\pi\chi K^{-1}\sin^2(\rho K^{1/2}), \end{aligned}$$

$$(24)\qquad \begin{aligned} \text{Vol}_\rho = \text{Vol} &+ A_\rho + M_1(2K)^{-1}\sin^2(\rho K^{1/2}) \\ &+ \pi\chi K^{-1}(\rho - K^{-1/2}\sin(\rho K^{1/2})\cos(\rho K^{1/2})). \end{aligned}$$

These are generalizations of those given in Theorem 2.

4. **Formulae for $K=0$.** In the preceding discussion it has been explicitly assumed that $K\neq 0$. To derive similar formulae for $K=0$ we can repeat the above discussion under this assumption; and we also get the same result by taking the limit of the formulae of Theorem 3 as $K\to 0$. The results are:

$$(25)\qquad A_\rho = \sum_{i=0}^{n} M_i\rho^{n-i},$$

$$(26)\qquad \text{Vol}_\rho = \text{Vol} + \sum_{i=0}^{n}(n-i+1)^{-1}M_i\rho^{n-i+1}.$$

In this case the Gauss-Bonnet theorem says that:

$$M_0 = \begin{cases} -\omega^n\chi' = \omega^n\chi/2, & n \text{ even}, \\ +\omega^n\chi', & n \text{ odd}. \end{cases} \tag{27}$$

Combining this with (25) and (26) and rearranging we have:

$$A_\rho = A + \sum_{i=1}^{n-1} M_{n-i}\rho^i + \rho^n \begin{cases} \omega^n\chi/2, & n \text{ even}, \\ \omega^n\chi', & n \text{ odd}, \end{cases} \tag{28}$$

and

$$\mathrm{Vol}_\rho = \mathrm{Vol} + A\rho + \sum_{i=2}^{n} \frac{M_{n-i+1}\rho^i}{i} + \frac{\rho^{n+1}}{n+1} \begin{cases} \omega^n\chi/2, & n \text{ even}, \\ \omega^n\chi', & n \text{ odd}. \end{cases} \tag{29}$$

In comparing these results with Theorem 2, we note that the M of Theorem 2 equals $M_1/2$ as here defined.

5. **Polar surfaces.** When $K>0$, the formulae (18), (19) and (20) suggest that we consider the "polar" surface to V^n; that is, the surface at a distance $\pi/(2K^{1/2})$ from V^n. Since this surface may lie outside the domain of validity of our coordinate system, the application of Theorem 3 will not in general give correct results. Formal application of the formulae of Theorem 3, however, does give answers which may be interpreted as the algebraic area and volume of the polar surface respectively. This means that in carrying out the integrations we have algebraically combined the positive and negative portions of the result and thus obtained their algebraic sum. Using a subscript, P, to indicate the polar surface, we find that:

$$A_P = M_0K^{-n/2}. \tag{30}$$

Thus for $n=1$:

$$L_P = (2\pi - FK)/K^{-1/2}. \tag{31}$$

And for $n=2$:

$$A_P = (2\pi\chi - AK)/K \quad \text{or} \quad A + A_P = 2\pi\chi/K. \tag{32}$$

This is a generalization of a result due to Blaschke [4] in which he assumes that V^2 is a topologic sphere.

The Gauss-Bonnet formula suggests that we consider the "volume" (here we mean algebraic volume) bounded by the two polar surfaces at distances $+\pi/(2K^{1/2})$ and $-\pi/(2K^{1/2})$ respectively. Designating this volume by the symbol $\mathrm{Vol}\,|_{-P}^{P}$ we find that for n even:

$$\mathrm{Vol}\,|_{-P}^{P} = 2(M_0C_0 + M_2C_2 + \cdots + M_nC_n)/K^{(n+1)/2}. \tag{33}$$

Or from the Gauss-Bonnet theorem it follows that:

$$\mathrm{Vol}\,\Big|_{-P}^{P} = \frac{1}{2}\,\frac{\omega^{n+1}}{K^{(n+1)/2}}\,\chi. \tag{34}$$

If V^n is a topologic sphere (n even) we see that the volume $\mathrm{Vol}\,|_{-P}^{P}$ equals the surface area of an $(n+1)$-dimensional sphere of radius $K^{-1/2}$.

The analogous result for n odd is obtained by considering the "doubly polar surface" to V^n, namely the one at a distance of $\pi/K^{1/2}$ from V^n. Designating the volume between V^n and this surface by $\mathrm{Vol}\,|_0^{2P}$ we find that for n odd:

$$\mathrm{Vol}\,\Big|_0^{2P} = 2(M_0C_0 + M_2C_2 + \cdots + M_{n-1}C_{n-1})/K^{(n+1)/2}. \tag{35}$$

Or from the Gauss-Bonnet theorem it follows that:

$$\mathrm{Vol}\,\Big|_0^{2P} = \frac{\omega^{n+1}}{K^{(n+1)/2}}\,\chi' - 2\,\mathrm{Vol} \tag{36}$$

or

$$2\,\mathrm{Vol} + \mathrm{Vol}\,\Big|_0^{2P} = \frac{\omega^{n+1}}{K^{(n+1)/2}}\,\chi'. \tag{37}$$

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HAVERFORD COLLEGE

A CHARACTERISTIC PROPERTY OF AFFINE COLLINEATIONS IN A SPACE OF K-SPREADS

BUCHIN SU

1. **Introduction.** In a recent paper[1] M. S. Knebelman has proved among other things that a necessary and sufficient condition which a mapping of an affinely connected space V_n upon itself shall satisfy in order that the covariant differentiation and the variation (the Lie derivative) of a tensor be interchangeable is that the mapping be an affine collineation. The present note deals with a similar problem in a space of K-spreads[2] by showing that the same condition is also characteristic of the isomorphic transformations.[3]

2. **Affine collineations.** Let

$$(1)\qquad \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^i_{jk}(x, p) p^j_\alpha p^k_\beta = 0 \qquad \left(p^j_\alpha = \frac{\partial x^j}{\partial u^\alpha}\right)$$

be the partial differential equations of the K-spreads in an N-dimensional space, where $i, j, k, \cdots = 1, 2, \cdots, N$; $\alpha, \beta, \cdots = 1, 2, \cdots, K$. The integrability conditions are assumed to be satisfied, namely,

$$R^i_{\cdot jkl} p^j_\alpha p^k_\beta p^l_\gamma = 0,$$

where we have placed

$$(2)\qquad \begin{aligned} R^i_{\cdot jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} - (\Gamma^i_{jk}|^r_m \Gamma^m_{sl} - \Gamma^i_{jl}|^r_m \Gamma^m_{sk}) p^s_r \\ + \Gamma^i_{sl}\Gamma^s_{jk} - \Gamma^i_{sk}\Gamma^s_{jl}, \end{aligned}$$

and

$$A^{\cdots}_{\cdots}|^r_l = \partial A^{\cdots}_{\cdots}/\partial p^l_r.$$

The conditions satisfied by the functions $\xi^i(x)$ such that the infinitesimal transformation

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[1] M. S. Knebelman, *On the equations of motions in a Riemann space*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 682–685.

[2] J. Douglas, *Systems of K-dimensional manifolds in an N-dimensional space*, Math. Ann. vol. 105 (1931) pp. 707–733.

[3] E. T. Davies, *On the isomorphic transformations of a space of K-spreads*, J. London Math. Soc. vol. 18 (1943) pp. 100–107.

$$\bar{x}^i = x^i + \xi^i(x)\delta t \tag{3}$$

shall determine an affine collineation are known to be

$$\xi^m|_{hk} + R^{\;m}_{\cdot hkl}\xi^l + \Gamma^m_{hk}|^\alpha_s\xi^s|_\alpha = 0 \tag{4}$$

with $\xi^s|_\alpha = \xi^s|_k p^k_\alpha$.

For simplicity, let us consider a tensor $X^{\cdot i}_{\cdot j}$ which depends on the p^l_α as well as the x^i. The covariant derivative of $X^{\cdot i}_{\cdot j}$ is defined by the equation

$$X^{\cdot i}_{\cdot j}|_k = \frac{\partial X^{\cdot i}_{\cdot j}}{\partial x^k} - X^{\cdot i}_{\cdot j}|^\alpha_m p^n_\alpha \Gamma^m_{nk} + X^{\cdot h}_{\cdot j}\Gamma^i_{hk} - X^{\cdot i}_{\cdot h}\Gamma^h_{jk}. \tag{5}$$

Denoting

$$X^{\cdot i}_{\cdot j}|_k|_l = X^{\cdot i}_{\cdot j}|_{kl}$$

we can readily show that

$$\begin{aligned} X^{\cdot i}_{\cdot j}|_k|^\alpha_l - X^{\cdot i}_{\cdot j}|^\alpha_l|_k &= (\delta^i_m X^{\cdot h}_{\cdot j} - \delta^h_j X^{\cdot i}_{\cdot m} - X^{\cdot i}_{\cdot j}|^\alpha_m p^h_\alpha)\Gamma^m_{hk}|^\alpha_l, \\ X^{\cdot i}_{\cdot j}|_{kl} - X^{\cdot i}_{\cdot j}|_{lk} &= (\delta^i_m X^{\cdot h}_{\cdot j} - \delta^h_j X^{\cdot i}_{\cdot m} - X^{\cdot i}_{\cdot j}|^\alpha_m p^h_\alpha)R^{\;m}_{\cdot hkl} \end{aligned} \tag{6}$$

with an evident generalization for any tensor.

3. **An extension of Knebelman's theorem.** We are in a position to generalize the result of Knebelman to the case of affine collineations in a space of K-spreads.

THEOREM. *A necessary and sufficient condition that a mapping of a space of K-spreads upon itself shall satisfy in order that the covariant differentiation of a tensor be interchangeable with the Lie derivative is that the mapping be an affine collineation of the space.*

To prove this, we have to recall the definition of the Lie derivative of the tensor $X^{\cdot i}_{\cdot j}$,

$$\Delta X^{\cdot i}_{\cdot j} = \lim_{\delta t \to 0} \frac{X^{\cdot i}_{\cdot j}(\bar{x}, \bar{p}) - \bar{X}^{\cdot i}_{\cdot j}(\bar{x}, \bar{p})}{\delta t}, \tag{7}$$

when the variables x^i are subjected to by (3).

It is readily shown that

$$\Delta X^{\cdot i}_{\cdot j} = X^{\cdot i}_{\cdot j}|_l\xi^l + X^{\cdot i}_{\cdot j}|^\alpha_l p^\gamma_\alpha \xi^l|_\gamma - X^{\cdot r}_{\cdot j}\xi^i|_r + X^{\cdot i}_{\cdot r}\xi^r|_j, \tag{8}$$

whence follows the relation

$$(\Delta X^{i}_{\cdot j})|^{\rho}_{k} - \Delta(X^{i}_{\cdot j}|^{\rho}_{k}) = 0. \tag{9}$$

That is, *the partial differentiation $\partial/\partial p^{k}_{\rho}$ of a tensor is always interchangeable with the Lie derivative.*

In virtue of (6), (8) and (9) we obtain

$$\begin{aligned}(\Delta X^{i}_{\cdot j})|_{h} - \Delta(X^{i}_{\cdot j}|_{h}) = (X^{i}_{\cdot j}|^{\alpha}_{m} p^{k}_{\alpha} + \delta^{k}_{j} X^{i}_{\cdot m} - \delta^{i}_{m} X^{k}_{\cdot j})\\ \cdot(\xi^{m}|_{hk} + R^{m}_{\cdot hkl}\xi^{l} + \Gamma^{m}_{hk}|^{\sigma}_{n}\xi^{n}|_{\sigma}),\end{aligned} \tag{10}$$

which is equal to zero when, and only when, ξ^{i} are solutions of (4). Thus we have completed the proof.

NATIONAL UNIVERSITY OF CHEKIANG

AN INEQUALITY CONCERNING POLYHEDRA[1]

LÁSZLÓ FEJES TÓTH

In 1897, at the mathematical competition of the Loránd Eötvös Mathematical and Physical Society, Professor L. Fejér, at the time still a student, noted the following interesting corollary of a well known elementary geometrical theorem of Euler:[2]

If R denote the radius of the circumscribed circle and r the radius of the inscribed circle of a given triangle, then

$$R \geqq 2r. \tag{1}$$

This is easily established, since according to the theorem of Euler mentioned above, if d denotes the distance between the centers of the circumscribed and inscribed circles, then

$$d^2 = R^2 - 2rR.$$

It follows that $R^2-2rR \geqq 0$, and therefore $R \geqq 2r$. Equality holds only if the two circles are concentric, that is, if the triangle is equilateral.

The problem of generalizing the above result to space was proposed by Professor L. Fejér. A young mathematician, I. Ádám, deported to Germany during the war—where all traces of him have been lost—found and communicated to Professor Fejér in 1943 a very simple proof of the above extremum property of the equilateral triangle. His proof, which may be immediately generalized to space, runs as follows:

If ρ is the radius of the circle passing through the midpoints of the sides of the triangle, then $\rho = R/2$, and all that need be shown is that ρ is at least equal to the radius of the inscribed circle. This follows from the fact that the inscribed circle is the smallest among all circles which have common points with all three sides of the triangle. Such a circle is, namely, the circle inscribed in a homothetic triangle containing the original one.

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[1] Lecture held in the seminar of Professor L. Fejér in April 1946.

[2] T. Radó, *On mathematical life in Hungary*, Amer. Math. Monthly vol. 39 (1932) pp. 85–90; J. Kürschák, *Matematikai versenytételek*, Szeged, 1929. A sharper inequality than (1) is contained in a problem proposed by M. Schreier, Jber. Deutschen Math. Verein. vol. 45 (1935) p. 196. L. J. Mordell [Középiskolai Matematikai és Fizikai Lapok vol. 11 (1935) pp. 145–146, see also Amer. Math. Monthly vol. 44 (1937) p. 252] has proved an analogous better inequality conjectured by P. Erdős: If R_1, R_2, R_3 are the distances of an inner point in a triangle from the vertices and r_1, r_2, r_3 the distances from the sides then $R_1+R_2+R_3 \geqq 2(r_1+r_2+r_3)$.

Equality in (1) can occur only if the inscribed circle passes through all three midpoints of the sides, that is, if the triangle is equilateral.

In this paper we generalize (1) and the analogous inequality for tetrahedra $R \geqq 3r$ to arbitrary convex polygons and polyhedra, respectively. Our main results are the following two theorems:

If R_n and r_n denote the radii of the least sphere containing, and the greatest sphere contained in, an n-verticed convex polyhedron, then

If R_n and r_n denote the radii of the least sphere containing, and the greatest sphere contained in, an n-faced polyhedron, then

$$(2) \qquad R_n/r_n \geqq 3^{1/2} \tan \frac{n}{n-2} \frac{\pi}{6} \cdot$$

Equality holds only for a regular tetrahedron, octahedron and icosahedron.

Equality holds only for a regular tetrahedron, hexihedron and dodecahedron.

Hence (2) is exact for $n=4$, 6, and 12. Furthermore, the inequality (2) gives an exact asymptotic estimate for large values of n, that is, (2) gives the exact value of lim inf $n(R_n-r_n)/R_n$.

It is worth mentioning that the regular hexahedron and dodecahedron are not the "best" polyhedra among the 8 or 20 verticed polyhedra and similarly the regular octahedron and icosahedron are not the best ones among the 8 or 20 faced polyhedra. This shows that in certain extremum problems for polyhedra they are the *trigonal faced*, in others the *trihedral verticed regular polyhedra* which play a distinguished roll. The natural question to transfer other well known extremum properties of the regular polygons to the trihedral verticed or trigonal faced regular polyhedra has—as far as I know—not been treated in literature, except for the few cases quoted in this paper.

When Professor Fejér mentioned the above problem to me, I noticed that the extremum property in question can be related to the following two well known extremum properties:

(1) Among all triangles inscribable in a given circle the equilateral triangle has the greatest area.

(2) Among all triangles circumscribable about a given circle, the equilateral triangle has the smallest area.

In other words:

(1a) Among all circles circumscribed about triangles having the same area, the circle circumscribed about the equilateral triangle is the smallest.

(2a) Among all circles inscribed in triangles having the same area, the circle inscribed in the equilateral triangle is the largest.

The circle circumscribed about a non-equilateral triangle is therefore greater, and the inscribed circle smaller, than the circles circumscribed about, and inscribed in, an equilateral triangle of the same area.

The following obvious generalization of the above results suggests itself: If t_n and T_n denote the areas of two ellipses the first of which is contained in an n-sided polygon of area t, and the second of which contains the same polygon, then

$$t_n \frac{n}{\pi} \tan \frac{\pi}{n} \leqq t \leqq T_n \frac{n}{2\pi} \sin \frac{2\pi}{n} \cdot$$

It follows that for any two ellipses satisfying the above condition,

$$\frac{T_n}{t_n} \geqq \cos^{-2} \frac{\pi}{n} \cdot$$

Equality can occur only for an affine regular n-sided polygon.

Corresponding to this result, (2) shall be proved under the following more general conditions:

THEOREM. *Let V_n and v_n denote the volumes of two ellipsoids, the first containing, the second contained in, a convex polyhedron having either n vertices or n faces. In both cases we have*

$$V_n/v_n \geqq 3^{3/2} \tan^2 \frac{n\pi}{(n-2)6} \cdot \tag{3}$$

This inequality is exact[3] *for $n=4$, 6, 12 and gives an exact asymptotic estimate for large values of n.*[4]

This theorem is a direct generalization of the following known result:[5] If a convex polyhedron with n faces (or n vertices) lies between two concentric spheres with radii r_n and R_n, then

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[3] Equality holds if the number of vertices is given for affine regular triangular-faced polyhedra, or if the number of faces is given for affine regular polyhedra with trihedral vertices.

[4] (3) gives the exact value of lim inf $n(V_n-v_n)/V_n$.

[5] L. Fejes, *Egy gömbfelület befedése egybevágó gömbsüvegekkel*, Matematikai és Fizikai Lapok vol. 50 (1943) pp. 40–46. Other generalizations of (4) are the two following equivalent theorems [L. Fejes, *Über einige Extremaleigenschaften der regulären Polyeder und des gleichseitigen Dreiecksgitters*, Annali della R. Scuola Normale Superiore di Pisa (under publication). L. Fejes, *Extremális pontrendszerek a síkban, a gömbfelületen és a térben*. Acta Sci. Math. et Nat. 23. Kolozsvár 1944]: If a polyhedron with n faces having only trihedral vertices contains a sphere with unit radius then the arithmetic mean of the distances $R_1, R_2, \cdots, R_{2n-4}$ between the

$$R_n/r_n \geqq 3^{1/2} \tan \frac{n\pi}{(n-2)6}\,. \tag{4}$$

The theorem for polyhedra with n vertices follows from the theorem for polyhedra with n faces, and conversely. This is a consequence of the following lemma.

LEMMA. *Let V_1, V_2, V_3 denote three ellipsoids, V_1 and V_3 being polar-reciprocals of each other with respect to V_2. Between the volumes*[6] *of the ellipsoids the following inequality holds:*

$$V_3/V_2 \geqq V_2/V_1. \tag{5}$$

Equality holds only if V_1, V_2, V_3 are concentric.

PROOF. It can be supposed that V_2 is the unit sphere, with center at the origin, and that the x, y, z, axes are parallel to the principal axes, $2a$, $2b$, $2c$, of V_1, respectively. Because, by the polar-reciprocity with réspect to V_2, V_1 is carried into an ellipsoid, it follows that V_1 contains the center of V_2 and thus if the coordinates of the center of V_2 are ξ, η, ζ, we have

$$|\xi| < a, \qquad \eta| < b, \qquad |\zeta| < c.$$

The reciprocity, applied to the tangent planes of V_1 at the end points of the axis of length $2a$, yields two points of V_3, the distance between which is given by

$$\frac{1}{a+\xi} + \frac{1}{a-\xi} = \frac{2a}{a^2-\xi^2} \geqq \frac{2}{a}\,.$$

Similar considerations applied to the other axes of V_1 yield three mutually perpendicular chords of V_3 whose lengths are not less than $2/a$, $2/b$, $2/c$, respectively. A fortiori V_3 has three mutually perpendicular diameters of length $2a'$, $2b'$, $2c'$ such that

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centre O of the sphere and the vertices satisfies the following inequality: (a) $(R_1+R_2+\cdots+R_{2n-4})/(2n-4) \geqq 3^{1/2}\tan(n/(n-2))(\pi/6)$.

If a polyhedron with n vertices bounded only by triangles is contained in a sphere with unit radius with center an inner point O of the polyhedron then the harmonic mean of the distances $r_1, r_2 \cdots, r_{n+4}$ between O and the faces satisfies the following inequality: (b) $(2n-4)/(1/r_1+1/r_2+\cdots+1/r_{2n-4}) \leqq (3^{1/2}/3)$ cot $(n/(n-2))(\pi/6)$.

[6] In what follows we denote the area or volume of a spherical or solid domain by the same letter.

$$8a'b'c' \geqq \frac{8}{abc}.$$

Let us replace one of the diameters of the octahedron with diameters $2a'$, $2b'$, $2c'$, for example, $2b'$, by the diameter $2b''$ conjugate with respect to V_2 to the diameter $2a'$ lying in the plane $a'b'$. Similarly let us replace $2c'$ by the diameter $2c''$ conjugate to the diametral plane $a'b''$. The volume of the original octahedron $4a'b'c'/3$ has been increased by both steps. The diameters of the octahedron thus obtained are pair by pair conjugate with respect to V_2 and thus if $\bar{a}$, $\bar{b}$, $\bar{c}$ denote the principal axes of V_2, we have

$$\frac{4}{3}\bar{a}\bar{b}\bar{c} \geqq \frac{4}{3}a'b'c' \geqq \frac{4}{3}\,\frac{1}{abc}.$$

Hence

$$V_1V_2 = \frac{4\pi}{3}abc\,\frac{4\pi}{3}\bar{a}\bar{b}\bar{c} \geqq \left(\frac{4\pi}{3}\right)^2 = V_2^2,$$

which proves the lemma.

Let us suppose now that the statement of the theorem concerning the number of vertices has been proved. Let P_n denote a convex n-faced polyhedron, v_n an ellipsoid contained in P_n, V_n an ellipsoid containing P_n. Taking polar reciprocals with respect to v_n, there corresponds to P_n a polyhedron with n vertices contained in v_n and containing the ellipsoid V_n' reciprocal to V_n. According to the lemma and our hypothesis

$$V_n/v_n \geqq v_n/V_n' \geqq 3^{3/2}\tan^3\frac{n}{(n-2)}\,\frac{\pi}{6},$$

which is the statement of the theorem for polyhedra with n faces. In a similar manner we may obtain the theorem for n-vertices from the theorem for n faces.[7]

We shall prove the statement of the theorem for polyhedra with n vertices.

Analogously to our considerations concerning the plane, our theorem is the consequence of two inequalities. Let $\bar{v}_n$ denote the volume of any n-verticed convex polyhedron contained in the unit sphere. Then[8]

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[7] Another interesting consequence of our lemma is that for the n-hedron (or for the polyhedron with n vertices) which has a minimal value of V_n/v_n the respective ellipsoids are concentric. This can be seen by making use twice of polar reciprocation.

[8] See the second and the third paper referred to in footnote 5.

$$(6)\qquad \bar{v}_n \leqq \frac{n-2}{6}\left(3-\cot^2\frac{n}{n-2}\,\frac{\pi}{6}\right)\cot\frac{n}{n-2}\,\frac{\pi}{6}\,.$$

Let V_n be the volume of any n-verticed convex polyhedron containing the unit sphere. Then[9]

$$(7)\qquad V_n \geqq \frac{3^{1/2}(n-2)}{2}\left(3\tan^2\frac{n}{n-2}\,\frac{\pi}{6}-1\right).$$

In the proof of (6) it can be supposed that the vertices C_i of $\bar{v}_n$ lie on the unit sphere S with centre O, and that all faces of $\bar{v}_n$ are triangular. If $C_iC_jC_k$ is one of the faces, the tetrahedron $OC_iC_jC_k$ shall be denoted by v_{ijk} and the spherical triangle $C_iC_jC_k$ by f_{ijk}. It will be shown that if the area of f_{ijk} is given, v_{ijk} has maximal volume if $C_iC_jC_k$ is equilateral.

The existence of the extremum being assured by Weierstrass' theorem, it is sufficient to show that, for instance, in the case $C_iC_k \neq C_kC_j$, the volume v_{ijk} can be increased. For that purpose let us move C_k on the Lexell circle passing through the points C_i', C_j' diametrically opposed to C_i, C_j respectively, and through the original position of C_k. The area of f_{ijk} remains unaltered and the volume of v_{ijk} takes its maximal value if $C_iC_k = C_kC_j$, since the height of the tetrahedron drawn from the point C_k is in every other case less.

This extremum property is expressed—making an elementary calculation—by the inequality

$$v_{ijk} \leqq \frac{1}{12}\left(3-\cot^2\frac{f_{ijk}+\pi}{6}\right)\cot\frac{f_{ijk}+\pi}{6}\,.$$

But the function of f_{ijk} which shall be denoted for the sake of brevity by $v(f_{ijk})$ is concave from below in the interval $(0, 2\pi)$ since

$$v'(f) = \frac{\cos (f+\pi)/3}{24\sin^4 (f+\pi)/6}$$

is a monotonically decreasing function of f in the interval $(0, 2\pi)$. Therefore—taking into account the fact that a convex polyhedron with triangular faces having n vertices has $2n-4$ faces—by adding the last inequalities for each face of $\bar{v}_n$, and by using Jensen's inequality[10] we obtain

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[9] The analogous estimate concerning polyhedra with n faces was found by M. Goldberg, *The isoperimetric problem for polyhedra*, Tohôku Math. J. vol. 40 (1935) pp. 226–236.

[10] Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math. vol. 30 (1906) pp. 175–193.

$$v_n = \sum v_{ijk} \leqq \sum v(f_{ijk}) \leqq (2n-4)v\left(\frac{4\pi}{2n-4}\right)$$

which was to be proved.

(7) can be proved in a similar manner. For this purpose we use the following elementary extremum property. The central projection from O of the spherical triangle f_{ijk} with given surface on a plane p which does not intersect S has minimal area if f_{ijk} is equilateral and p is tangential to S at the center of f_{ijk}. Accordingly, if the area of a face of V_n is denoted by t_{ijk}, and its projection on S is denoted by f_{ijk}, we have

$$t_{ijk} \geqq \frac{3^{3/2}}{4}\left(3\cot^2\frac{2\pi - f_{ijk}}{6} - 1\right).$$

The function $t(f_{ijk})$ being convex from below in $(0, 2\pi)$, we have

$$V_n \geqq \frac{1}{3}\sum t_{ijk} \geqq \frac{1}{3}\sum t(f_{ijk}) \geqq \frac{2n-4}{3}\,t\left(\frac{4\pi}{2n-4}\right).$$

Now it follows from (6) and (7) that if the ellipsoid with volume v_n is contained in a polyhedron with n vertices with volume V, and the ellipsoid with volume V_n contains the same polyhedron, we have

$$\begin{aligned} v_n \frac{3}{4\pi}\,\frac{3^{1/2}(n-2)}{2}&\left(3\tan^2\frac{n}{n-2}\,\frac{\pi}{6} - 1\right) \\ &\leqq V \leqq V_n\frac{3}{4\pi}\,\frac{n-2}{6}\left(3 - \cot^2\frac{n}{n-2}\,\frac{\pi}{6}\right)\cot\frac{n}{n-2}\,\frac{\pi}{6}. \end{aligned}$$

Thus (3) has been proved. The cases of equality can be immediately established. The meaning of the estimate for large values of n shall be illustrated by an application.

As a corollary to our theorem we can prove that if $n > 2$ spherical segments (of one base) with equal surfaces f_n cover the whole surface of the sphere of unit radius, we have

$$f_n \geqq 2\pi\left(1 - \frac{3^{1/2}}{3}\cot\frac{n}{n-2}\,\frac{\pi}{6}\right). \tag{8}$$

The "density" d of this system of spherical segments is accordingly given by

$$d = \frac{nf_n}{4\pi} \geqq \frac{n}{2}\left(1 - \frac{3^{1/2}}{3}\cot\frac{n}{n-2}\,\frac{\pi}{6}\right).$$

Since the right side is greater than $2\cdot 3^{1/2}\pi/9$,[11] for every value of $n>2$, we have, for the density, d, of any system of congruent spherical segments smaller than the hemisphere covering the whole surface of the sphere,

$$d > \frac{2\cdot 3^{1/2}\pi}{9} \quad (=1.209\cdots). \tag{9}$$

This remarkable inequality implies that of R. Kershner[12] which expresses the fact that the density of any system of congruent circles covering the plane is not less than $2\cdot 3^{1/2}\pi/9$.

For $n=3$, (8) expresses the simple fact that the surface of the sphere cannot be covered by three spherical segments which are smaller than the hemisphere. Therefore it is sufficient to restrict ourselves to $n\geqq 4$. Since the spherical segments cover the sphere of unit radius, the n-hedron P_n determined by the planes of the bases of the spherical segments is contained in S. Further the heights of the spherical segments being $f_n/2\pi$, P_n contains the sphere with radius $1-f_n/2\pi$. Thus according to our theorem

$$1/(1-f_n/2\pi) \geqq 3^{1/2}\tan\frac{n}{n-2}\frac{\pi}{6}$$

which is equivalent to (8).

An analogous result holds for the density d of an arbitrary system consisting of $n>2$ congruent segments of the unit sphere having no common inner points.[13] In this case we have

$$d \geqq \frac{n}{2}\left(1-\frac{1}{2}\sin^{-1}\frac{n}{n-2}\frac{\pi}{6}\right) > \frac{3^{1/2}\pi}{6} \quad (= 0.907\cdots).$$

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[11] See the third paper in footnote 5.

[12] R. Kershner, *The number of circles covering a set*, Amer. J. Math. vol. 61 (1939) pp. 665–671.

[13] See L. Fejes, *Über eine Abschätzung des kürzesten Abstandes zweier Punkte eines auf einer Kugelfläche liegenden Punktsystems*, Jber. Deutschen Math. Verein vol. 53 (1943) pp. 65–68, and the third paper referred to in footnote 5.

A NOTE ON LOCAL CONNECTIVITY

EDWARD G. BEGLE

Two neighborhoods of a point are involved in the definition[1] of local connectivity: a space T is p-LC at a point x if every neighborhood U of x contains a neighborhood V of x such that any continuous p-sphere in V bounds a continuous $(p+1)$-cell in U. T is p-LC if it is p-LC at every point, and it is LC^n if it is p-LC for $0 \leqq p \leqq n$.

For the case $p=0$, it is well known that there is an equivalent definition: a space T is 0-LC if every point has arbitrarily small neighborhoods V such that any continuous 0-sphere in V bounds a continuous 1-cell in V. But for $p>0$, Borsuk and Mazurkiewicz have shown by an example[2] that these two definitions are not equivalent.

Hence the question arises as to the relative size of V with respect to U in the first definition. At first glance, the Borsuk-Mazurkiewicz example would seem to indicate that V must be considerably smaller than U. This, however, is not the case.

THEOREM. *If a space T is* LC^n, *then each point of T has arbitrarily small neighbrhoods V such that any continuous p-sphere,* $0 \leqq p \leqq n$, *in V bounds a continuous $(p+1)$-cell in V.*

PROOF. Let U be a neighborhood of a point x of T such that any continuous 0-sphere in U bounds a continuous 1-cell in U. Let A be the class of all neighborhoods V of x such that any continuous p-sphere, $0<p\leqq n$, in V bounds a continuous $(p+1)$-cell in U. Since T is LC^n, A is not vacuous. Order the elements of A by inclusion. Since the continuous image of a sphere is a compact set, the union of the elements of any simply ordered subset of A is again an element of A. Hence, by Zorn's lemma, A contains a maximal element, V_0.

We assert that $V_0 \supset U$. If not, let y be a point of the open set $U-V_0$, and let W be a neighborhood of y, $W \subset U-V_0$, such that any continuous p-sphere, $0<p\leqq n$, in W bounds a continuous $(p+1)$-cell in U. Since $p>0$, any continuous p-sphere in $V_0 \cup W$ is either in V_0 or in W, so $V_0 \cup W$ is an element of A, which contradicts the maximality of V_0.

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[1] S. Lefschetz, *Locally connected and related sets.* I, Ann. of Math. vol. 35 (1934) pp. 118–129.

[2] K. Borsuk and S. Mazurkiewicz, *Sur les rétractes absolus indécomposables*, C.R. Acad. Sci. Paris vol. 199 (1934) pp. 110–112.

Now, by the original choice of U, V_0 has the property required by the theorem.

We remark that the same proof holds, with trivial modifications, for homology local connectivity.

It is an open question whether the neighborhood V can be required to be connected.

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UNCONDITIONAL CONVERGENCE IN BANACH SPACES

S. KARLIN

Introduction. This note investigates an apparent generalization of unconditionally convergent series $\sum x_i$ in weakly complete Banach spaces. A series of elements with x_i in E is said to be unconditionally convergent if for every variation of sign $\epsilon_l = \pm 1$, $\sum_1^\infty \epsilon_l x_l$ is convergent. This formulation of the definition of unconditional convergence is equivalent to that given by Orlicz [4].[1] We call $\sum x_i$ unconditionally summable if there exists a finite row Toeplitz matrix (b_{ik}) such that for every variation of sign $\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l x_l$ converges. The fact that unconditional summability implies unconditional convergence is established in this note. Finally, applications to orthogonal functions are presented.

Preliminary lemmas. In what follows, b_{ik} will denote an arbitrary finite row Toeplitz matrix.

LEMMA 1. *If* $S_n(\theta) = \sum_1^n a_l r_l(\theta)$ *converges to an essentially bounded function* $f(t)$, *then* $\left|\sum_1^n a_n r_n(\theta)\right| \leq c$ *almost everywhere.* ($r_n(\theta)$ *denote the Rademacher functions.*)

PROOF. This is an immediate consequence of the result that

$$(1)\qquad \left(\int_0^1 \left(\max_{1\leq n\leq m}\left|\sum_1^n a_l r_l(\theta)\right|\right)^p d\theta\right)^{1/p} \leq C\left(\int_0^1 |S_m(\theta)|^p d\theta\right)^{1/p}, \qquad 1 \leq p \leq \infty,$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

for all m [3]. For, applying (1) for $p=\infty$, we get

$$\left|\sum_1^m a_l r_l(\theta)\right| \leqq \max_n \left|\sum_1^n a_l r_l(\theta)\right| \leqq C\left|\sum_1^\infty a_l r_l(\theta)\right| \leqq cf(\theta) \leqq \gamma.$$

LEMMA 2. *If* $\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l$ *converges for every* $\epsilon_l = \pm 1$, *then* $\sum_1^\infty |a_l| < \infty$.

PROOF. We first establish that if $A_i = \sum_{k=1}^{n_i} A_{ik}\epsilon_k$ converges for every variation of sign $\epsilon_k = \pm 1$, then $\sum_{k=1}^{n_i} |A_{ik}| \leqq C$ for every i. If we observe that $\{(A_{ik}) = x_i\}_i$ is a sequence of elements in (l) (space of absolutely convergent series), then the hypothesis implies that x_i converges weakly for all functionals f of the form $f = \{\epsilon_k\}$. A result of Banach [1] implies x_i are strongly convergent and hence $\|x_i\| = \sum_{k=1}^{n_i} |A_{ik}| \leqq C$. We now complete the proof. Since

$$\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l = \sum_{l=1}^{n_i} \epsilon_l \left[a_l \sum_{k=l}^{n_i} b_{ik}\right] = \sum_{l=1}^{n_i} \epsilon_l A_{il}$$

converges, we have, in view of the preceding, that

$$\left|\sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l\right| \leqq \sum_{l=1}^{n_i} \left|a_l \sum_{k=l}^{n_i} b_{ik}\right| = \sum_{l=1}^{n_i} |A_{il}| \leqq C. \tag{2}$$

If we replace the ϵ_l by $r_l(\theta)$, then the hypothesis states that $\sigma_i(\theta)$ converges for every θ. In virtue of a known result [6], this implies that $S_m(\theta) = \sum_1^m a_l r_l(\theta)$ converges almost everywhere. This fact combined with (2) yields easily by Lemma 1 that almost everywhere

$$\left|\sum_1^m a_n r_n(\theta)\right| \leqq C, \qquad n = 1, 2, \cdots.$$

Using the independence of the Rademacher functions, we obtain for $|\lambda_k| \leqq 1$ that

$$\sum_{n=1}^m a_n \lambda_n = \int_0^1 \left(\sum_1^m a_n r_n(\theta)\right) \prod_1^m (1 + \lambda_k r_k(\theta))\,d\theta$$
$$\leqq C \int_0^1 \prod_1^m (1 + \lambda_k r_k(\theta))\,d\theta \leqq C.$$

Choosing $\lambda_n = \operatorname{sign} a_n$, we get $\sum |a_n| < \infty$, which completes the proof.

The hypothesis requiring every set $\{\epsilon_l\}$ in the lemma is necessary. For there exist numbers a_l with $\sum a_l^2 < \infty$, $\sum |a_l| = \infty$, and $\sum a_l r_l(\theta)$ convergent almost everywhere, for example $a_l = 1/l$.

Another result needed in the same direction is the following:

LEMMA 3. *If for almost all variation of sign*

$$\left|\sum_{k=1}^{n_i} b_{ik}\sum_{l=1}^{k}\epsilon_l a_l\right| \leq C,$$

then $\sum |a_l| < \infty$.

PROOF. In view of the hypothesis and the proof of the preceding lemma, it will be sufficient to show that $\sigma_i = \sum_{k=1}^{n_i} b_{ik}\sum_{l=1}^{k}\epsilon_l a_l$ converges almost everywhere. We proceed to show this. The hypothesis implies that for $\sigma_i(\theta) = \sum_{k=1}^{n_i} b_{ik}\sum_{l=1}^{k} r_l(\theta) a_l$, we have for any $p > 1$

$$\int_0^1 |\sigma_i(\theta)|^p d\theta \leq \gamma.$$

From a result of Banach and Saks [5], we infer

$$\lim_{n,m\to\infty}\int_0^1\left|\frac{1}{m}\sum_{k=1}^{m}\sigma_{i_k}(\theta) - \frac{1}{n}\sum_{k=1}^{n}\sigma_{i_k}(\theta)\right|^p d\theta = 0. \tag{3}$$

This can be represented as a new Toeplitz matrix E operating on $\sum_1^\infty a_l r_l(\theta)$ with $E(S_m(\theta)) = \{\sigma_n'\} = \{(1/n)\sum_{k=1}^{n}\sigma_{i_k}\}$. Consequently, (3) states that

$$\int_0^1 |\sigma_n'(\theta) - \sigma_m'(\theta)|^p d\theta \to 0. \tag{4}$$

But this yields the existence of a subsequence $\sigma'_{n_k}(\theta)$ which converges almost everywhere. As above, this can be represented as $\{\sigma'_{n_k}(\theta)\} = E'\{\sigma_n'(\theta)\} = E'E\{S_n(\theta)\}$ where $E'E$ is a new Toeplitz matrix. Hence, we have exhibited a Toeplitz matrix which sums $S_m(\theta)$. As in Lemma 1 this implies that $S_m(\theta)$ converges almost everywhere. Q.E.D.

The theorems. We now establish several results on unconditional convergent series in a Banach space E.

We assume that E is weakly complete.

THEOREM 1. *If for* $x_i \in E$, $\sum x_i$ *is summable weakly by a Toeplitz matrix for every variation of sign, that is,*

$$\sigma_i = \sum_{k=1}^{n_i} b_{ik}\sum_{l=1}^{k}\epsilon_l x_l$$

converges weakly as elements, then $\sum x_l$ *is unconditionally convergent.*

PROOF. In view of the fact that E is weakly complete, it is sufficient to demonstrate that $\sum |f(x_l)| < \infty$ for every functional f [see [4]). If f denotes an arbitrary functional and $a_l = f(x_l)$, the hypothesis implies that, for every variation of sign $\epsilon_l = \pm 1$, $\sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l$ converges. If we apply Lemma 2, we obtain that

$$\sum | a_l | < \infty \quad \text{or} \quad \sum | f(x_l) | < \infty .$$

The following similar result can be established with the use of Lemma 3.

THEOREM 2. *If E is weakly complete and $\| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l x_l \| \leqq C$ for almost all variation of signs, then $\sum x_l$ is unconditionally convergent.*

A final result in this connection is the following. We assume that E is weakly complete.

THEOREM 3. *If for every rearrangement there exists a positive Toeplitz matrix with $\sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} x_{\pi(l)}$ convergent for x_l in E, then $\sum x_l$ converges unconditionally.*

PROOF. If $f(x_l) = a_l$, we assert that $\sum |a_l| < \infty$. Otherwise there exists a rearrangement with $\sum a_{\pi(l)} = \infty$ and $\sum_1^m a_{\pi(l)} > 0$ for every m. This implies that, for any positive Toeplitz matrix, $\sum_{k=1}^{n_i} b_{ik} \sum_1^k a_{\pi(l)} \to \infty$ which contradicts the hypothesis. The remainder of the proof is straightforward.

Applications. We shall now apply these results to complete orthogonal systems of functions. It is customary in treating bounded orthogonal systems to assume that the Lebesque kernel is summable by a finite row Toeplitz matrix, that is

$$\int_0^1 \left| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \phi_l(t)\phi_l(\theta) \right| dt \leqq C$$

[2] almost everywhere in θ. We make this assumption and establish the following theorem.

THEOREM 4. *If for every variation of sign $\epsilon_l = \pm 1$, $\{\epsilon_l a_l\}$ are the Fourier coefficients with respect to $\{\phi_k(t)\}$ of a function in $L^p (p \geqq 1)$, then $\{d_l a_l\}$ are the Fourier coefficients of a function in L^p for every sequence of numbers d_k with $|d_k| \leqq 1$.*

We first demonstrate a lemma.

LEMMA. *If $\sum x_l$ converges unconditionally, then $\| \sum_1^n d_l x_l \| \leqq C$ for all d_l with $|d_l| \leqq 1$.*

PROOF. The hypothesis implies that

$$\left\| \sum_{1}^{m} \epsilon_l x_l \right\| \leqq C \tag{5}$$

for all m and $\epsilon_l = \pm 1$. For each m and $l = 1, \cdots, m$ with $|d_l| \leqq 1$ there exists a functional f_m with $|f_m| = 1$ such that

$$\left\| \sum_{l=1}^{m} d_l x_l \right\| = f_m\left(\sum_{1}^{m} d_l x_l \right). \tag{6}$$

We obtain now using (5) and (6)

$$\begin{aligned} \left\| \sum_{1}^{m} d_l x_l \right\| &= \left| \sum_{1}^{m} d_l f_m(x_l) \right| \leqq \sum_{1}^{m} | f_m(x_l) | \\ &= \sum_{1}^{m} \epsilon_l f_m(x_l) = f_m\left(\sum_{1}^{m} \epsilon_l x_l \right) \\ &\leqq | f_m | \left\| \sum_{1}^{m} \epsilon_l x_l \right\| \leqq C. \end{aligned}$$

PROOF OF THEOREM 4. Let $\sigma_n(t) = \sum_{k=1}^{n} b_{nk} \sum_{l=1}^{k} \epsilon_l a_l \phi_l(t)$, then in virtue of the criterion that a sequence of real numbers be the Fourier coefficients of a function in L^p [2], we conclude that $\sum a_l \phi_l(t)$ is unconditionally summable in L^p. Applying Theorem 1, we obtain that $\sum a_l \phi_l(t)$ is unconditionally convergent. The above lemma implies that $\sum_{1}^{m} d_l a_l \phi_l(t)$ is uniformly bounded in L^p for any fixed set of d_l. Applying the criterion for determining Fourier coefficient in L^p gives us the result.

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PRINCETON UNIVERSITY

ON A CERTAIN CLASSIFICATION OF RINGS AND SEMIGROUPS

DOV TAMARI

In his paper *Linear equations in non-commutative fields* (Ann. of Math. vol. 32 (1931) pp. 463–477) Professor Oystein Ore defines regularity and irregularity of rings and an "order of irregularity" in such a way that regularity becomes irregularity of order 1. Herewith the following problem is proposed: Do irregular rings of order $n>1$ really exist? If so of what type are they? In this note these questions will be answered. A classification on this line yields nine different types, for which explicit examples are given. This classification turns out to be essentially one of semigroups. For the so-called "ringlike domains," that is, domains having one distributive law only, the position is otherwise and will be treated in detail elsewhere.

The first section of this paper contains the general considerations, the second one the examples.

1. General considerations. According to Ore we call a ring without divisors of zero $\mathfrak{R}$ *left-regular* if any two elements $a, b \in \mathfrak{R}$ have a nontrivial[1] common left multiple (C.L.M.), that is, there exists at least one pair of elements $x, y \in \mathfrak{R}$, not both 0, such that $xa=yb$ $=\text{C.L.M.}(a, b)$. Otherwise we call $\mathfrak{R}$ *left-irregular*. In the same way *right-regularity and right-irregularity* are defined. Another approach to this problem is suggested by the concepts of linear dependence and independence. For the sake of generality we do not exclude rings with nontrivial[2] divisors of zero and define:

If $a_1b_1+\cdots+a_nb_n=\sum a_ib_i=0$, $a_i, b_i \in \mathfrak{R}$, where at least one $b_i \neq 0$, we say that the n elements a_i are *linearly right dependent* (*lin. r. dep.*), and if at least one $a_i \neq 0$, that the n elements b_i are *linearly left dependent* (*lin. l. dep.*). But if for a given set $a_1, \cdots,$ $a_n \in \mathfrak{R}$ it follows from the equation $a_1x_1+\cdots+a_nx_n=\sum a_ix_i$ $=0$ $(x_i \in \mathfrak{R})$ that all $x_i=0$, we say that the set $a_1, \cdots, a_n$ is *linearly right independent* (*lin. r. ind.*). Similarly *linear left independence* (*lin. l. ind.*) is defined.

Obviously a set containing a left (right) divisor of zero is lin. r. (l.) dep.; a set containing a subset, lin. dep. on one side, is itself lin. dep. on the same side. Any subset of a lin. ind. set is therefore also lin.

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[1] There exist always the trivial common "one-sided" multiples $0\cdot a=0\cdot b$ and $a\cdot 0=b\cdot 0$ and the "mixed" multiple $xa=by$ with $x=b$, $y=a$.

[2] 0 shall be called a trivial divisor of zero.

indep. on the same side as the set. A set containing equal elements is lin. dep. on both sides. Therefore a lin. r. (l.) ind. set contains only unequal elements, which are not l. (r.) divisors of zero.[3]

If we can find at least one set of n elements $a_1, \cdots, a_n \in \mathfrak{R}$, which is lin. r. indep., and if any set of $(n+1)$ elements of $\mathfrak{R}$ is lin. r. dep., we say, according to Ore, that $\mathfrak{R}$ is of *right order* (*of "irregularity"*) n. Similarly we define the *left order* of $\mathfrak{R}$. But if for any positive integer n there exists in $\mathfrak{R}$ a set of n lin. l. (r.) ind. elements, we say, $\mathfrak{R}$ is of *infinite left* (*right*) *order* and write n_l resp. $n_r = \infty$. We may therefore classify the rings into types (n_l, n_r), where n_l denotes the left, n_r the right order of the ring. Thus for example a ring of type (1, 1) without divisors of zero is regular on both sides in the sense of Ore. Irregularity means that n_l and/or $n_r > 1$. The order zero, that is, n_l resp. $n_r = 0$, means of course that all elements of $\mathfrak{R}$ are right resp. left divisors of zero.

THEOREM. *There exist no rings of finite* (*left or right*) *order* $n > 1$.

PROOF.[4] Assume $\mathfrak{R}$ to be of right order $n_r > 1$. Then there exist at least two lin. r. ind. elements $a, b \in \mathfrak{R}$, that is, $ax+by=0$ implies $x=y=0$. Then the 4 elements aa, ab, ba, bb will also be lin. r. ind., because by using the associativity of multiplication and the right distributive law[3] $aa\cdot x+ab\cdot y+ba\cdot z+bb\cdot t=a(ax+by)+b(az+bt)=0$ implies $ax+by=az+bt=0$ and hence $x=y=z=t=0$. Similarly the 8 elements aaa, aab, $\cdots$, bbb and generally all the 2^n elements, obtained by forming all possible "words" of length n using the 2 "letters" a and b, can be proved to be lin. r. ind. Indeed, if c_i $(i=1, \cdots, \nu=2^{n-1})$ are the above defined products of length $(n-1)$, the 2^n products of length n are given by ac_i, bc_i $(i=1, \cdots, \nu)$. Therefore $\sum ac_i\cdot x_i+\sum bc_i\cdot y_i=c\sum c_ix_i+b\sum c_iy_i=0$ implies $\sum c_ix_i = \sum c_iy_i=0$. If we assume the lin. r. independence of the c_i, $x_i=y_i=0$ is implied and therefore the lin. r. ind. of the 2^n elements ac_i, bc_i is proved by induction. As the same holds for left indep., our theorem is proved.

COROLLARY. *There exist at most the following* 9 *types of rings*:

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[3] In view of later developments, it is worth while to note that all these remarks follow either directly from the distributive laws or from their consequences $0\cdot r=0$, $r\cdot 0=0$, for any $r\in\mathfrak{R}$. $a(b+c)=ab+ac$ is the right distributive law, $(a+b)c=ac+bc$ the left one.

[4] This theorem can also be obtained as a corollary of a more general theory concerning modules with rings as operator domains. Such modules seem to have been considered independently by Paul Dubreil (see Mathematical Reviews vol. 7 (1946) pp. 2, 3).

$(0, 0), (0, 1), (0, \infty), (1, 0), (\infty, 0), (1, 1), (1, \infty), (\infty, 1), (\infty, \infty)$.[5]

That rings of each of these types really exist will be shown by examples in §2.

This result permits us to formulate our classification of rings in the following form: $\mathfrak{R}$ is of r. (l.) order 0 only when all its elements are l. (r.) divisors of zero. Otherwise $\mathfrak{R}$ is of r. (l.) order 1 or ∞. It is of r. (l.) order 1 when every pair of elements has a nontrivial C.R.(L.)M. It is of order ∞ when not every pair of elements has a nontrivial C.R.M. In this form we see that our second approach is not much more general than our first. It is also seen that our classification of rings is one of semigroups only, because "divisor of zero" and "common multiple" are notions based on multiplication only. A semigroup, namely, is a system with one composition (multiplication), which is (1) general, (2) unique and (3) associative. It remains only to define "divisors of zero" for a semigroup $\mathfrak{S}$ as follows: $a \in \mathfrak{S}$ is a right (left) *divisor of zero* in $\mathfrak{S}$, if there exists a pair $x, y \in \mathfrak{S}$, $x \neq y$, such that $xa = ya$ $(ax = ay)$.[6] Therefore a is not a r. (l.) divisor of zero, if $xa = ya$ $(ax = ay)$ implies $x = y$. Thus $\mathfrak{S}$ is a semigroup without divisors of zero, right and/or left, if the right and/or left cancellation law (that is, $ba = ca$ and/or $ab = ac$ implies $b = c$) holds in $\mathfrak{S}$. Now if we consider a ring as a semigroup with respect to multiplication the above defined concept of divisor of zero becomes identical with that usually defined for rings.[7]

In this connection it is of interest to point out some differences between semigroups and rings: (1) In a ring with divisors of zero there must exist right as well as left divisors of zero; but in semigroups it may occur that all elements are right divisors of zero (that is, $n_l = 0$) with not a single one being a left divisor of zero. (2) In a ring of order 0, say $n_l = 0$, there exists for every pair of elements at least one nontrivial C.L.M. $= 0$; but in semigroups it may occur that $n_l = 0$ with no C.L.M. whichever existing. (3) As a semigroup may be finite, but not of order 0 or 1 (for rings this is impossible), it seems more appropriate to use the letter i (to suggest "irregular") instead of the symbol ∞ in the notation for types of semigroups. This is especially necessary in view of a more general theory of "ringlike domains," as will be shown elsewhere.

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[5] As to rings without divisors of zero, only such of the last type (∞, ∞) are properly irregular in the sense that they can not be immersed in a quotient field according to the construction given in the previously mentioned paper of Ore, while the others permit this construction, which requires regularity on one side only. This does not exclude the immersibility into other kinds of embracing fields. The analogous remark holds for semigroups too.

[6] In the case $\mathfrak{S}$ contains a zero element the latter is called a trivial divisor of zero.

[7] Every divisor of zero in the multiplicative semigroup is also a divisor of zero in the ring and conversely. For trivial ones this is clear. For nontrivial ones this follows by the distributive laws: $ac = bc$, $a \neq b$, $c \neq 0$ implies $(a-b)c = 0$, $a-b \neq 0$, $c \neq 0$. $dc = 0$, $d \neq 0$, $c \neq 0$ implies, for each x, $d((c+x)-x) = 0$ and hence $d(c+x) = dx$, $c+x \neq x$ and, for each y, $((d+y)-y)c = 0$ and hence $(d+y)c = yc$, $d+y \neq y$.

2. **Examples.** We start by constructing a semigroup without divisors of zero $\mathfrak{F}$ of type (i, i). Any "*word*" composed by the 2 letters x, y shall be an element of $\mathfrak{F}$, and nonidentical "words" represent different elements. The composition, defined by connecting one word to the other, is obviously (1) general, (2) unique, (3) associative, and (4) complies with the cancellation laws. No C.R.(L.)M.'s exist except in the case of one word being a right (left) "end" (that is, "divisor" or "part") of its pair. Therefore $\mathfrak{F}$ is of the required type.

Consider all elements of $\mathfrak{F}$ as the basis of a module $\mathfrak{M}$ over a ring $\mathfrak{R}$, which may be chosen as the field $\{0, 1\}$ of residue classes (mod 2). Thus each nonzero element of $\mathfrak{M}$ has the form $\sum f_i$, where the f_i's form an arbitrary finite set of different elements of $\mathfrak{F}$. Defining multiplication in $\mathfrak{M}$ by the distributive laws, $\mathfrak{M}$ becomes a ring which can be called the "*semigroup ring*" of $\mathfrak{F}$ over $\mathfrak{R}$, written $\mathfrak{R}(\mathfrak{F})$. We call the sums $\sum f_i$ "*polynomials*," those consisting of one term "*monomials*." The "*degree*" of a monomial $f \in \mathfrak{F}, f \in \mathfrak{R}(\mathfrak{F})$ is the number of its letters and is denoted by $\delta(f)$ A polynomial p containing only monomials of equal degree δ is called a "*homogeneous*" polynomial of degree $\delta = \delta(p)$. Any polynomial P is a sum of monomials. P is also a sum of homogeneous polynomials p_i of different degrees, that is, $P = \sum_{i=1}^{n} p_i$ with $\delta(p_1) < \delta(p_2) < \cdots < \delta(p_n) = \delta(P)$, the *degree of* P. By multiplying terms of highest degree $\delta(P)$, $\delta(Q)$ of $P \neq 0$, $Q \neq 0$ we obtain every product of degree $\delta(P) + \delta(Q)$ only once. This yields exactly all terms of highest degree $\delta(PQ)$ in PQ, which therefore cannot vanish, giving the rule $\delta(PQ) = \delta(P) + \delta(Q)$. From this it follows as for ordinary polynomials that our ring $\mathfrak{R}(\mathfrak{F})$ does not contain divisors of zero. $\mathfrak{R}(\mathfrak{F})$ is of the type (∞, ∞): to show this it is enough to find 2 monomials which have neither C.R.M. nor C.L.M. Indeed, by choosing the generators x, y themselves whatever the polynomials P, $Q \in \mathfrak{R}(\mathfrak{F})$, it is obvious that $Px \neq Qy$ $(xP \neq yQ)$, because Px (xP) is composed of monomials ending (beginning) with x only, while all monomials of Qy (yQ) end (begin) with y.

Next we construct a semigroup without divisors of zero $\mathfrak{G}$ of type $(1, i)$ consisting of all symbols $x^{\alpha}y^{\beta}$ $(\alpha, \beta = 0, 1, 2, \cdots)$. We define equality and composition by: $x^{\alpha_1}y^{\beta_1} = x^{\alpha_2}y^{\beta_2}$ only when $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and conversely. $x^{\alpha_1}y^{\beta_1} \cdot x^{\alpha_2}y^{\beta_2} = x^{\gamma}y^{\beta_1+\beta_2}$, $\gamma = \alpha_1 + 2^{\beta_1}\alpha_2$. The composition is (1) general, (2) unique, (3) associative, and (4) complies with the cancellation laws, as may be easily verified.[8] To any 2 elements $g_i = x^{\alpha_i} y^{\beta_i} \in \mathfrak{G}$ $(i=1, 2)$, we can find 2 elements $z_i = x^{\xi_i} y^{\eta_i}$ $(i=1, 2)$, such that $z_1 g_1 = z_2 g_2$; for example, if we take $\eta_1 = \beta_2$, $\eta_2 = \beta_1$, $\xi_1 = 2^{\beta_1}\alpha_2$, $\xi_2 = 2^{\beta_2}\alpha_1$, the C.L.M. (g_1, g_2) becomes $x^{\lambda} y^{\beta_1+\beta_2}$ with $\lambda = 2^{\beta_1}\alpha_2 + 2^{\beta_2}\alpha_1$. But generally the g_i have no C.R.M., because $g_i z_i = x^{\alpha_i + \xi_i} y^{\beta_i + \eta_i}$, and because for $\beta_1, \beta_2 \neq 0$ and $\alpha_1 \not\equiv \alpha_2 \pmod 2$ no ξ_i, η_i can be found such that $g_1 z_1 = g_2 z_2$; for example xy and x^2y certainly have no C.R.M. $\mathfrak{G}$ is therefore of the type $(1, i)$.

Similarly, as before, we construct a semigroup ring $\mathfrak{R}(\mathfrak{G})$ and write the general element $P \in \mathfrak{R}(\mathfrak{G})$ as a polynomial in y using the left distributive law: $P = \sum_{i=0}^{\mu} p_i(x) y^i$, where the p_i are polynomials in x, $p_\mu(x) \neq 0$ and $\mu = \delta_y(P)$, the "*degree of P in y.*" Note the multiplication rule: $y^{\rho} p(x) = p(x^{2^{\rho}}) y^{\rho}$. As may be easily verified $\mathfrak{R}(\mathfrak{G})$ does not contain divisors of zero. The monomials xy and x^2y have no C.R.M. in $\mathfrak{R}(\mathfrak{G})$. Indeed from $xyP = x^2yQ$ it would follow that the different monomials of xyP are equal in some order to those of x^2yQ, which is impossible. Therefore $n_r = \infty$.

To prove $n_l = 1$ we take advantage of some results contained in O. Ore's paper *Theory of non-commutative polynomials* (Ann. of Math. vol. 34 (1933) pp. 480–508) and consider the ring $\mathfrak{R}^*$ of the noncommutative polynomials $P^* = \sum r_i y^i$ where $r_i = r_i(x) = p_i(x)/p_i'(x)$ are elements of the field of all rational functions of x over $\mathfrak{R}$. The multiplication in $\mathfrak{R}^*$ shall be defined by $yr = \bar{r}y + r' = r(x^2)y$; that is,

$$\overline{r(x)} = r(x^2), \qquad r'(x) = 0.^{9}$$

Thus we specialize Ore's operations in such a way that $\mathfrak{R}(\mathfrak{G})$ may be identified with a subring of $\mathfrak{R}^*$. We know that for every 2 elements $P, Q \in \mathfrak{R}(\mathfrak{G}) \subset \mathfrak{R}^*$ we can find $S^*, T^* \in \mathfrak{R}^*$, not both 0, such that $S^*P = T^*Q = \text{C.L.M.}\ (P, Q) \in \mathfrak{R}^*$. By left-hand multiplication with a suitable polynomial $p(x)$ we get $pS^* = S$, $pT^* = T$, $S, T \in \mathfrak{R}(\mathfrak{G})$ and therefore $SP = TQ = \text{C.L.M.}(P, Q) \in \mathfrak{R}(\mathfrak{G})$. By the way, the ring $\mathfrak{R}^*$ itself is also of the type $(1, \infty)$. We need only to show that $n_r = \infty$. It is sufficient to prove that, for example, $xy \cdot P^* \neq x^2y \cdot Q^*$ for every pair $P^*, Q^* \in \mathfrak{R}^*$, not both 0. Indeed, from $xyP^* = x^2yQ^*$ it would follow that $xy(\sum r_i(x)y^i) = x^2y(\sum s_i(x)y^i)$, that is, $\sum r_i(x^2)y^{i+1} = x\sum s_i(x^2)y^{i+1}$, that is, $r_i(x^2) = xs_i(x^2)$ and hence, for at least one index i, $x = r_i(x^2)/s_i(x^2) = t(x^2)$, that is, x is a rational function of x^2, which is impossible.

Now we construct a semigroup $\mathfrak{H}$ of type $(0, i)$ by taking all the symbols (words) $f\in\mathfrak{F}$ (see first example) and adding the symbols ϵ and $f\epsilon$ (that is, a word of $\mathfrak{F}$ with an ϵ "hung on"). We define equality by identity and composition by connecting words, with the additional provision that an ϵ not standing at the end (that is, belonging to the first (left) factor) shall be suppressed. In this way we obtain again one and only one symbol of $\mathfrak{H}$, that is, the composition is general and unique; its associativity is obvious too. All elements of $\mathfrak{H}$ are right divisors of zero, because for $f\in\mathfrak{F}\subset\mathfrak{H}$ we have $f\epsilon\neq f$, but $f\epsilon\cdot h=fh$ for every $h\in\mathfrak{H}$. But $\mathfrak{H}$ has no left divisors of zero because $h_1\neq h_2$ implies, for every $h\in\mathfrak{H}$, $hh_1\neq hh_2$. The two special elements x, $y\in\mathfrak{H}$ have no C.R.M. because $xh\neq yh$ for every $h\in\mathfrak{H}$. $\mathfrak{H}$ is therefore indeed of type $(0, i)$.

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[8] The semigroup could also have been defined by the 2 generators x and y and the defining relation $yx = x^2y$.

[9] Compare loc. cit. p. 481, equation (3).

As in the previous examples we construct the semigroup ring $\mathfrak{R}(\mathfrak{H})$. Its elements have the forms $R=P(x, y)+Q(x, y)\cdot\epsilon=P+Q\epsilon$, where P, $Q\in\mathfrak{R}(\mathfrak{F})$. The multiplication rule is $R_1R_2=(P_1+Q_1\epsilon)(P_2+Q_2\epsilon)$ $=(P_1+Q_1)P_2+(P_1+Q_1)Q_2\epsilon=(P_1+Q_1)R_2$. $R_1R_2=0$, R_1, $R_2\neq 0$ implies $(P_1+Q_1)P_2=(P_1+Q_1)Q_2=0$, that is, $P_1+Q_1=0$, that is, $P_1=Q_1$, that is, $P_1\equiv Q_1 \pmod 2$. Therefore every element $R_2\in\mathfrak{R}(\mathfrak{H})$ is a right divisor of zero, that is, $n_l=0$. But $n_r=\infty$, because the 2 elements x, $y\in\mathfrak{R}(\mathfrak{H})$ have no C.R.M.: $xR_1\neq yR_2$ for any R_1, $R_2\in\mathfrak{R}(\mathfrak{H})$. $\mathfrak{R}(\mathfrak{H})$ is therefore of type $(0, \infty)$.

We consider the semigroup $\mathfrak{I}$ consisting of the symbols ϵ, x^n and $x^n\epsilon$ $(n=1, 2, \cdots)$ with identity and composition defined as for $\mathfrak{H}$. Every element of $\mathfrak{I}$ is a right divisor for zero, but none a left one; any 2 elements have a C.R.M. Therefore $\mathfrak{I}$ is of the required type (0, 1). Considering again the semigroup ring $\mathfrak{R}(\mathfrak{I})$ we see that its elements have the form $R=p(x)+q(x)\epsilon$ with the same multiplication rule as in $\mathfrak{R}(\mathfrak{H})$ and therefore $n_l=0$. But $n_r=1$, because for any $R_i=p_i+q_i\epsilon$ $(i=1, 2)$ we can find $U_i=s_i+t_i\epsilon$ with $R_1U_1=R_2U_2$; for example, choose $t_1=t_2=0$, $s_1=p_2+q_2$, $s_2=p_1+q_1$.

For the type (0, 0) we may construct an example, different from the trivial $\{0\}$ containing the zero element only, by taking any finite or infinite set of different symbols x_i, one of them being "0." This set becomes a semigroup $\mathcal{J}$ by defining multiplication by $x_ix_j=0$, that is, every product is 0. $\mathcal{J}$ is of type (0, 0), and so is the semigroup ring $\mathfrak{R}(\mathcal{J})$, as is easily seen.

Conclusion: As the type (1, 1) is that of the most common rings and as by simple symmetry our examples prove also the existence of rings (semigroups) of the types $(\infty, 1)$, $(\infty, 0)$ (resp. $(i, 1)$, $(i, 0)$) and (1, 0), we have proved that rings and semigroups of every one of the 9 possible types really exist.

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ON THE REPRESENTATION, IN THE RING OF p-ADIC INTEGERS, OF A QUADRATIC FORM IN n VARIABLES BY ONE IN m VARIABLES[1]

IRMA MOSES

Introduction. In this paper, we relate the existence of p-adically integral, linear transformations taking a quadratic form f in m variables into a quadratic form g in n variables with the representation of g by f rationally without essential denominator. Before stating our result, we introduce some terminology and recall some known theorems on the subject.

We denote by R, R_∞, and R_p respectively the rational field, the real field, and the p-adic field for p an arbitrary, fixed prime. We also designate the ring of rational integers by J and the ring of p-adic integers by J_p. We recall the definition that a form f, with matrix in J, represents a form g, with matrix in J, *rationally without essential denominator*, if, for every positive, rational integer q, f may be taken into g by a linear transformation whose elements are rational numbers with denominators relatively prime to q.

We assume throughout this paper that any considered transformation is linear and that the matrix of any considered quadratic form is nonsingular and has elements in J, unless otherwise specified. We shall feel free to phrase theorems and proofs either in terms of the matrix of a form or in terms of the form itself.

It was proved by Helmut Hasse [1, pp. 205–224][2] that if f and g are quadratic forms with the same number of variables, the existence of transformations in all R_p and in R_∞, each taking f into g, implies the existence of such a transformation in R. He later [2, pp. 12–24] extended the theorem to the case where f and g do not necessarily contain the same number of variables.[3] Then C. L. Siegel [5, pp. 678–680] proved that if f and g contain the same number of variables, the existence of transformations in all J_p and in R_∞, each taking f into g, implies that f represents g rationally without essential denominator. We now wish to extend this theorem of Siegel to the case where f

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[1] The material of this paper comes from a thesis, written under the direction of Professor Burton W. Jones, and presented to the Graduate School of Cornell University for the degree of Doctor of Philosophy.

[2] Numbers in brackets refer to the references cited at the end of the paper.

[3] The reader is also referred to a proof by C. L. Siegel [6, p. 549].

and g do not necessarily contain the same number of variables. Siegel's proof must be modified to give the extension, as it uses, to a large degree, square matrices and their inverses, which do not exist in case f and g contain different numbers of variables. We now state the extension theorem formally. Its proof and a corollary will then occupy the remainder of the paper.

THEOREM. *Let S and T be symmetric, nonsingular matrices in J, of orders m and n, respectively. If there exists a transformation in each J_p and a transformation in R_∞, each taking S into T, then S represents T rationally without essential denominator.*

1. Canonical forms for quadratic forms with matrices in J_2. In the proof of the theorem, we use canonical forms for quadratic forms with matrices in J_2. B. W. Jones [**3**, pp. 726–727] and Gordon Pall [**4**, pp. 35–38] have established canonical forms, modulo an arbitrary, fixed power of 2, for quadratic forms with matrices in J. By slight modifications of their respective theorems and proofs, we obtain the following results, designated as Lemmas 1, 2, and 3.

LEMMA 1. *Any quadratic form in s variables, with matrix in J_2, is equivalent in J_2 to a form,*

$$\phi = 2^{e_1}\theta_1 + \cdots + 2^{e_k}\theta_k,$$

where $\theta_1, \cdots, \theta_k$ are quadratic forms, each with matrix in J_2 having a 2-adic unit determinant, and each in variables different from those of the remaining forms; and where the e_i are in J $(0 \leqq e_1 < e_2 < \cdots < e_k)$.

LEMMA 2. *Let θ be a quadratic form in t variables, with matrix in J_2 having a 2-adic unit determinant. Then either θ is equivalent in J_2 to a form of the type*

$$a_1x_1^2 + \cdots + a_tx_t^2,$$

where the a_i are units in J_2; or θ is equivalent in J_2 to a form of the type

$$2(b_1x_1^2 + c_1x_1x_2 + d_1x_2^2) + \cdots + 2(b_rx_{t-1}^2 + c_rx_{t-1}x_t + d_rx_t^2),$$

where the b_i and c_i are units in J_2, and the d_i are in J_2.

LEMMA 3. *Every form $2bx_1^2+2cx_1x_2+2dx_2^2$, in which c is a unit in J_2, and b and d are in J_2, is equivalent in J_2 either to*

$$(1.1) \qquad 2x_1x_2 \quad \text{or} \quad 2x_1^2 + 2x_1x_2 + 2x_2^2,$$

according as bd is not a unit or is a unit.

We now combine Lemmas 1, 2, and 3 to see that for any quadratic form with matrix in J_2, there exists an equivalent form in J_2 of one of the following kinds: (1) a diagonal matrix; (2) a matrix with powers of 2 times the matrices of binary quadratic forms of type (1.1) on the principal diagonal and 0's elsewhere; and (3) a matrix which is a mixture of (1) and (2), containing matrices of both unary and binary quadratic forms on its principal diagonal and 0's elsewhere.

2. **The main lemma for the proof of the theorem.** In this section, we state and prove the main lemma used in the proof of the theorem. We shall use of the results of §1, as well as the following two theorems from a paper of Siegel [6, pp. 536, 538], designated as Lemmas 4 and 5.

LEMMA 4. *Let S and T be symmetric matrices in J, of orders m and n, respectively. Let P designate any one of the fields R, R_∞, and R_p. Then if $C_0' SC_0 = T$ is a representation of T by S in P, then each other representation $C'SC = T$ in P, for which $(C_0' SC - T)^{-1}$ exists, can be written in the form,*

$$C = C_0 + 2M(N - M'SM)^{-1}M'SC_0, \tag{2.1}$$

where N is an n by n skew-symmetric matrix in P and M is an m by n matrix in P. If, conversely, N is a skew-symmetric matrix in P and M is an arbitrary matrix in P, for which $(N - M'SM)^{-1}$ exists, then (2.1) *furnishes a solution of $C'SC = T$.*

LEMMA 5. *For every symmetric matrix in J_p, there exists an automorph in J_p of determinant -1.*

We wish now to prove the following lemma:

LEMMA 6. *Let a prime p be given. Let S in J and T in J_p be nonsingular, symmetric matrices, of orders m and n, respectively. If B in R_p and B_p in J_p are transformations taking S into T, then there is an automorph of T in J_p, say A_p, such that $\bar{B}_p = B_pA_p$ takes S into T and $|B'S\bar{B}_p - T| \neq 0$.*

We prove Lemma 6 by induction on n, finding it necessary to prove it first for $n = 1$ and $n = 2$.

Case I. $n = 1$. When $n = 1$, T and $B'SB_p$ are scalars. If $B'SB_p \neq T$, we choose $\bar{B}_p = B_p$; if $B'SB_p = T$, we choose $\bar{B}_p = -B_p$. Since T is nonsingular, we then surely have $|B'S\bar{B}_p - T| \neq 0$.

Case II. $n = 2$. If $|B'SB_p - T| \neq 0$, we set $\bar{B}_p = B_p$. If $|B'SB_p - T| = 0$, we may take $\bar{B}_p = -B_p$, providing that $|-B'SB_p - T| \neq 0$. If both $|B'SB_p - T| = 0$ and $|-B'SB_p - T| = 0$, we may add the expansions of the determinants to get $|B'SB_p| = -|T|$. In this case, we use Lemma 5 to find an automorph of T in J_p, say L_p, with $|L_p| = -1$. Then, since $|T| \neq 0$,

$$|B'SB_pL_p| = -|B'SB_p| \neq -|T|.$$

Thus, at least one of $|B'SB_pL_p - T|$ and $|-B'SB_pL_p - T|$ is different from zero, and we can define B_p accordingly.

Case III. $n \geqq 3$. If p is odd, we assume the lemma true for $n-1$ and proceed to show it holds for n. By a well known theorem,[4] since p is odd, there exists a unimodular transformation, say C_p, in J_p, taking T into a form,

$$Q_a = \begin{pmatrix} Q_{a0} & 0 \\ 0 & q_a \end{pmatrix},$$

where Q_{a0} is an $n-1$ by $n-1$ matrix in J_p and q_a is a scalar in J_p.

If p is even, we use the results of §1 to obtain a canonical form in J_2 for T in J_2. The first possibility mentioned there, (1), gives us the same kind of form Q_a that we have just indicated for an odd p, and we handle (1) exactly as the case for p odd. To treat (2), we proceed with an induction proof, assuming the theorem for $n-2$ and proving it for n. We rely here upon the fact that the lemma has been proved for both $n=1$ and $n=2$. For the third possibility, we may surely make an induction proof, for which, at each stage of the induction, either the treatment for (1) or that for (2) will be suitable.

We consider now the proof for (2), wherein the theorem is assumed true for $n-2$ and is to be proved for n. From the previous discussion we know there exists a unimodular transformation, C_2, in J_2, taking T into a form Q_b which is schematically either

$$\text{(i)}\quad 2^k \begin{pmatrix} Q_{b0}/2^k & 0 \\ 0 & \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix} \end{pmatrix} \quad \text{or} \quad \text{(ii)}\quad 2^k \begin{pmatrix} Q_{b0}/2^k & 0 \\ 0 & \begin{matrix} 2 & 5 \\ 5 & 12 \end{matrix} \end{pmatrix},$$

where Q_{b0} is an $n-2$ by $n-2$ matrix in J_2 and k is a non-negative, rational integer. Form (ii) follows directly from (1.1), since the unimodular matrix

$$\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

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[4] For a proof of this theorem, see, for example, C. L. Siegel [6, p. 535]. In the statement of Siegel's lemma, R_p should be replaced by G_p.

takes

$$\begin{pmatrix} 2 & 5 \\ 5 & 12 \end{pmatrix} \text{ into } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We proceed now, regardless of the parity of p. Let Q designate appropriately Q_a or Q_b, Q_0 either Q_{a0} or Q_{b0}, and q the appropriate one of

$$q_a, \quad 2^k\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \text{and} \quad 2^k\begin{pmatrix} 2 & 5 \\ 5 & 12 \end{pmatrix}.$$

We now define $V = BC_p$ and $V_p = B_pC_p$, noting that $V'SV = V_p'SV_p = Q$. If we partition V and V_p into $V = (U, u)$ and $V_p = (W, w)$, where u and w each designate a unicolumnar matrix in case q is a scalar and each a duocolumnar matrix in case q is a 2 by 2 matrix, we see readily that $U'SU = W'SW = Q_0$, that $U'Su = W'Sw = 0$ and that $u'Su = w'Sw = q$.

Then by the hypothesis of our induction, with B replaced by U, B_p by W, and T by Q_0, we see that there exists an automorph, say H, in J_p of Q_0, such that for $\tilde{W} = WH$, it is true that

$$(2.2) \qquad | U'S\tilde{W} - Q_0 | \neq 0.$$

We take $\tilde{w} = wL$, where L is an automorph of q in J to be chosen later, and $\tilde{V}_p = (\tilde{W}, \tilde{w})$. Then $W'Sw = 0$ implies $H'W'SwL = 0$; that is, $\tilde{W}'S\tilde{w} = 0$. Thus, since surely $\tilde{W}'S\tilde{W} = H'Q_0H = Q_0$ and $\tilde{w}'S\tilde{w} = q$, we have $\tilde{V}_p'S\tilde{V}_p = Q$. Now, schematically,

$$(2.3) \qquad \begin{aligned} | V'S\tilde{V}_p - Q | &= \left| \begin{pmatrix} U' \\ u' \end{pmatrix} S(\tilde{W}, \tilde{w}) - Q \right| \\ &= \begin{vmatrix} U'S\tilde{W} - Q_0 & U'S\tilde{w} \\ u'S\tilde{W} & u'S\tilde{w} - q \end{vmatrix}. \end{aligned}$$

If q is of form q_a, we take L successively equal to $+1$ and -1. Thus we first assume $\tilde{w} = w$, whence expansion of the above determinant, (2.3), by the last column will give us

$$(u'Sw - q) \, | U'S\tilde{W} - Q_0 | + K,$$

where K is a linear combination of the elements of w. If, on the other hand, we assume that $\tilde{w} = -w$, we see that the above determinant becomes

$$(-u'Sw - q) \, | U'S\tilde{W} - Q_0 | - K.$$

The sum of these two determinants is $-2q | U'S\tilde{W} - Q_0 |$, which is nonzero, since the nonsingularity of T implies $q \neq 0$ and (2.2) tells us that $|U'S\bar{W}-Q_0| \neq 0$. Thus, at least one of the two determinants is different from zero.

If q is of form (i), namely,

$$2^k\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},$$

we take L successively equal to

$$\text{(a)}\quad \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \text{(b)}\quad \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},$$

$$\text{(c)}\quad \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, \quad \text{and} \quad \text{(d)}\quad \begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}.$$

If q is of form (ii), namely,

$$2^k\begin{pmatrix}2 & 5\\ 5 & 12\end{pmatrix},$$

we take L successively equal to

$$\text{(a}'\text{)}\quad \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \text{(b}'\text{)}\quad \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},$$

$$\text{(c}'\text{)}\quad \begin{pmatrix}1 & 5\\ 0 & -1\end{pmatrix}, \quad \text{and} \quad \text{(d}'\text{)}\quad \begin{pmatrix}-1 & -5\\ 0 & 1\end{pmatrix}.$$

That the L's are indeed automorphs of the respective q's may be verified directly.

If we add the determinants, obtained from (2.3), for the values of L given by (a), (b), (c), and (d), we find their sum is $3\cdot 2^{2k+2}|U'S\bar{W}-Q_0|$. If we add the determinants, obtained from (2.3), for the values of L given by (a′), (b′), (c′), and (d′), we find their sum is $-2^{2k+2}|U'S\bar{W}-Q_0|$. From (2.2), $|U'S\bar{W}-Q_0| \neq 0$, so that at least one of the determinants in the first sum and one of the determinants in the second sum are different from zero.

We have thus shown that in case q is of the form

$$q_0, \quad 2^k\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}, \quad \text{or} \quad 2^k\begin{pmatrix}2 & 5\\ 5 & 12\end{pmatrix},$$

we may choose $\bar{w}$ appropriately so that

$$|V'S\bar{V}_\nu - Q| \neq 0. \tag{2.4}$$

Now

$$\tilde{V}_p = (W, w)\begin{pmatrix} H & 0 \\ 0 & L \end{pmatrix},$$

or denoting

$$\begin{pmatrix} H & 0 \\ 0 & L \end{pmatrix}$$

by D, we have $\tilde{V}_p = V_pD$, where D is an obvious automorph of Q in J_p.

We then have

$$V'S\tilde{V}_p - Q = V'SV_pD - Q = C_p'B'SB_pC_pD - Q. \tag{2.5}$$

Multiplying (2.5) on the left by $(C_p')^{-1}$ and on the right by C_p^{-1}, and using (2.4), we have

$$\left| B'SB_p(C_pDC_p^{-1}) - T \right| \not\equiv 0.$$

Noting that $C_pDC_p^{-1}$ is an automorph of T in J_p, we define $\tilde{B}_p = B_p(C_pDC_p^{-1})$, and this completes the basis for our induction.

3. **Proof of the theorem.** To prove the theorem, we shall use Lemmas 4 and 6, as well as the theorem of Hasse, referred to in the introduction. We proceed with the proof.

Since the hypotheses of the theorem satisfy the conditions of Hasse's theorem, there exists a rational transformation, say B, taking S into T. Since T in J is in each J_p, Lemma 6 is applicable, so that there is a $\tilde{B}_p$ in each J_p, such that $\tilde{B}_p'S\tilde{B}_p = T$ and

$$\left| B'S\tilde{B}_p - T \right| \not\equiv 0. \tag{3.1}$$

Now let q be any given, positive, rational integer and p a prime factor of q. Then using Lemma 4, by virtue of (3.1), we see that for each p, there exists a skew-symmetric matrix N_p in R_p and a matrix M_p in R_p such that

$$\tilde{B}_p = B + 2M_p(N_p - M_p'SM_p)^{-1}M_p'SB. \tag{3.2}$$

If β is an arbitrarily large rational integer, by the Chinese Remainder Theorem, we can find a matrix M in R, satisfying the congruence $M \equiv M_p \pmod{p^\beta}$, for all prime factors p of q, and a skew-symmetric matrix N in R, satisfying the congruence $N \equiv N_p \pmod{p^\beta}$. Now Lemma 4 implies the existence of $(N_p - M_p'SM_p)^{-1}$ for each p; hence, $|N_p - M_p'SM_p| \neq 0$. We then have, for β sufficiently large,

$$|N - M'SM| = |N_p - M_p'SM_p| \not\equiv 0 \pmod{p^\beta}$$

for any p. Thus, $|N-M'SM| \neq 0$.

Now we define a matrix $\bar{B}$ in R,

$$\bar{B} = B + 2M(N - M'SM)^{-1}M'SB. \tag{3.3}$$

Since $\bar{B}_p$ has p-adic integers as elements, then, by virtue of (3.2) and (3.3), for β sufficiently large, $\bar{B}$ is p-adically integral for all prime factors p of q. This means that the denominators of the elements of $\bar{B}$ are relatively prime to q. Finally, if we use (3.3) in Lemma 4, we see that $\bar{B}$ takes S into T.

4. An application of the theorem.

We now prove an immediate corollary of the theorem.

COROLLARY. *If S and T are symmetric, nonsingular matrices in J, of orders m and n, respectively, and if there exist transformations in all J_p and in R_∞, taking S into T, then there exists a rational integer q, prime to any given rational integer r, such that there is a transformation in J taking S into q^2T.*

Let r be given. Then, according to the theorem, S may be taken into T by a transformation, B, in R, the denominators of the elements of which are relatively prime to r. If we denote the least common multiple of the denominators of the elements of B by q, then $(q, r) = 1$, and

$$B'SB = T, \qquad qB'SqB = q^2T.$$

Surely qB is in J.

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THE ROLE OF THE CENTER IN THE THEORY OF DIRECT DECOMPOSITIONS

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It has been noticed for a long time that the center plays a fundamental part in the theory of direct decompositions of operator groups and loops. In particular it has been found that the existence of isomorphic refinements of direct decompositions can be assured by imposing conditions which refer solely to the center.[1] It is the object of the present note to give an explanation for these phenomena by proving quite generally that the validity of the refinement theory in an operator loop is a consequence of the validity of this theory in the center.[2]

In our discussion of operator loops and their direct decompositions we shall use the notations and fundamental definitions which we introduced in a previous paper (Baer [1]) and which are, of course, quite analogous to those customarily used in the theory of operator groups. The refinement theory, however, which we developed recently differs materially from previous statements of the theory, and thus we restate it here in the form best suited to our present purposes.

DEFINITION. *If*

$$\text{(i)} \qquad L = A \oplus B = D \oplus E$$

are direct decompositions of the M-loop L into direct sums of two M-subloops, if the refinements

$$\text{(ii)} \quad A = A' \oplus A'', \quad B = B' \oplus B'', \quad D = D' \oplus D'', \quad E = E' \oplus E''$$

satisfy the conditions

$$\text{(iii)} \quad \begin{aligned} A' \oplus B' &= B' \oplus D' = D' \oplus E' = E' \oplus A' (= L'), \\ A'' \oplus D'' &= D'' \oplus E'' = E'' \oplus B'' = B'' \oplus A''(= L''), \end{aligned}$$

then the decompositions (ii) *constitute canonical refinements of the pair* (i) *of direct decompositions of L.*

Using this definition our refinement theory may be expressed briefly in the form of the following proposition.

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[1] See Baer [1] for bibliographical references and a survey of the pertinent facts, in particular the theorems of Kořinek and Kurosh. Numbers in brackets refer to the bibliography at the end of the paper.

[2] A more precise formulation of this statement will be given immediately below.

POSTULATE E. *If the M-subloop S of the M-loop L is a direct summand of the M-loop L, then any pair of direct decompositions of S into two components possesses canonical refinements.*

We have to justify our contention that the Postulate E is an adequate expression of a refinement theory.

1. In Baer [1] we have shown that the Postulate E is satisfied under rather general hypotheses; it is valid in particular whenever the hypotheses are valid that have previously been needed for proving refinement theorems.

2. In Baer [1] we have shown that Postulate E implies the existence of exchange isomorphic refinements to any two given direct decompositions of L into a finite number of direct summands. Here we say that the direct decompositions

$$L = A(1) \oplus \cdots \oplus A(n) \quad \text{and} \quad L = B(1) \oplus \cdots \oplus B(m)$$

are exchange isomorphic, if $m = n$ and if there exists a permutation $i \rightarrow i'$ of the integers from 1 to n such that

$$L = A(1) \oplus \cdots \oplus A(i-1) \oplus B(i') \oplus A(i+1) \oplus \cdots \oplus A(n) \quad \text{for every } i.$$

It is easy to verify, and has been shown in Baer [1], that this implies the center isomorphy of $A(i)$ and $B(i')$ for every i.

3. The Postulate E constitutes nothing but the requirement that decompositions of the special form $L = A \oplus B \oplus F = D \oplus E \oplus F$ with $A \oplus B = D \oplus E$ possess refinements that are exchange isomorphic in a very strict sense.

The principal result that we are going to obtain may now be stated in a more precise form than in the introductory paragraph:

Postulate E *is satisfied by the M-loop L whenever it is satisfied by the M-center of L.*

(The center of L consists, as usual, of all the elements in L which commute with every element in L and which associate with every pair of elements in L; and the M-center M of L is the uniquely determined greatest M-subgroup of the center of L, that is, the M-center is the sum of all the admissible subgroups of the center of L.)[3]

NOTATION. If S is an M-subloop of the M-loop L, then $Z(S)$ is the M-center of S.

LEMMA. *If*

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[3] It will become apparent in the course of our discussion that we could have substituted for $Z(L)$ a certain, generally proper, M-subgroup of $Z(L)$.

$$L = A \oplus B = D \oplus E \tag{1}$$

are direct decompositions of the M-loop L, *and if the decompositions*

$$A = A' \oplus A'', \quad B = B' \oplus B'', \quad D = D' \oplus D'', \quad E = E' \oplus E'' \tag{2}$$

are canonical refinements of the pair (1) *of direct decompositions of* L, *then*

$$\begin{aligned} A' &= A \cap [D \oplus Z(E')], & A'' &= A \cap [E \oplus Z(D'')], \\ B' &= B \cap [E \oplus Z(D')], & B'' &= B \cap [D \oplus Z(E'')], \\ D' &= D \cap [A \oplus Z(B')], & D'' &= D \cap [B \oplus Z(A'')], \\ E' &= E \cap [B \oplus Z(A')], & E'' &= E \cap [A \oplus Z(B'')]. \end{aligned} \tag{3}$$

PROOF. It follows from the definition of canonical refinements that

$$\begin{aligned} &A' \oplus B' = B' \oplus D' = D' \oplus E' = E' \oplus A' \,(= L'), \\ &A'' \oplus D'' = D'' \oplus E'' = E'' \oplus B'' = B'' \oplus A''(= L''). \end{aligned} \tag{4}$$

From $E' \oplus D' = E' \oplus A'$ one deduces that there exists to every element a in A' one and only one element d in D' such that $a \equiv d$ modulo E'. There exists one and only one element e in L such that $a = d + e$; and it follows from $a \equiv d$ modulo E' that e belongs to E'. But the mapping of a in A' upon d in D' is known[4] to be a center isomorphism; and thus e actually belongs to $Z(E')$. Thus we have shown that $A' \leqq D' \oplus Z(E')$; and from $D'' \oplus E'' = D'' \oplus A''$ one deduces likewise that $A'' \leqq E'' \oplus Z(D'')$. From these inequalities one infers immediately that

$$\begin{aligned} A' &\leqq A \cap [D' \oplus Z(E')] \leqq A \cap [D \oplus Z(E')] \leqq A \cap [D \oplus E'], \\ A'' &\leqq A \cap [E'' \oplus Z(D'')] \leqq A \cap [E \oplus Z(D'')] \leqq A \cap [E \oplus D'']. \end{aligned} \tag{5}$$

It follows from (4), (2), (1) that[4]

$$L = L' \oplus L'' = D' \oplus E' \oplus D'' \oplus A'' = D \oplus E' \oplus A'',$$

implying that $A'' \cap (D \oplus E') = 0$. Consequently it follows from $A' \leqq A \cap (D \oplus E') \leqq A = A' \oplus A''$ that

$$\begin{aligned} A \cap (D \oplus E') &= A' \oplus [A'' \cap A \cap (D \oplus E')] \\ &= A' \oplus [A'' \cap (D \oplus E')] = A' \end{aligned}$$

and $A \cap (E \oplus D'') = A''$ is shown likewise. Hence the inequalities (5) are actually equalities, proving the first two equations (3) of the lemma; the rest is verified likewise. We note that we have proved slightly more than we intended to prove, namely the validity of the following equations:

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[4] This fact as well as many related assertions are very well known in the theory of groups; a verification for loops may be found in various places, for example Baer [1].

$$(6)\quad \begin{aligned} A' &= A \cap [D' \oplus Z(E')] = A \cap [D \oplus Z(E')] = A \cap [D \oplus E'], \\ A'' &= A \cap [E'' \oplus Z(D'')] = A \cap [E \oplus Z(D'')] = A \cap [E \oplus D'']. \end{aligned}$$

The following fact is easily verified:[4] If $L = A(1) \oplus \cdots \oplus A(n)$ is a direct decomposition of the M-loop L, then $Z(L) = Z(A(1)) \oplus \cdots \oplus Z(A(n))$ is a direct decomposition of the M-center of L. The latter decomposition will be referred to as *the center decomposition induced by the former decomposition.*

THEOREM 1. *Two canonical refinements of the pair* $L = A \oplus B = D \oplus E$ *of direct decompositions of the M-loop L are equal if, and only if, they induce the same center decompositions.*

This is an almost immediate consequence of the equations (3) of the preceding lemma.

THEOREM 2. *Suppose that* (1) *are direct decompositions of the M-loop L, that*

$$(\mathrm{Z}.1)\qquad Z(L) = Z(A) \oplus Z(B) = Z(D) \oplus Z(E)$$

are the center decompositions induced by (1), *and that*

$$(\mathrm{Z}.2)\qquad \begin{aligned} Z(A) &= A^* \oplus A^{**}, & Z(B) &= B^* \oplus B^{**}, \\ Z(D) &= D^* \oplus D^{**}, & Z(E) &= E^* \oplus E^{**} \end{aligned}$$

are refinements of the decompositions (Z.1). *Then the refinements* (Z.2) *are canonical refinements of* (Z.1) *if, and only if, they are induced in the M-center* $Z(L)$ *of L by canonical refinements of* (1).

PROOF. It is almost obvious that canonical refinements of (1) induce in the center canonical refinements of (Z.1). Thus the refinements (Z.2) are certainly canonical refinements of (Z.1) if they are induced in the center by canonical refinements of (1).

Assume now that the refinements (Z.2) are canonical refinements of (Z.1). Then we infer from the definition that

$$(3)\quad \begin{aligned} Z^* &= A^* \oplus B^* = B^* \oplus D^* = D^* \oplus E^* = E^* \oplus A^*, \\ Z^{**} &= A^{**} \oplus D^{**} = D^{**} \oplus E^{**} = E^{**} \oplus B^{**} = B^{**} \oplus A^{**}. \end{aligned}$$

Clearly $Z(L) = Z^* \oplus Z^{**}$. Following the suggestions inherent to the lemma we define:

$$(4)\qquad \begin{aligned} A' &= A \cap (D \oplus E^*), & A'' &= A \cap (E \oplus D^{**}),\\ B' &= B \cap (E \oplus D^*), & B'' &= B \cap (D \oplus E^{**}),\\ D' &= D \cap (A \oplus B^*), & D'' &= D \cap (B \oplus A^{**}),\\ E' &= E \cap (B \oplus A^*), & E'' &= E \cap (A \oplus B^{**}). \end{aligned}$$

It is clear that they are all normal M-subloops of L. In order to prove that they constitute the desired canonical refinements of (1) we need the M-endomorphisms α, β, and δ, ϵ which are uniquely determined by the requirements

$$\begin{aligned} x\alpha &= x \text{ for } x \text{ in } A, & B\alpha &= 0, & x\beta &= x \text{ for } x \text{ in } B, & A\beta &= 0;\\ x\delta &= x \text{ for } x \text{ in } D, & E\delta &= 0, & x\epsilon &= x \text{ for } x \text{ in } E, & D\epsilon &= 0. \end{aligned}$$

These endomorphisms induce[4] in the M-center $Z(L)$ of L endomorphisms with completely analogous properties which we designate by the same symbols.

From $B^* \oplus A^* = B^* \oplus D^*$ it follows that there exists to every element x in D^* one and only one element y in A^* such that $x \equiv y$ modulo B^*; and it is easily seen that $y = x\alpha$. One verifies now that α induces an isomorphism of D^* upon A^*; and in this way one proves the following contentions:[4]

$$(5)\qquad \begin{aligned} &\text{an isomorphism of } D^* \text{ upon } A^* \text{ and of } E^{**} \text{ upon } A^{**} \text{ is induced by } \alpha,\\ &\text{an isomorphism of } E^* \text{ upon } B^* \text{ and of } D^{**} \text{ upon } B^{**} \text{ is induced by } \beta,\\ &\text{an isomorphism of } A^* \text{ upon } D^* \text{ and of } B^{**} \text{ upon } D^{**} \text{ is induced by } \delta,\\ &\text{an isomorphism of } B^* \text{ upon } E^* \text{ and of } A^{**} \text{ upon } E^{**} \text{ is induced by } \epsilon. \end{aligned}$$

We prove first:

$$(6)\qquad A' \cap A'' = B' \cap B'' = D' \cap D'' = E' \cap E'' = 0.$$

If the element x belongs to the cross cut of A' and A'', then x is in A and so $x\beta = 0$. Since x is in A', there exist uniquely determined elements r, s in D and E^* respectively such that $x = r + s$, and since x is in A'', there exist elements u, v in D^{**} and E respectively such that $x = u + v$. Since r and u are in D, s and v are in E, and since $L = D \oplus E$, it follows that $r = u$ is in D^{**} and $s = v$ is in E^*. Thus x belongs to $D^{**} \oplus E^*$. But β induces an isomorphism of $D^{**} \oplus E^*$ upon B, by (5); and $x\beta = 0$. Hence $x = 0$, proving $A' \cap A'' = 0$; and the remainder of (6) is verified likewise.

$$(7)\qquad A = A' \oplus A'', \quad B = B' \oplus B'', \quad D = D' \oplus D'', \quad E = E' \oplus E''.$$

It follows from (6) that the compositum of A' and A'' is their direct sum; and hence it follows from (4) that

$$A' \oplus A'' \leqq A.$$

It has been shown elsewhere (Baer [1]) that $\alpha\delta\beta$ maps L into part of its M-center. Consider an element x in A. Then $x = x\alpha$ and $x\delta\beta = x\alpha\delta\beta$ is therefore an element in $Z(L)$. But $x\delta\beta$ certainly belongs to $L\beta = B$. Hence $x\delta\beta$ is an element in $Z(B) = B^* \oplus B^{**}$; and thus there exist uniquely determined elements b^* and b^{**} in B^* and B^{**} respectively such that $x\delta\beta = b^* + b^{**}$. We infer from (5) the existence of uniquely determined elements e^* and d^{**} in E^* and D^{**} respectively such that $e^*\beta = b^*$ and $d^{**}\beta = b^{**}$. Since $x\delta$ belongs to D and $x\epsilon$ belong to E, there exist uniquely determined elements d and e in D and E respectively such that $x\delta = d + d^{**}$ and $e = x\epsilon + e^*$. There exist finally uniquely determined elements x' and x'' such that $x' + e^* = d$ and $x'' = e + d^{**}$. It is clear that x' is in $D \oplus E^*$ and that x'' is in $E \oplus D^{**}$. Furthermore

$$\begin{aligned} b^* + b^{**} &= x\delta\beta = (d + d^{**})\beta = d\beta + d^{**}\beta = (x' + e^*)\beta + b^{**} \\ &= (x'\beta + b^*) + b^{**} \end{aligned}$$

so that $x'\beta = 0$. Hence x' is in A and therefore in A'. Likewise we infer from $x\beta = 0$ and the fact that b^* and b^{**} belong to the center that

$$\begin{aligned} x''\beta &= (e + d^{**})\beta = e\beta + d^{**}\beta = (x\epsilon + e^*)\beta + d^{**}\beta \\ &= (x\epsilon\beta + b^*) + b^{**} = x\epsilon\beta + (b^* + b^{**}) \\ &= x\epsilon\beta + x\delta\beta = x\beta = 0, \end{aligned}$$

proving that x'' belongs to A and therefore to A''. Considering finally that e^* and d^{**} belong to the center we find that

$$\begin{aligned} x + e^* &= (x\delta + x\epsilon) + e^* = x\delta + (x\epsilon + e^*) = (d + d^{**}) + e \\ &= ((x' + e^*) + d^{**}) + e = (x' + e^*) + (e + d^{**}) \\ &= (x' + e^*) + x'' = (x' + x'') + e^* \end{aligned}$$

or $x = x' + x''$ so that x belongs to $A' \oplus A''$, proving $A = A' \oplus A''$; the other contentions (7) are verified likewise.

$$\begin{aligned} &A^* \leqq A' \leqq D' \oplus E^*, &&A^{**} \leqq A'' \leqq D^{**} \oplus E'', \\ &B^* \leqq B' \leqq E' \oplus D^*, &&B^{**} \leqq B'' \leqq E^{**} \oplus D'', \\ &D^* \leqq D' \leqq A' \oplus B^*, &&D^{**} \leqq D'' \leqq A^{**} \oplus B'', \\ &E^* \leqq E' \leqq B' \oplus A^*, &&E^{**} \leqq E'' \leqq B^{**} \oplus A''. \end{aligned} \tag{8}$$

Clearly $A^* \leqq A \cap (D + A^*) \leqq A \cap (D + Z^*) = A \cap (D + D^* + E^*) = A \cap (D \oplus E^*) = A'$; $A^{**} \leqq A''$, and so on, are verified likewise. Every element in A' has, by (4), the form $a=d+e^*$ for a in A, d in D and e^* in E^*. Thus $0=a\beta=d\beta+e^*\beta$. It follows from (5) that $e^*\beta$ belongs to $E^*\beta=B^*$, proving that $d\beta$ is in B^*. Hence $d=d\alpha+d\beta$ belongs to the cross cut D' of D and $A\oplus B^*$, proving that a is in $D'\oplus E^*$ or $A'\leqq D'\oplus E^*$; and the remaining inequalities (8) are verified in a like manner.

$$\begin{aligned} A' \oplus B' &= B' \oplus D' = D' \oplus E' = E' \oplus A', \\ A'' \oplus D'' &= D'' \oplus E'' = E'' \oplus B'' = B'' \oplus A''. \end{aligned} \tag{9}$$

Since $A'\leqq A$ and $B'\leqq B$, we may infer $A'\cap B'=0$ from $A\cap B=0$; $D'\cap E'=0$ is seen likewise. Furthermore we have

$$B' \cap D' = B \cap (A \oplus B^*) \cap (E \oplus D^*) \cap D = B^* \cap D^* = 0$$

by (3); $E'\cap A'=0$ is verified likewise.

Now we deduce from (3) and (8) that

$$\begin{aligned} A' \oplus B' &\leqq D' + E^* + B' \leqq D' + D^* + E^* + B' \\ &\leqq D' + D^* + B^* + B' \leqq D' \oplus B' \\ &\leqq D' + E' + D^* \leqq D' \oplus E' \\ &\leqq A' + B^* + E' \leqq A' + A^* + B^* + E' \\ &\leqq A' + A^* + E^* + E' \leqq A' \oplus E' \\ &\leqq A' + B' + A^* \leqq A' \oplus B', \end{aligned}$$

and the first set of equalities (9) is an immediate consequence of this series of inequalities. The second set of equations (9) is proved likewise.

The normal M-subloops, defined by (4), constitute by (7) and (9) canonical refinements of the decompositions (1). These refinements induce in the M-center $Z(L)$ of L just the refinements (Z.2) of (Z.1), since, by (8), we have for instance $A^*\leqq Z(A')$, $A^{**}\leqq Z(A'')$, and $A^*\oplus A^{**}=Z(A)=Z(A'\oplus A'')=Z(A')\oplus Z(A'')$. This completes the proof of Theorem 2.

COROLLARY 1. *The direct decompositions* (1) *of the M-loop L possess canonical refinements if, and only if, the induced center decompositions* (Z.1) *possess canonical refinements. Mapping direct decompositions of L upon the induced center decompositions effects a one-to-one correspondence between the canonical refinements of* (1) *and the canonical refinements of* (Z.1).

These important propositions are immediate consequences of Theorems 1 and 2.

THEOREM 3. *Postulate* E *is satisfied by the M-loop L if, and only if, the following condition is satisfied by* $Z(L)$:

(Z.E) *If* $A \oplus B = D \oplus E$ *is a direct summand of the M-loop L, then the induced direct decompositions* $Z(A) \oplus Z(B) = Z(D) \oplus Z(E)$ *possess canonical refinements.*

This is an almost immediate consequence of the first assertion of Corollary 1.

COROLLARY 2. *Postulate* E *is satisfied by the M-loop L whenever it is satisfied by the abelian M-group* $Z(L)$.

This is an immediate consequence of Theorem 3, since $Z(A) \oplus Z(B)$ is a direct summand of $Z(L)$, whenever $A \oplus B$ is a direct summand of L.

It should be noted that we cannot claim the converse of Corollary 2 to be true. For instance, it may very well happen that L is indecomposable, though Postulate E is not satisfied by $Z(L)$.

Finally the question may be asked whether it might be possible to prove a like theorem for weaker forms of the refinement theory than the one concentrated in Postulate E. An example, due to Kurosh [1], shows the impossibility of substituting center isomorphy for exchange isomorphy. He has constructed a group G, an ordinary group generated by four elements, with the property that $G = A \oplus B = D \oplus E$ where A, B, D, E are indecomposable, E is infinite cyclic and neither A nor B is commutative. Thus G possesses direct decompositions into indecomposable components which are not even isomorphic. But the center of G is a free abelian group of rank 2, and any two direct decompositions of such a free abelian group of rank 2 into indecomposable direct summands are center isomorphic, though not exchange isomorphic (in particular the induced center decompositions are not exchange isomorphic).

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NONASSOCIATIVE VALUATIONS

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In a paper entitled *Noncommutative valuations*, Schilling [8][1] proved that if an algebra of finite order over its center is relatively complete in a valuation (where the value group of the nonarchimedean, exponential valuation is not assumed to be commutative) then the value group is commutative. A similar type of theorem, proved by Albert [1], states that if an algebra of finite order over a field is an ordered algebra, then the algebra is itself commutative.

In the present paper, the notions of valuation and ordered ring are carried over in the obvious fashion to the case of nonassociative algebras (the set of values of a valuation are no longer required to lie in an ordered group, but only in an ordered loop). In this situation, an analogue of Schilling's result remains true with an added hypothesis: If an algebra of finite order has a unity quantity and has a valuation inducing a rank one valuation of the base field, then the value loop of the algebra is commutative, associative, and archimedean-ordered. No completeness is needed. In particular, every valuation of an algebra (with a unity quantity) of finite order over an algebraic number field has a group of real numbers for its value loop.

However, in general the obvious extensions of both Schilling's and Albert's results are false, since there are noncommutative, nonassociative algebras of arbitrary finite order over a field which have valuations with nonassociative value loops and which are ordered algebras. Examples of such algebras are obtained by proving that a necessary and sufficient condition for an ordered loop L to be the value loop of some algebra of finite order is that some subgroup of the center of L have finite index in L. We construct some such loops in §3. All such loops will be determined in another paper.

1. **Ordered loops.** An ordered loop L is a set of elements $x, y, z, \cdots$ on which are defined a binary function, $+$, and a binary relation, $>$, with the following properties:

(1) $+$ is a single-valued function on LL to L.

(2) If x and y are in L, then there exist unique elements u and v in L such that $u+x=y$ and $x+v=y$.

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

(3) There is an element 0 in L such that $0+x=x+0=x$ for every x in L.

(4) If $x>y$ and $y>z$ then $x>z$.

(5) For each x and y in L, one and only one of the three possibilities $x>y$, $y>x$, $x=y$ holds.

(6) If $x>y$ then for any z in L, $x+z>y+z$, $z+x>z+y$.

We shall use the notations $y-x$ and $-x+y$ for the quantities u and v, respectively, defined in (2).

Ordered loops have most of the elementary properties of ordered groups (in fact, lattice-ordered quasigroups, if suitably defined, have a theory closely paralleling that of l-groups). In particular, we list without proof the following lemmas, true in any ordered loop.

(7) If $x>y$ then $x-z>y-z$ and $-z+x>-z+y$.

(8) If $x>y$ and $x'\geqq y'$ or if $x\geqq y$ and $x'>y'$ then $x+x'>y+y'$, $x-y'>y-x'$ and $-y'+x>-x'+y$.

If n is a positive integer and x is an element of a loop we define the symbol nx inductively by writing $1x=x$, $nx=(n-1)x+x$; we call nx the nth right multiple of x.

(9) If $nx>ny$ then $x>y$.

(10) If $x\neq 0$ then every element of L obtained by adding a finite number of summands, each equal to x (with any association), is different from zero.

Let us also list a few definitions usually made in the study of loops. The center of a loop L is the set of all c in L that commute and associate with all elements of L. More precisely (cf. Albert [2, p. 516]) the elements c are characterized by the property that for all x and y of L,

$$c+(x+y)=(c+x)+y=x+(y+c).$$

An equivalent statement is that the center of L consists of all elements of L invariant under the group of inner mappings of L (cf. Bruck [4, p. 257]). The definition of the center of a loop leads readily to the theorem that if c is in the center of L, then the loop generated by c is an abelian group and is contained in the center of L.

A normal subloop N of a loop L is a subloop invariant under the inner mapping group of L. Equivalently,

$$N+(x+y)=(N+x)+y=x+(y+N)$$

for every x and y in L. The characteristic property of a normal subloop N is the existence of a quotient loop L/N, formed in the usual way, to which L is homomorphic. Clearly any loop contained in the center of L is a normal subloop of L.

An element x is positive if $x>0$. An ordered loop is discrete if it has a smallest positive element (note that this terminology differs from that of Krull [7, p. 171]).

Two ordered loops are order-isomorphic if there is a one-to-one correspondence between them which preserves the operation $+$ and the relation $>$.

A subset H of an ordered loop L is an isolated subloop or l-ideal in case H is a normal subloop and if H contains, with x and y, every z such that $x>z>y$ (an equivalent definition is obtained by taking $y=0$). Just as in the case of abelian groups, the characteristic property of an isolated subloop is the existence of a natural ordering of the quotient loop L/H, so that if $x>y$ in L, then $x+H\geq y+H$ in L/H (cf. Birkhoff [3, p. 310]).

An ordered group is archimedean-ordered in case it has no isolated subgroups besides zero and the whole group. The principal property of archimedean-ordered groups is that every such group is order-isomorphic with an additive group of real numbers (Cartan [5]).

Our study of nonassociative valuations will be based on the following two theorems on ordered loops.

THEOREM 1. *Let L be a discrete ordered loop having a least positive element, e. Then e is in the center of L and so generates an associative, commutative, normal subloop G of L. Furthermore, G is an isolated subloop of L.*

PROOF. We must show that for all x, y in L,

$$e+(x+y)=(e+x)+y=x+(y+e). \tag{11}$$

First, there is no element of L between $x+y$ and $e+(x+y)$. For if $x+y<z<e+(x+y)$ then $0<z-(x+y)<e$, contradicting the hypothesis that e is the least positive element of L. Similarly there is no element between $x+y$ and $(e+x)+y$, nor between $x+y$ and $x+(y+e)$ so that of the three elements in (11), all are greater than $x+y$ but none is greater than any other. Hence they are equal. This proves that e is in the center of L so that the loop G generated by e is an abelian group and is also in the center of L. If this group is not isolated, there is an x in L but not in G and there is a positive integer n with $0<x<ne$. Let n be chosen as the smallest such positive integer. Then $(n-1)e<x<ne$, $0<x-(n-1)e<e$, contrary to our hypothesis on e.

THEOREM 2. *Let L be an ordered loop containing an archimedean-ordered group G in its center. Suppose there is a positive integer n such* *that G contains the nth right multiple of every element of L. Then L is commutative, associative and archimedean-ordered.*

PROOF. First, suppose G is discrete. Then G is order-isomorphic with the additive group of rational integers and we may prove that L also has a least positive element. For consider the set of all nx with $x>0$ in L. This is a set of positive elements in G and so there is a smallest one, say ne. By (9) this determines a unique $e>0$ in L which, again by (9), is a least positive element of L. By Theorem 1, e generates an isolated abelian group H in the center of L. The intersection $G\cap H$ is then an isolated subgroup of G and is not zero since it contains ne. Since G is archimedean-ordered, $G\cap H=G$ and $H\supset G$. But consider the mapping of L onto the ordered quotient loop L/H. The group G maps onto zero so that for every y in L/H, we have $ny=0$. Then by (10), it follows that $y=0$, $L/H=0$, $L=H$, which proves Theorem 2 in this case.

Second, suppose G has no least positive element. In this case, using the fact that G is order-isomorphic with an additive group G' of real numbers, we shall set up an order-isomorphism between L and a group L' of real numbers. We already have an isomorphism between G and G'. Let the correspondent in G' of an element u of G be denoted by u'. Then if x belongs to L, define

$$A(x) = [\text{all } u' \text{ in } G' \text{ such that } u \geqq x],$$
$$B(x) = [\text{all } u' \text{ in } G' \text{ such that } u \leqq x].$$

Since we are assuming that G' is not discrete it follows in the usual fashion that for each x, g.l.b. $A(x)=$l.u.b. $B(x)$. We define a correspondence on L to the real numbers as follows:

$$x \to x' = \text{g.l.b. } A(x) = \text{l.u.b. } B(x). \tag{12}$$

If x is in G we then have two definitions for x' which obviously coincide. By the standard methods it can now be proved that (12) is a homomorphism so that the set L' of all x' is a group of real numbers. Then if $x'=y'$ we have $(nx)'=nx'=ny'=(ny)'$; but nx and ny are in G so that by hypothesis $nx=ny$ and by (9), $x=y$. Thus (12) is an isomorphism. Finally, if $x>y$ then $A(x)\subset A(y)$, $x'\geqq y'$ and $x'\neq y'$ by the one-to-one character of (12). It follows that (12) is an order-isomorphism, which completes our proof.

2. Valuations. Let R be a ring (not necessarily associative). A valuation of R is a function V which associates to each nonzero element of R an element of an ordered loop L with the properties

$$V(a+b) \geqq \min\,[V(a),\ V(b)], \tag{13}$$

$$V(ab) = V(a) + V(b), \tag{14}$$

whenever a, b and $a+b$ are nonzero elements of R. We define $V(0)=\infty$ where the symbol ∞ is to have the properties $\infty + x = x + \infty = \infty$, $\infty > x$ for all x in L. Then (13) and (14) hold for all a, b in R. We denote by $V(R)$ the set of all $V(a)$ for $a \neq 0$ in R.

Such a valuation, even though L is nonassociative, has most of the elementary properties of an ordinary valuation. In particular, $V(a_1, \cdots, a_t) = \min\,[V(a_1), \cdots, V(a_t)]$ unless for some pair of distinct subscripts i and j, $V(a_i) = V(a_j)$.

Note that a ring R with a valuation V has no divisors of zero, for if $ab=0$ then $\infty = V(ab) = V(a)+V(b)$ and at least one of $V(a)$, $V(b)$ must be ∞. If R is a division ring (that is, for every $a \neq 0$ and b of R there exist unique elements c and d in R such that $ac = da = b$), and if in addition R has a unity quantity e such that $ea = ae = a$ for all a in R, then $V(R)$ is an ordered loop called the value loop. This is the case in the following theorem.

THEOREM 3. *Let A be an algebra (not necessarily associative) of order n over a field F and let A contain a unity quantity. If A has a valuation V then V induces a valuation on F such that $V(F)$ is an ordered abelian group G. If G is archimedean-ordered, then the value loop $L = V(A)$ is in fact associative, commutative and archimedean-ordered.*

PROOF. Since A is an algebra of finite order over F without divisors of zero, it follows that A is a division ring and that L is an ordered loop. Since the elements of F commute and associate with the elements of A, G will be in the center of L. Our proof will then merely consist in showing that for every x of L, $(n!)x$ belongs to G, so that Theorem 2 will immediately give the desired result.

LEMMA 1. *L/G is a finite loop whose order is not greater than n.*

PROOF. Suppose there are $n+1$ distinct elements of L/G and hence $n+1$ elements $x_1, \cdots, x_{n+1}$ of L that are incongruent modulo G. Then $x_i = V(a_i)$ for some a_i of A. But there exist α_i in F, not all zero, with $\sum \alpha_i a_i = 0$. Then $\infty = V(\sum \alpha_i a_i)$ and since $\infty \neq \min\,[V(\alpha_i a_i)]$, we must have $V(\alpha_i a_i) = V(\alpha_j a_j)$ for some i and j with $i \neq j$. That is, $V(\alpha_i) + x_i = V(\alpha_j) + x_j$, $x_i = -V(\alpha_i) + V(\alpha_j) + x_j$, $x_i \equiv x_j \pmod{G}$. This is a contradiction.

LEMMA 2. *If K is a finite loop whose order is not greater than n, then for every x of K, $n!x = 0$.*

PROOF. First, for every x there is a positive integer $n_x \leqq n$ such that $n_x x = 0$. For there certainly is a pair of distinct positive integers p_x and q_x, both not greater than n, such that $p_x x = q_x x$. And by subtracting x's from the right, one at a time, we arrive at $n_x x = 0$, where $n_x = |p_x - q_x|$.

Second, if n_x divides a positive integer p, then $px = 0$. This is proved by writing $p = n_x q$ and making an induction on q. Hence $p = n!$, being divisible by n_x for every x, has the property that $px = 0$ for all x.

Lemmas 1 and 2 together imply that for every x of L, $n!x$ maps into zero in L/G and so is in G. Then Theorem 2 immediately asserts the truth of Theorem 3, as predicted.

3. **Examples of ordered loops.** According to Lemma 1, L is an ordered loop containing the ordered group G in its center in such a way that the index of G in L is finite. In this section we shall demonstrate that for some nonarchimedean groups G there actually exist nonassociative ordered loops L in the relation to G described above.

In particular, let G be an ordered loop with an isolated subloop H and let G/H be discrete. Denote by e an element of G whose image in G/H is the least positive element of G/H. Next, let K be a cyclic group of order n generated by an element b. We shall construct an ordered, nonassociative loop L as a loop extension of G by K. L consists of all ordered pairs (k, g) with k in K and g in G. Addition in L is defined by the rule

$$(k_1, g_1) + (k_2, g_2) = (k_1 + k_2, g_1 + g_2 + f[k_1, k_2])$$

where $f(k_1, k_2)$ is a function on KK to G with $f(0, k) = f(k, 0) = 0$ for all k of K. This makes L a loop with the identity element $(0, 0)$. The loop G is embedded in L in the sense that the set of all $(0, g)$ is isomorphic with G. We make the following restrictions on the "factor set" $f(k_1, k_2)$ that will allow us to order L. If m_1 and m_2 are nonnegative integers, both less than n, then we demand that $f(m_1 b, m_2 b)$ be in H or in $e + H$ according as $m_1 + m_2 < n$ or $m_1 + m_2 \geqq n$. Clearly if $H \neq 0$ we may still choose the function f so that L is nonassociative and noncommutative. Regardless of this fact, we can order L by defining $(m_1 b, g_1) > (m_2 b, g_2)$ in case

$$g_1 + H > g_2 + H,$$

or

$$g_1 + H = g_2 + H \quad \text{and} \quad n > m_1 > m_2 \geqq 0,$$

or

$$g_1 + H = g_2 + H, \quad m_1 = m_2 \quad \text{and} \quad g_1 > g_2.$$

This is clearly a linear, transitive ordering of L which preserves the ordering of G. A straightforward application of the definition shows that L is actually an ordered loop under this ordering.

It is a trivial exercise to prove that the commutative and associative laws hold in L whenever one of the summands is in the center of G. Thus if G is an abelian group, it is in the center of L and L is an example of an ordered loop such as we were searching for.

4. **Examples of nonassociative valuations.** Let L be any ordered loop and D any ring (not necessarily associative) without divisors of zero. Then we shall exhibit a ring R with a valuation V with $V(R)=L$ and with a residue class ring[1] isomorphic with D. The ring R is constructed in the usual fashion as the ring of "power-series" with exponents in L and coefficients in D. More precisely, R consists of all functions f on L to D with the property that the set $L(f)$, consisting of all x in L for which $f(x)\neq 0$, is well-ordered by the ordering of L. If f and g are in R we define $f+g$ to be the function whose value at each x of L is $f(x)+g(x)$ and the product fg to be the function whose value at x is[2] $\sum f(y)g(z)$, the summation extending over all pairs (y, z) with $y+z=x$. The summation is meaningful because for a given x there is only a finite number of pairs (y, z) for which y is in $L(f)$, z is in $L(g)$ and $y+z=x$. The proof of this fact as well as the proofs that $L(f+g)$ and $L(fg)$ are well-ordered and that R is actually a ring are essentially identical with the proofs given by Hahn [6, pp. 647ff.]. A similar comment applies to the theorem that if D is a division ring, then so also is R. In particular if D has a unity quantity 1, then R has a unity quantity e with the characteristic property that $e(0)=1$, $e(x)=0$ if $x\neq 0$. If we define $V(f)$ for any f of R to be the smallest element of $L(f)$, we have a function on R to L with $V(R)=L$. In fact this is a valuation, the proof of (13) and (14) proceeding exactly as in the associative case. To prove that the residue class ring is isomorphic to D, remark that the ring Q consists of all f in R for which $f(x)=0$ when $x<0$ and that the prime ideal P consists of all f in R for which $f(x)=0$ when $x\leqq 0$. The correspondence $f\rightarrow f(0)$ is then an isomorphism of Q/P and D.

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[1] If Q is the set of all x in R with $V(x)\geqq 0$ and P is the set of all x with $V(x)>0$, then Q is a ring in which P is a prime ideal. The residue class ring Q/P is called the residue class ring of R.

[2] This is a very special type of power series ring, but more generality here would be unwarranted.

THEOREM 4. *If we choose D to be a field and L to be an ordered loop such as was constructed in* §3 *(L contains a group G in its center with L/G finite), the resultant power-series ring R is an algebra of finite order over a field F and has a unity quantity. It also has a valuation V with a value loop L. The order of R over F is the index of G in L so that by suitable choice of L we can find algebras of arbitrary order over F with non-associative valuations.*

PROOF. Let F be the set of all f in R such that $L(f)\subset G$. By the classical result, F is a field (Hahn [6]). Let $x_1, \cdots, x_n$ be a maximal set of elements of L incongruent modulo G and let $f_1, \cdots, f_n$ be the elements of R defined by the equations $f_i(x_i)=1$, $f_i(x)=0$ if $x\neq x_i$. If $g(x)$ is any element of R, $gf_i(x)=\sum g(y)f(z)$, the sum ranging over all y of $L(g)$ and z of $L(f_i)$ with $y+z=x$. Since $L(f_i)$ consists of the single point x_i, we have the identity

$$gf_i(x) = g(x - x_i).$$

If f is any element of R, let $L_i(f)$ be the set of all x in $L(f)$ with $x\equiv x_i$ (mod G). Then each $L_i(f)$ is well-ordered and $L(f)$ is the union of the disjoint sets $L_i(f)$ $(i=1, \cdots, n)$. Define $f^i(x)$ to be the function which is equal to $f(x)$ on $L_i(f)$ but zero elsewhere and $g_i(x)$ to be $f^i(x+x_i)$. Then f^i and g_i are elements of R, $f^i=g_if_i$ and

$$f = \sum_{i=1}^{n} f^i = \sum_{i=1}^{n} g_if_i. \tag{15}$$

Since $g_i(x)\neq 0$ implies that $x+x_i$ is in $L_i(f)$ and hence congruent with x_i modulo G, we know that $g_i(x)$ is zero unless $x\equiv 0$ (mod G), so that $L(g_i)\subset G$ and g_i is in F. Hence (15) expresses the fact that R is a linear space of order not greater than n over F. In fact the f_i are linearly independent over F, else $\sum g_if_i=0$ with the g_i in F, $V(g_if_i) = V(g_jf_j)$ for some pair of distinct indices i, j, which implies that $x_i\equiv x_j$ (mod G), a contradiction. This completes the proof of Theorem 4.

If, in particular, D is the field of real numbers, the ring R is an ordered division ring[4] if we define $f>0$ to mean $f(V[f])>0$ in D. The proof is trivial.

THEOREM 5. *There exist noncommutative and (necessarily) nonassociative ordered algebras of arbitrary finite order over a field.*

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[4] An ordered division ring is a ring, linearly ordered by a relation $>$, such that $a>0$ and $b>0$ imply $a+b>0$ and $ab>0$.

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ON THE ROOTS OF A POLYNOMIAL AND ITS DERIVATIVE

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Let $r_1, r_2, \cdots, r_n$ be the roots of a polynomial $f(z)$ with complex coefficients, and let $R_1, R_2, \cdots, R_{n-1}$ be the roots of its derivative. N. G. de Bruijn[1] has proved that

$$\frac{1}{n}\sum_{j=1}^{n} | I(r_j) | \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} | I(R_j) |, \tag{1}$$

when $f(z)$ has real coefficients; he raises the question whether this holds in general. We prove that this inequality holds when $f(z)$ has complex coefficients; also that (1) is an equality only when the roots of $f(z)$ are not both above and below the real axis. An immediate consequence of this is that if $D_l(z)$ represents the (positive) distance from z to any straight line l in the complex plane, then

$$\frac{1}{n}\sum_{j=1}^{n} D_l(r_j) \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} D_l(R_j), \tag{2}$$

with the equality holding only when the r_j are not located on both sides of l. Further, if for any point A in the complex plane, $D_A(z)$ represents the distance from z to A, then

$$\frac{1}{n}\sum_{j=1}^{n} D_A(r_j) \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} D_A(R_j), \tag{3}$$

with the equality holding only when all the r_j lie on a half line emanating from A.

If A is taken as the origin, we have

$$\frac{1}{n}\sum_{j=1}^{n} | r_j |^m \geqq \frac{1}{n-1}\sum_{j=1}^{n-1} | R_j |^m$$

with $m=1$. This inequality, with $m=1, 2, 3, \cdots$, has been established by H. E. Bray[2] for the special case in which $f(z)$ is a real

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[1] *On the zeros of a polynomial and of its derivative*, K. Akademie van Wetenschappen, Proceedings vol. 49 (1946) pp. 1037–1044. *Added in proof*: In a second paper by de Bruijn and T. A. Springer, *On the zeros of a polynomial and of its derivative* II, Ibid. vol. 50 (1947) pp. 264–270, the results of the present paper are obtained, and the inequality following (3) is obtained for any $m \geqq 1$.

[2] *On the zeros of a polynomial and its derivative*, Amer. J. Math. vol. 53 (1931) pp. 864–872.

polynomial with non-negative real roots.

If (3) is applied with A located successively at the roots of $f(z)$, and if these inequalities are summed, and if this process is repeated with A located successively at the roots of $f'(z)$, it is seen that the two resulting inequalities imply that

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|r_i - r_j| \geqq \frac{1}{(n-1)^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}|R_i - R_j|,$$

with equality only if all roots of $f(z)$ are equal. It seems likely that this inequality holds with the factors $1/n^2$ and $1/(n-1)^2$ replaced by $1/C_{n,2}$ and $1/C_{n-1,2}$, but we have not been able to prove this.

We now begin the proof of (1).

Lemma 1. *In case all the roots of $f(z)$ lie on one side of or on the real axis, relation* (1) *holds with the equality sign. In case the roots lie both above and below the real axis, but the roots of $f'(z)$ lie on one side of or on the real axis, relation* (1) *holds with the inequality sign.*

Proof. We recall Gauss' theorem that the roots of $f'(z)$ lie inside or on the convex polygon determined by the roots of $f(z)$; and they are on the polygon only when $f(z)$ has a multiple root or when all the roots of $f(z)$ lie on a line. Hence in the first statement in the lemma, the roots of $f'(z)$ lie on one side of or on the real axis. By relating the sum of the roots of a polynomial to the coefficients of the two highest powers, we see that

$$\frac{1}{n}\sum r_j = \frac{1}{n-1}\sum R_j,$$

whence

$$\frac{1}{n}\sum_{j=1}^{n} I(r_j) = \frac{1}{n-1}\sum_{j=1}^{n-1} I(R_j), \tag{4}$$

which proves the first statement in the lemma. The second statement is also a consequence of (4), because in that case all terms on the right side of (4) have like signs, whereas there are mixed signs in the sum on the left.

Lemma 2. *Inequality* (1) *is equivalent to*

$$\frac{1}{n}\sum{}' |I(r_j)| \geqq \frac{1}{n-1}\sum{}' |I(R_j)|, \tag{5}$$

where $\sum{}'$ designates the sum over the roots lying below the real axis.

Equality in (1) *implies equality in* (5), *and conversely.*

Proof. Subtract (4) from (1). We note in passing that the addition of (4) to (1) implies a result (5) with $\sum'$ representing the sum over the roots above the real axis.

We now proceed by induction, and assume that a result corresponding to (5) holds for all polynomials of degree less than n, with equality only when the roots are not on both sides of the real axis. (Note that this is true for quadratic polynomials.) Then we suppose, using Lemmas 1 and 2, that for some polynomial $f(z)$ of degree n with roots r_j on both sides of the real axis and derivative roots also on both sides

$$F = \frac{1}{n}\sum{}' \left| I(r_j) \right| - \frac{1}{n-1}\sum_{j=1}^{q} \left| I(R_j) \right| \leqq 0, \tag{6}$$

where $R_1, R_2, \cdots, R_q$ are the roots of $f'(z)$ which lie below the real axis. We show that these assumptions lead to a contradiction.

In addition to the hypotheses just stated, we shall use induction on the number of roots of $f(z)$ lying above the real axis, and assume first that $f(z)$ has exactly one such root, say r_1: thus the first sum in F ranges over $j=2, 3, \cdots, n$. We consider what happens to F in (6) when r_1 is moved slightly, the other roots of $f(z)$ remaining fixed.

The first possibility is that $\sum_{j=1}^{q} \left| I(R_j) \right|$ for exactly these R_j does not change no matter in what direction the root r_1 is given a slight motion from its original position. Consider the roots R_j of $f'(z)$, each R_j being an analytic function[3] of r_1. More precisely, if r_1 moves to a value r in the neighborhood, then the R_j move to positions S_j given by

$$r = r_1 + t^{\alpha}, \qquad S_j = R_j + b_{1j}t + b_{2j}t^2 + \cdots, \tag{7}$$

α being a positive integer. For sufficiently small values of t, our assumption that

$$\sum_{j=1}^{q} \left| I(R_j) \right| = \sum_{j=1}^{q} \left| I(S_j) \right| \tag{8}$$

is the same as $\sum_{j=1}^{q} I(R_j) = \sum_{j=1}^{q} I(S_j)$, because the R_j and S_j are below the real axis. Thus for small values of the complex number t,

$$I\left(\sum_{j=1}^{q} b_{1j}t + \sum_{j=1}^{q} b_{2j}t^2 + \cdots \right) = 0$$

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[3] Cf. G. A. Bliss, *Algebraic functions*, Amer. Math. Soc. Colloquium Publications, vol. 16, Theorem 13.1, p. 32.

and consequently $\sum b_{1j}=0$, $\sum b_{2j}=0$, and so on. By analytic continuation (8) holds when r_1 is moved to a position r on the real axis. But in this case those roots of $f'(z)$ which were on or above the real axis, namely $R_{q+1}, \cdots, R_{n-1}$, have moved to positions $S_{q+1}, \cdots, S_{n-1}$ below the real axis, so that we have

$$\frac{1}{n}\sum_{j=1}^{n}|I(r_j)| - \frac{1}{n-1}\sum_{j=1}^{n-1}|I(S_j)| < 0,$$

which contradicts Lemma 1.

Next suppose that $\sum_{j=1}^{q}|I(R_j)|$ changes when r_1 is moved to a neighboring position. Then (7) implies that r_1 can be moved in such a direction as to decrease $\sum_{j=1}^{q} I(R_j)$ and thus decrease F in (6), because

$$I\left(\sum_{j=1}^{q} S_j\right) = I\left(\sum_{j=1}^{q} R_j\right) + I\left(\sum_{j=1}^{q} b_{1j}t\right) + I\left(\sum_{j=1}^{q} b_{2j}t^2\right) + \cdots.$$

Let us move r_1 along a path so as to decrease F. It is clear that such a path will not lead to the real axis, because of Lemma 1, so we need treat only the possibilities of r_1 moving to infinity along a path in the upper half plane.

Consider, then, the polynomial $f(z)$ with roots $r, r_2, r_3, \cdots, r_n$, where $|r|$ is very large relative to $|r_j|$, $j=2, 3, \cdots, n$. If $g(z)$ is the polynomial with roots $r_2, r_3, \cdots, r_n$ then

$$f'(z) = g(z) + (z-r)g'(z). \tag{9}$$

Consider any fixed circle which has center at a root of $g'(z)$, but which does not pass through any root of $g'(z)$. On the circumference of such a circle, $|(z-r)g'(z)| > |g(z)|$ for sufficiently large r. By Rouché's theorem each such circle contains as many roots of $f'(z)$ as of $g'(z)$. Thus if $t_1, t_2, \cdots, t_{n-2}$ are the roots of $g'(z)$, all roots except one of $f'(z)$ can be written as

$$t_1 + \epsilon_{1r},\ t_2 + \epsilon_{2r}, \cdots, t_{n-2} + \epsilon_{n-2,r}$$

where every ϵ_{jr} tends to 0 as r tends to infinity. If the other root of $f'(z)$ is denoted by R, then since

$$\frac{1}{n}(r + r_2 + r_3 + \cdots + r_n) = \frac{1}{n-1}\left\{R + \sum_{j=1}^{n-2}(t_j + \epsilon_{jr})\right\},$$

we see that R is approximately $(n-1)r/n$, with a bounded error as r tends to infinity.

We first discuss the case where R is above the real axis. By our induction hypothesis applied to the polynomial $g(z)$,

$$\frac{1}{n-1}\sum_{j=2}^{n}|I(r_j)| \geqq \frac{1}{n-2}\sum_{j=1}^{n-2}|I(t_j)|. \tag{10}$$

Turning to $f(z)$, we see that the function F in (6) has the value

$$\frac{1}{n}\sum_{j=2}^{n}|I(r_j)| - \frac{1}{n-1}\sum_{j=1}^{n-2}|I(t_j+\epsilon_{jr})|, \tag{11}$$

since we are assuming that R has positive imaginary part. But the ϵ_{jr} are arbitrarily small, so (10) implies that (11) is positive, and this contradicts our assumption that F is negative.

If, on the other hand, R is below the real axis, we argue as follows. Since the roots of $g(z)$ are on or below (with at least one below) the real axis, by Gauss' theorem a particular root t_j of $g'(z)$ is on the real axis only if it is a multiple root of $g(z)$, in which case it is a multiple root of $f(z)$ and the corresponding $\epsilon_{jr}=0$. Thus we can take r sufficiently large so that the roots $t_j+\epsilon_{jr}$ of $f'(z)$ are all on or below the real axis. Hence all the roots of $f'(z)$ lie on or below the real axis, and we use the second part of Lemma 1 to complete the proof.

Having completed the proof of (5) in case there is exactly one root of $f(z)$ with positive imaginary part, let us now use induction and assume that (5) holds for all polynomials of degree n with $k-1$ roots above the real axis. Then consider any polynomial of degree n with k roots, $r_1, r_2, \cdots, r_k$, above the real axis. We proceed as in the case where there is exactly one root with positive imaginary part. We move r_1 to the real axis if (a) such motion does not alter $\sum_{j=1}^{q} I(R_j)$; otherwise (b) we move r_1 along a path which decreases this sum. In case (a) the problem is reduced to that of a polynomial with $k-1$ roots having positive imaginary part: likewise in case (b) when r_1 moves along a path crossing or touching the real axis. (Note that if by such motion of r_1, either one or more of the R_j for $j\leqq q$ move to positions above the real axis or one or more of the R_j for $j>q$ move to positions below the real axis, such alterations work in our favor in decreasing F.) Finally, if in case (b) r_1 moves to infinity in the upper half plane, we handle the problem in the following way.

As before we use the notation r for the new location of r_1, and $t_j+\epsilon_{jr}$ $(j=1, 2, \cdots, n-2)$ and R for the roots of $f'(z)$. In case R is above the real axis, we argue as above, with (10) and (11) suitably altered.

In case R is below the real axis, the argument for the case of exactly one root with positive imaginary part is inadequate. If $\sum''$ designates the sum over those of the indicated roots which lie above the real axis, then

$$(12)\qquad 0<\frac{1}{n-1}\sum_{j=2}^{n}{}''\left|I(r_j)\right|-\frac{1}{n-2}\sum_{j=1}^{n-2}{}''\left|I(t_j)\right|,$$

by the induction hypothesis applied to $g(z)$ and the remark following the proof of Lemma 2. Define a by

$$a=\sum_{j=2}^{n}{}''\left|I(r_j)\right|-\sum_{j=1}^{n-2}{}''\left|I(t_j)\right|.$$

Clearly $a>0$, by (12), and a is independent of r, R, and the ϵ_{jr}. Choose r sufficiently large so that $a>n\sum_{j=1}^{n-2}\left|\epsilon_{jr}\right|$. Then (12) implies that

$$\begin{aligned}0&<(n-1)\sum_{j=2}^{n}{}''\left|I(r_j)\right|-n\sum_{j=1}^{n-2}{}''\left|I(t_j)\right|-a\\&<(n-1)\sum_{j=2}^{n}{}''\left|I(r_j)\right|-n\sum_{j=1}^{n-2}{}''\left|I(t_j)\right|-n\sum_{j=1}^{n-2}\left|\epsilon_{jr}\right|\\&<(n-1)\sum_{j=2}^{n}{}''\left|I(r_j)\right|-n\sum_{j=1}^{n-2}{}''\left|I(t_j+\epsilon_{jr})\right|.\end{aligned}$$

This is the inequality we want for the roots of $f(z)$ and $f'(z)$ provided we add $\left|I(r)\right|$ to the first sum, which works in our favor. This completes the proof of (5) and (1).

To prove (2), we observe that if the roots of $f(z)$ are translated or rotated about the origin, the roots of $f'(z)$ undergo an identical translation or rotation. This remark not only proves (2) but also shows that we need prove (3) only when A is the origin. Next we note that if $D(\theta, r)$ denotes the distance from r to a line through the origin with direction angle θ, then

$$\int_0^{2\pi}D(\theta,r)d\theta=4\left|r\right|.$$

Thus (3) is obtained from (2) by integration over the direction angle. Since there is equality in (2) only if all the r_j are on the same side of l, we see that there is equality in (3) only if the r_j lie on a half line out of A.

REMARK. Let $f(z)$ have exactly one root, say r_1, below the real axis; let r_2 be the root, apart from r_1, with least imaginary part. Then there is no root of $f'(z)$ below the line

$$y = I(r_1) + \frac{1}{n} I(r_2 - r_1).$$

This can be seen by translating all roots a distance $I(r_1)$ in the direction of the negative imaginary axis, and by application of (5). It can also be obtained directly from the well known relation $\sum_j (R - r_j)^{-1} = 0$.

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NOTE ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC FUNCTIONS

J. L. WALSH

By a limiting process, a theorem recently proved by the writer can be generalized, and yields a new result with interesting applications which we wish to record here. We take as point of departure[1] the following theorem.

THEOREM 1. *Let the region R of the extended (x, y)-plane be bounded by a finite number of mutually disjoint Jordan curves $C_0, C_1, C_2, \cdots, C_n$. Let the function $u(x, y)$ be harmonic in R, continuous in the corresponding closed region, equal to zero on C_0 and to unity on $C_1, C_2, \cdots, C_n$. Denote by R_0 the region bounded by C_0 containing the curves $C_1, C_2, \cdots, C_n$ in its interior; define noneuclidean straight lines in R_0 as the images of arcs of circles orthogonal to the unit circle, when R_0 is mapped conformally onto the interior of the unit circle.*

If Π is any non-euclidean convex region in R_0 which contains all the curves $C_1, C_2, \cdots, C_n$, then Π also contains all critical points of $u(x, y)$ in R.

We extend Theorem 1 by admitting arcs of C_0 on which $u(x, y)$ is prescribed to take the value unity, and also by admitting the intersection of curves $C_1, C_2, \cdots, C_n$ with C_0:

THEOREM 2. *Let the region R be bounded by the whole or part of the Jordan curve C_0, and by mutually disjoint Jordan arcs or curves $C_1, C_2, \cdots, C_n$ in the closed interior of C_0; some or all of the latter arcs or curves may have points in common with C_0. Let a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$ of C_0 belong to the boundary of R and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in R, and take continuously the boundary values unity on $C_1, C_2, \cdots, C_n, \alpha_1, \alpha_2, \cdots, \alpha_m$ and zero in the remaining boundary points of R, except that in points common to C_0 and $C_1 + C_2 + \cdots + C_n$ and in end points of the α_j, no continuous boundary value is required. Denote by R_0 the region bounded by C_0 containing R, and define non-euclidean straight lines in R_0 by mapping R_0 onto the interior of a circle. If Π is any closed region in the closure of R_0 which is non-euclidean convex and which contains $C_1 + C_2 + \cdots + C_n + \alpha_1 + \alpha_2 + \cdots + \alpha_m$, then Π contains all critical point of $u(x, y)$ in R.*

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[1] Bull. Amer. Math. Soc. vol. 52 (1946) pp. 346–347.

Theorem 2 may be proved by mapping R_0 onto the interior of the unit circle; we retain the original notation. The region R can be approximated by a region R' bounded by C_0 and by Jordan curves $C_1', C_2', \cdots, C_n', \alpha_1', \alpha_2', \cdots, \alpha_m'$ in R_0 which are mutually disjoint and disjoint with C_0 and which respectively approximate $C_1, C_2, \cdots, C_n, \alpha_1, \alpha_2, \cdots, \alpha_m$. Let the function $u'(x, y)$ be harmonic in R', continuous in the corresponding closed region, zero on C_0 and unity elsewhere on the boundary of R'. Then as R' suitably approaches R, the variable function $u'(x, y)$ approaches $u(x, y)$ throughout R, uniformly on any closed set interior to R; we omit the proof. Any critical point of $u(x, y)$ interior to R is a limit point of critical points of the variable function $u'(x, y)$, so Theorem 2 follows from Theorem 1.

A further general result has recently been established[3] for the case $n=0$, which constructs Π in R_0 not by joining the ends of each arc of C_0 in the complement of the set α_j by a non-euclidean line but by similarly joining the ends of each double arc composed of an α_j plus one of the adjoining arcs of C_0 complementary to the set $\alpha_1+\alpha_2+\cdots+\alpha_m$. It is still true (we shall refer to this result as Theorem 3) that Π contains all critical points in R of the corresponding harmonic function $u(x, y)$.

Theorem 3 is more powerful than Theorem 2 for the case $n=0$, but requires for its application essentially the use of a specific conformal map, and the latter quality may be an advantage or a disadvantage. It is an indication of the power of Theorem 2 that in the application of it to a given configuration, with or without the auxiliary use of conformal mapping, there may obviously be some arbitrariness in the notation, especially as to what shall be chosen as the region R_0. So far as convenience is concerned, it is desirable to choose simple configurations, where the totality or useful subset of non-euclidean lines is easily determined. It is also well to choose R_0 in such a way that the point set $C_1+\cdots+C_n+\alpha_1+\cdots+\alpha_m$ is as nearly non-euclidean convex as possible. But *if the aim is precision, the larger R_0 the better*, as we proceed to indicate in a special but typical case.

In Theorem 2, let C_0 be the unit circle in the z-plane, $n=2$, $m=0$, with C_1 and C_2 mutually disjoint Jordan arcs whose end points lie on C_0 and whose interior points lie interior to C_0. Let the subregion R of the interior of C_0 be bounded by C_1, C_2, and two appropriate arcs of C_0. In the actual application of Theorem 2, we can choose R_0 as the interior of C_0, or as the region R_1 containing R bounded by C_1

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[3] J. L. Walsh, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 18–20.

and a suitable arc of C_0, or as the region R_2 containing R bounded by C_2 and a suitable arc of C_0, or as R. We now show as a general indication but without a complete rigorous proof that among these choices *the most precise results are obtained by choosing R_0 as the interior of C_0.*

Map (for instance) the region R onto the interior of the unit circle in the w-plane. Let α_z be an arbitrary arc of C_0 belonging to the boundary of R, which corresponds to the arc α_w in the w-plane. Let α_z' be the circular arc having the same end points as α_z, orthogonal to α_z, and whose interior points lie interior to C_0; we assume α_z' to lie in the closure of R. Let α_w' be the circular arc having the same end points as α_w, orthogonal to α_w, and whose interior points lie in $|w| < 1$. The arcs α_z' and α_w' determine the respective non-euclidean geometries in the z-plane and w-plane, and it follows from a general theorem due to R. Nevanlinna[3] that *the region bounded by α_z and α_z' contains every point of the image of the region bounded by α_w and α_w'*. Corresponding to every arc α_z belonging to the boundary of R and on which the prescribed boundary value of $u(x, y)$ is zero, and to the adjacent arc α_z', with no point of C_1 or C_2 in the lens-shaped region between α_z and α_z' there exists in the w-plane an arc α_w whose end points correspond to those of α_z under the conformal map, such that the interior points of the arc α_w' lie interior to the lens-shaped region bounded by α_w and the image of α_z'. It follows that if we neglect arcs α_z' that cut C_1 or C_2 in R, then in this particular case Theorem 2 can be more favorably applied by choosing R_0 as the interior of C_0, that is to say, as large as possible.

The remark just made is of fairly general application. Moreover, in the specific case used, the interior of the given C_0 may be enlarged, without altering R or $u(x, y)$, by adding to R_0 regions adjacent to the arcs A of the given C_0 bounded by the end points of C_1 and C_2, the arcs A not being part of the boundary of R. Indeed, we may even adjoin an infinitely many sheeted logarithmic Riemann surface along each arc A; this is equivalent to mapping onto the interior of the unit circle the region R plus auxiliary regions, so that with the omission of two exceptional points the circumference of the unit circle corresponds only to that part of the boundary of R on which the prescribed boundary value of $u(x, y)$ is zero. The image of C_1 (and likewise of C_2) under this map is a Jordan curve which except for a single point lies interior to the unit circle.

Still another instructive kind of conformal map can be used, namely to map $R+C_1+C_2$ onto the interior of the unit circle in such a way

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[3] Eindeutige analytische Funktionen, Berlin, 1936, p. 51.

that C_1 and C_2 correspond to radial slits, while the part of the boundary of R on which $u(x, y)$ has the prescribed boundary value zero corresponds to the whole circumference less two points. Here the region II of Theorem 2 may degenerate to a line segment.

Another indication of the power of Theorem 2 is the following. Let C_0 be the unit circle, $r_0=1$ with C_1 a concentric circle of radius $r_1<1$; let an arc α (not the whole circle) of C_0 contain all the arcs α_j. By a conformal map of the universal covering surface of R onto the unit circle and application of Theorem 3 extended to the case of an infinite number of arcs, α_j, it follows (loc. cit. footnote 2) that in the original plane no critical points of $u(x, y)$ lie in the annulus $r_1<r<r_1^{1/2}$; a second annulus $r_2<r<1$ free from critical points can also be determined by this method. By Theorem 2, any circle cutting C_0 orthogonally in two points of the complement of α and containing in its interior no point of α or of C_1 contains in its interior no critical point of $u(x, y)$. In all, these conclusions may leave only a very small subregion of R as the portion in which the critical points of $u(x, y)$ lie.

We continue with a generalization of this result, a further application of Theorem 2:

THEOREM 4. *Let R be a region bounded by the whole of the Jordan curve C_0, by the whole or part of the Jordan curve C_1 disjoint from C_0, and by mutually disjoint Jordan arcs or curves $C_2, C_3, \cdots, C_n$ in the closed interior of the annulus R_0 bounded by C_0 and C_1; some or all of the latter arcs and curves are permitted to have points in common with C_1, but none with C_0. Let a finite number of arcs $\beta_1, \beta_2, \cdots, \beta_m$ of C_1 be part of the boundary of R and mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in R, and take continuously the boundary value zero on $C_0+\beta_1+\beta_2+\cdots+\beta_m$ and unity in the remaining boundary points of R, except that in points common to the β_j and $C_2+\cdots+C_n$ and in end points of the β_j, no continuous boundary value is required.*

If $\omega(z, C_0, R_0)$ denotes the harmonic measure of C_0 in the point z with respect to the annular region R_0, then for constant μ the largest region $\omega(z, C_0, R_0)>\mu\geqq 1/2$ which contains no points of $C_2+\cdots+C_n$ contains no critical points of $u(x, y)$.

Theorem 4 is proved by mapping onto the unit circle the universal covering surface of R_0, and by applying a slight generalization of Theorem 2. We omit the proof.

We turn now to a generalization of Theorems 2 and 4, in a more general situation. Let R be a region bounded by the whole or part of the mutually disjoint Jordan curves, $C_1, C_2, \cdots, C_k$ (which together bound a region R_0) and by mutually disjoint Jordan arcs or curves $C_{k+1}, \cdots, C_n$ in the closure of R_0; some or all of the latter arcs or curves may have points in common with $C_1+C_2+\cdots+C_k$. Let a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$ of $C_1+C_2+\cdots+C_k$ belong to the boundary of R and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in R, and take continuously the boundary values unity on $C_{k+1}+\cdots+C_n+\alpha_1+\cdots+\alpha_m$ and zero in the remaining boundary points of R, except that in points common to $C_1+\cdots+C_k$ and $C_{k+1}+\cdots+C_n$ and in end points of the α_j, no continuous boundary value is required. In studying the location of the critical points of $u(x, y)$, in order to apply Theorem 2 (in generalized form), it is natural to map onto the interior of the unit circle the universal covering surface of R_0. Any non-euclidean convex region in the unit circle containing all image points of the set $C_{k+1}+\cdots+C_n+\alpha_1+\cdots+\alpha_m$ contains all critical points of the transform of $u(x, y)$. But here we have a large choice; we may change the notation so that any subset of the arcs or curves $C_{k+1}, \cdots, C_n$ belongs to the set $C_1, C_2, \cdots, C_k$; each choice of the subset yields a new region R_0, a new conformal map, a new noneuclidean geometry, a new application of Theorem 2 (generalized), and a new conclusion.

Throughout the present note we have studied in detail harmonic functions which for a simply connected region R_0 take on the values zero (on arcs of the boundary of R_0) and unity (on arcs of the boundary or curves in R_0). By the same methods one can also study harmonic functions which take on the values zero (on arcs of the boundary of R_0), unity (on arcs of the boundary or curves in R_0), and minus unity (on arcs of the boundary or curves in R_0); the results generalize those previously obtained by the writer (loc. cit.) and can be still further generalized to regions of higher connectivity by a conformal map of the universal covering surfaces of such regions.

HARVARD UNIVERSITY

THE CRITICAL POINTS OF LINEAR COMBINATIONS OF HARMONIC FUNCTIONS

J. L. WALSH

In various extremal problems of function theory the critical points of linear combinations of Green's functions and harmonic measures are of significance.[1] The object of the present note is to indicate some of the more immediate results concerning the location of such critical points, for both simply and multiply connected regions.

The first configuration to be studied is a simple one. In the z-plane let C be the unit circle and let α_1 be an arc of C, whose initial and terminal points are a_1 and b_1 respectively, the positive direction chosen as counterclockwise. Let $\omega(z, \alpha_1, R)$ denote generically the harmonic measure of α_1 in the point z of R with respect to the region R; that is to say, ω is the function which is harmonic and bounded in R, continuous in the corresponding closed region except in the end points of α_1, equal to unity in the interior points of α_1 and to zero in the interior points of the boundary arcs complementary to α_1. The reader will verify the equation

$$\omega(z, \alpha_1, |z| < 1) = \frac{1}{\pi}\left[\arg (z - b_1) - \arg (z - a_1) - \frac{1}{2}\alpha_1\right]. \tag{1}$$

If ζ is a point interior to C: $|\zeta| < 1$, Green's function for R: $|z| < 1$ with pole in the point ζ can be written

$$g(z, \zeta, R) = \log |1 - \bar{\zeta} z| - \log |z - \zeta|. \tag{2}$$

An arbitrary linear combination of g and ω with real constant coefficients is $U(z) = \lambda g + \mu\omega$, which is the real part of the analytic function

$$f(z) = \lambda \log \frac{1 - \bar{\zeta} z}{z - \zeta} + \frac{\mu}{i\pi}\left[\log (z - b_1) - \log (z - a_1) - \frac{i\alpha_1}{2}\right].$$

The critical points of $U(z)$ are precisely the critical points of $f(z)$, namely the zeros of

$$if'(z) = \frac{i\lambda}{z - 1/\bar{\zeta}} - \frac{i\lambda}{z - \zeta} + \frac{\mu}{\pi}\left[\frac{1}{z - b_1} - \frac{1}{z - a_1}\right]. \tag{3}$$

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[1] See, for instance, M. Schiffer, Amer. J Math. vol. 68 (1946) pp. 417–448; L. V. Ahlfors, Duke Math. J. vol. 14 (1947) pp. 1–11.

We study the zeros of $f'(z)$ by setting up a field of force suggested by the individual terms in the second member of (3); it is more convenient to replace each term by its conjugate. If particles are chosen to repel with a force equal to the mass divided by the distance, the force at z due to a positive unit particle at b_1 is $1/(\bar{z}-\bar{b}_1)$; the force at z due to particles at a_1 and b_1 of respective masses $-\mu/\pi$ and $+\mu/\pi$ is the conjugate of the last term of (3). We introduce also *skew particles*, of such a nature that a skew particle at z_0 exerts a force at z equal to the mass divided by the distance, in the direction $\arg (z-z_0) +\pi/2$ for positive mass and in the reverse direction for negative mass.[3] The force at z due to skew particles of respective masses λ and $-\lambda$ at the points ζ and $1/\bar{\zeta}$ is the conjugate of the sum of the first two terms of (3); the critical points of $U(z)$ are then precisely the positions of equilibrium in the field of force due to the set of four particles.

The function $U(z)$ is invariant under conformal transformation of the interior of C into itself; we study for convenience the origin O as a possible critical point. The force at O due to the particles at b_1 and a_1 is μ/π times the sum of the vectors b_1O and Oa_1, which equals μ/π times the vector b_1a_1. The force at O due to the skew particles at ζ and $1/\bar{\zeta}$ is in magnitude λ times the difference $(1/\bar{\zeta})-\zeta$, and if $\lambda>0$ is exerted in the direction $-\arg \zeta+\pi/2$. The point O can be a position of equilibrium only if these two forces act in opposite directions, so if $\lambda\mu<0$, the point ζ lies between O and α_1 on the radius bisecting the arc α_1; if $\lambda\mu>0$, the point O can be a position of equilibrium only if ζ lies on this same diameter but is separated by O from α_1. An easy topological discussion, or a further study of the field of force, shows that $U(z)$ has at most a unique critical point interior to C. We express this entire result no longer requiring the position of equilibrium to lie at the center of C; moreover the result applies to an arbitrary Jordan region R, with non-euclidean geometry defined in R by mapping R onto the interior of the unit circle:

THEOREM 1. *If α_1 is an arc of the Jordan curve C whose interior is R, and if ζ is an arbitrary point of R, then for $\lambda\mu<0$ the unique critical point if any of $U(z)=\lambda g(z, \zeta, R)+\mu\omega(z, \alpha_1, R)$ in R lies on the non-euclidean line through ζ bisecting α_1 (in the sense that the angles subtended at ζ by the two parts of α_1 are equal), but does not lie between ζ and α_1; for $\lambda\mu>0$, the unique critical point of $U(z)$ if any in R lies on this same non-euclidean line between ζ and α_1.*

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[3] If a right forearm, thumb away from the plane, points from z_0 to z, then the sense of the force exerted at z by a positive skew particle at z_0 is indicated by the fingers at right angles to the forearm. Thus a positive skew particle can be considered as a *right* skew particle, and similarly a negative skew particle as a *left* skew particle.

We now generalize Theorem 1 so as to permit any finite number of Green's functions and harmonic measures:

THEOREM 2. *Let R be a region bounded by a Jordan curve C, let $\zeta_1, \zeta_2, \cdots, \zeta_m$ be points of R and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be arcs of C. We form the linear combination with real constant coefficients*

$$U(z) = \sum_{k=1}^{m} \lambda_k g(z, \zeta_k, R) + \sum_{k=1}^{n} \mu_k \omega(z, \alpha_k, R). \tag{4}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_{m'}$ and $\mu_1, \mu_2, \cdots, \mu_{n'}$ be positive, and the remaining λ_k and μ_k be negative.

All non-euclidean lines L (if any) for R each of which separates the points $\zeta_1, \zeta_2, \cdots, \zeta_{m'}$ and the arcs $\alpha_1, \alpha_2, \cdots, \alpha_{n'}$ from the points $\zeta_{m'+1}, \cdots, \zeta_m$ and the arcs $\alpha_{n'+1}, \cdots, \alpha_n$ lie in a locus Π which is a subregion of R in which no critical points of $U(z)$ lie. Each such line L separates a subset R_1 of $R+C$ from a subset R_2 of $R+C$, where R_1 is non-euclidean convex and contains the points ζ_k $(k=1, 2, \cdots, m')$ and the arcs α_k $(k=1, 2, \cdots, n')$, and R_2 is non-euclidean convex, disjoint from R_1, and contains all the remaining points ζ_k and arcs α_k. If the non-euclidean line L separates all the points ζ_k $(k=1, 2, \cdots, m')$ not on L and all the interior points of the arcs α_k $(k=1, 2, \cdots, n')$ from all the remaining points ζ_k not on L and all the interior points of the remaining arcs α_k, then no point of L in R is a critical point of $U(z)$ unless all ζ_k lie on L and n is zero.

In Theorem 2 the arcs α_k may overlap and may have end points in common.

In the proof of Theorem 2, let R be the interior of the unit circle, let L be the vertical diameter, let the points ζ_k $(k=1, 2, \cdots, m')$ not on L and the arcs α_k $(k=1, 2, \cdots, n')$ except perhaps end points lie to the right of L; the remaining ζ_k not on L and the remaining arcs α_k except perhaps end points lie to the left of L. We consider the point $z=0$ as a possible position of equilibrium in the field of force corresponding to that used in proving Theorem 1, where we now combine the fields for all m Green's functions and for all n harmonic measures involved; the positions of equilibrium in R are precisely the critical points of $U(z)$ in R. The force at O due to each pair of particles at end points of an arc α_k $(k=1, 2, \cdots, m')$ has a nonvanishing component directed vertically downward, as has the force at O due to each pair of particles at end points of an arc α_k $(k=n'+1, n'+2, \cdots, n)$. Likewise the force at O due to each pair of skew particles at points ζ_k and $1/\bar{\zeta}_k$ not on L has a nonvanishing component directed vertically downward, whether $k \leq m'$ or $k > m'$; for pairs of skew particles on L, the force exerted at O is horizontal. Thus the total force at O cannot be zero, and O cannot be a position of equilibrium nor a critical point of $U(z)$. This completes the proof of Theorem 2 except for fairly obvious geometric relationships which are left to the reader.

In Theorem 2 the case $m'=m$, $n'=n$ is not excluded:

COROLLARY 1. *Under the conditions of Theorem 2, suppose* $m'=m$, $n'=n$. *If* Π *is a non-euclidean convex closed region in the closure of* R *which contains all the* ζ_k *and* α_k, *then* Π *contains also all critical points of* $U(z)$ *in* R. *More explicitly, if a non-euclidean line* L *passes through a point* z *of* R, *if all the points* ζ_k *not on* L *lie on one side of* L, *and if all the arcs* α_k *except perhaps end points lie on this same side of* L, *where if all* ζ_k *lie on* L *we assume* $n>0$, *then* z *is not a critical point of* $U(z)$.

In the case under Corollary 1 with $\mu_1=\mu_2=\cdots=\mu_n=1$ and the α_k disjoint, a non-euclidean convex region Π_1 can be defined in $R+C$ containing the arcs α_k (loci $U(z)=1$) and also the loci $U(z)=1$ interior to R; one of the latter loci must clearly separate each ζ_k from C. It is then known[3] that Π_1 contains all critical points of $U(z)$ in R; but the minimal Π_1 always contains the minimal region Π, and may be larger than Π, so Corollary 1 is sharper in this particular situation than the previous result.

Likewise in the more general situation of Theorem 2 itself, with $|\mu_1|=|\mu_2|=\cdots=|\mu_n|=1$ and the α_k disjoint, a result similar to Theorem 2 is known (loc. cit.) involving the arcs α_k (loci $U(z)=\pm 1$) and the loci $U(z)=\pm 1$ interior to R; one of the latter separates each ζ_k from C. Theorem 2 is again sharper in this particular situation than the previous result.

In certain cases, Theorem 2 itself can be somewhat improved:

COROLLARY 2. *In Theorem* 2 *with* $m'=m$, $n'=n=1$, *no critical point of* $U(z)$ *lies on a non-euclidean line* L *which cuts* α_1 *but separates all the* ζ_k *from the end points of* α_1.

In Theorem 2 *with* $m'=0$, $n'=n=1$, *any subregion of* R *bounded by a subarc* α *of* α_1 *and by a non-euclidean line joining the end points of* α, *and which contains no point* ζ_k, *is free from critical points of* $U(z)$.

Corollary 2 is intended to be suggestive rather than exhaustive of the method. The proof, which is left to the reader, follows closely the proof of Theorem 2.

A new interpretation of Corollary 2 is found by denoting by β_1 the arc of C complementary to α_1, and by setting $\omega(z,\alpha_1,R)=1-\omega(z,\beta_1,R)$. The critical points of $U(z)$ are the same as those of the function

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[3] Walsh, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 191–195.

$$U(z) - \mu_1 = \sum_{k=1}^{m} \lambda_k g(z, \zeta_k, R) - \mu_1 \omega(z, \beta_1, R), \qquad \mu_1 > 0,$$

to which the application of Theorem 2 yields Corollary 2.

In comparison with all previous existing theorems, Theorem 2 is especially novel in that it applies to harmonic functions which assume in addition to zero *more than two* distinct values on prescribed point sets.

If a nonconstant harmonic function $U(z)$ is the uniform limit of a sequence of harmonic functions $U_p(z)$, the critical points of $U(z)$ are precisely the limits of the critical points of the $U_p(z)$. It is thus clear that Theorem 2 applies under suitable conditions when the two sums in (4) are replaced by the limits of similar sums. *If for each function of a uniformly convergent sequence* $U_p(z)$ *the set* R_1 *of Theorem* 2 *contains all the points* ζ_k *and all the arcs* α_k *corresponding to positive coefficients* λ_k *and* μ_k, *and if the set* R_2 *contains all the points* ζ_k *and all the arcs corresponding to negative coefficients* λ_k *and* μ_k, *then the region* II *if independent of* p *contains no critical point of the limit* (*assumed nonconstant*) *of the sequence*. We proceed to apply this remark, by proving the following corollary.

COROLLARY 3. *Let* R *be a Jordan region bounded by a curve* C, *let the function* $U_1(z)$ *be bounded and continuous on* C *except perhaps for a finite number of discontinuities, and let* $U(z)$ *be harmonic and bounded in* R, *and continuous and equal to* $U_1(z)$ *at every point of continuity of* $U_1(z)$ *on* C. *Let all points of* C *at which* $U_1(z)$ *is positive be contained in an arc* α *of* C, *and all points of* C *at which* $U_1(z)$ *is negative be contained in an arc* β *of* C *disjoint from* α. *Then all critical points of* $U(z)$ (*assumed not identically zero*) *in* R *lie in the sets* $\omega(z, \alpha, R) \geqq 1/2$, $\omega(z, \beta, R) \geqq 1/2$.

Corollary 3 is an immediate consequence of the representation

$$U(z) = \int_0^1 U_1(t) d\omega(z, t, R) \tag{5}$$

and of the corresponding expression for $U(z)$ as the limit of a sum. If the arcs α and β are minimal, the locus II is bounded by the arcs $\omega(z, \alpha, R) = 1/2$, $\omega(z, \beta, R) = 1/2$.

An easy immediate proof of Corollary 3 can be given by the usual form of Poisson's integral for the unit circle C; it is again convenient to choose the origin as the point studied, with the axis of imaginaries separating the intervals of C on which $U_1(z)$ is positive from the intervals of C on which $U_1(z)$ is negative; assume the latter intervals to lie to the left of that axis. Direct computation shows the partial derivative of $U(z)$ at the origin in the positive horizontal direction to be positive:

$$U'(x) = \frac{1}{\pi}\int_0^{2\pi} U_1 \cos\theta \, d\theta > 0,$$

so the origin is not a critical point of $U(z)$. This new proof (valid also in three dimensions) has the advantage over the previous one of slightly generalizing Corollary 3 by not requiring $U_1(z)$ to vanish on an entire arc of C. Theorem 2 itself can be proved by a similar procedure. Nevertheless the original proof of Theorem 2 is to be considered as intuitively suggestive and useful, as applying at once in the important case of linear combinations of harmonic measures, and as exhibiting relations with other proofs on the location of critical points.

Essentially a special case of Corollary 3 occurs if $U_1(z)$ is non-negative on C:

COROLLARY 4. *Let R be a Jordan region bounded by a curve C, let the function $U_1(z)$ be bounded, non-negative, and continuous on C except perhaps for a finite number of discontinuities, and let $U(z)$ be harmonic and bounded in R, continuous and equal to $U_1(z)$ at every point of continuity of $U_1(z)$ on C. Then any non-euclidean convex region of $R+C$ which contains all points of C at which $U_1(z)$ is positive contains also all critical points of $U(z)$ in R. Thus if $U_1(z)$ vanishes on an arc β of C, the region $\omega(z, \beta, R) > 1/2$ contains no critical points of $U(z)$.*

Just as the second sum of (4) can be replaced by the integral in (5), namely the limit of such a sum, so the first sum in (4) may be replaced by the sum of any number of functions of a certain class K, namely the class of functions each the uniform limit in a suitable subregion of R of sums of the type $\sum_{k=1}^{m}\lambda_k g(z, \zeta_k, R)$. To the class K belong: (a) Green's functions $g(z, \zeta_k, R)$; (b) any function $U(z)$ harmonic in a closed multiply connected subregion R_1 of R bounded by C and by a set of disjoint analytic Jordan curves Γ interior to R,[4] equal to zero on C and to unity on Γ. Thus if ν denotes the exterior normal for R_1, we have

$$U(z) = \frac{1}{2\pi}\int_\Gamma \left(g\frac{\partial U}{\partial \nu} - U\frac{\partial g}{\partial \nu}\right) ds + \frac{1}{2\pi}\int_C \left(g\frac{\partial U}{\partial \nu} - U\frac{\partial g}{\partial \nu}\right) ds,$$

which in the present case reduces to

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[4] The requirement that Γ be *interior* to R can be weakened (loc. cit.).

$$U(z) = \frac{1}{2\pi}\int_\Gamma g\,\frac{\partial U}{\partial \nu}\,ds, \qquad \frac{\partial U}{\partial \nu} > 0,$$

obviously of class K. More generally, there belongs to class K any function which can be expressed

$$U(z) = \int_\Gamma g(z, t, R)V(t)\,ds_t, \tag{6}$$

where the function $V(t)$ is integrable on Γ. Theorem 2 thus extends to apply to any sum of a function of class K and a function of type (5).

In this extension of Theorem 2, proper attention must of course be given to the signs of the functions $U_1(z)$ in (5) and $V(t)$ in (6), and the question of the latter seems not to be easy. We note explicitly that ***not every function** $U(z)$ **harmonic and positive in** R_1, **assuming continuous values zero on** C **and positive on** Γ, **admits a representation*** (6) ***where** $V(t)$ **is non-negative on*** Γ. We establish this result by display of a counter example. Let R_2 be the annular region bounded by the circles C: $|z| = 1$ and C_2: $|z| = r^2 < 1$, which are mutually inverse in the circle C_1: $|z| = r$. Let Green's function for R_2 be $g(z, \zeta, R_2)$; we consider the function $U_2(z) = g(z, r, R_2) + g(z, -r, R_2)$, which has (as study of the level curves indicates) precisely two critical points z_1 and z_2 interior to R_2; this set of two points is symmetric in C_1 and in the axis of reals, and we set $z_1 = ir$, $z_2 = -ir$. A suitably chosen circle γ orthogonal to C, having its center on the axis of imaginaries, contains in its interior z_1 but no point of C_2. Let Γ denote the sum of C_2 and the two level curves Γ_1: $U_2(z) = N$ (>0), where N is chosen so large that Γ_1 is exterior to γ. The function $V_p(z)$ harmonic and continuous in the region R_1 bounded by C and Γ, equal to zero on C, to N on Γ_1, to $1/p$ on C_2, approaches uniformly in R_1 as p becomes infinite the limit $U_2(z)$. Moreover $V_p(z)$ is positive in R_1 and also on Γ. The critical points of $U_2(z)$ in R_1 are the limits of those of $V_p(z)$ in R_1. If we assume $V_p(z)$ to admit a representation of form

$$V_p(z) = \int_\Gamma g(t, z, R)W_p(t)\,ds,$$

where $W_p(t)$ is non-negative on Γ, it follows that γ, containing no point of Γ, contains no critical point of $V_p(z)$ and contains no critical point of $U_2(z)$, contrary to construction.[5]

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[5] A contradiction can be obtained here also by considering the integral of the normal derivative of $V_p(z)$ over a curve in R_1 for which C_2 is interior but Γ_1 is exterior.

Theorem 2 and the suggested generalizations apply primarily to a simply connected region R, but if we start with a multiply connected region, functions harmonic there may be studied by mapping a covering surface onto a simply connected region. In preparation for this study we prove the following theorem.

THEOREM 3. *Let the function* $V(z)$ *be harmonic in the upper half-plane* R, *except that in the neighborhood of each point* $\rho^n\alpha$ (α *in* R, $\rho>1$, $n=\cdots, -2, -1, 0, 1, 2, \cdots$) *the sum* $V(z)+\log|z-\rho^n\alpha|$ *is harmonic. Let* $V(z)$ *be continuous and equal to zero at each finite point of the axis of reals other than* $z=0$, *and satisfy the functional equation* $V(\rho z)=V(z)$. *Then we have the representation*

$$V(z) = \sum_{-\infty}^{\infty} \log \left| \frac{z-\rho^n\bar{\alpha}}{z-\rho^n\alpha} \right|, \tag{7}$$

a series which converges uniformly in any closed bounded region of the plane not containing a point $\rho^n\alpha$ *or the origin.*

Each term of the series in (7) is Green's function for R with pole in a point $\rho^n\alpha$. The function $V(z)$ is closely related to analytic functions introduced by Pincherle.

As n becomes positively infinite we write

$$\log \left| \frac{z-\rho^n\bar{\alpha}}{z-\rho^n\alpha} \right| = \log \left| \frac{1-z/\rho^n\bar{\alpha}}{1-z/\rho^n\alpha} \right|,$$

which has approximately the value $(\bar{\alpha}-\alpha)z/\rho^n\alpha\bar{\alpha}$, and as n becomes negatively infinite we write

$$\log \left| \frac{z-\rho^n\bar{\alpha}}{z-\rho^n\alpha} \right| = \log \left| \frac{1-\rho^n\bar{\alpha}/z}{1-\rho^n\alpha/z} \right|,$$

which has approximately the value $\rho^n(\alpha-\bar{\alpha})/z$; the convergence of the series in (7) follows. Denote the sum of the series by $V_1(z)$, which then is harmonic at every finite point of the plane other than the origin and the points $\rho^n\alpha$ and $\rho^n\bar{\alpha}$. The function $V_1(z)$ is continuous and vanishes on the axis of reals ($z\neq 0$).

In the formula for $V_1(z)$ we replace z by ρz, and we have

$$\frac{\rho z-\rho^n\bar{\alpha}}{\rho z-\rho^n\alpha} = \frac{z-\rho^{n-1}\bar{\alpha}}{z-\rho^{n-1}\alpha},$$

so the two series for $V_1(z)$ and $V_1(\rho z)$ are identical except for notation: $V_1(\rho z)=V_1(z)$, and this equation is true for all values of z for which $V_1(z)$ is defined.

The function $V(z)-V_1(z)$ is harmonic, when suitably defined in the points $\rho^n\alpha$, at every point of R, and is continuous and equal to zero on the axis of reals $(z\neq 0)$. From the equation $V(\rho z)-V_1(\rho z)=V(z)-V_1(z)$ it follows that this difference is bounded in the neighborhoods of the origin and of the point at infinity in R, hence vanishes identically in R, and equation (7) is established. The essence of Theorem 3 can be summarized in the statement that $V(z)$ *is of class* K, in the region R_1 consisting of R with the points $\rho^n\alpha$ deleted.

Theorem 3 is to be used in proving the following theorem.

THEOREM 4. *Let R be the annular region bounded by disjoint Jordan curves C_1 and C_2, and let $\zeta_1, \zeta_2, \cdots, \zeta_m$ be points of R. Let the function $U(z)$ be harmonic in R except in the points ζ_k, let the sum $U(z)+\lambda_k \log|z-\zeta_k|$ with $\lambda_k>0$ be harmonic throughout the neighborhood of ζ_k when suitably defined at ζ_k, and let $U(z)$ be bounded in the neighborhood of C_1 and C_2, continuous and non-negative at all except perhaps a finite number of points of C_1 and C_2.*

If the function $U(z)$ vanishes at all points of C_j, $j=1$ or 2, then any region $\omega(z, C_j, R)>\mu$ $(\geqq 1/2)$ which contains no point ζ_k contains no critical point of $U(z)$.

If the universal covering surface of R is mapped onto the interior of a circle, and if non-euclidean geometry is defined in R by means of the map, then any non-euclidean convex set of $R+C_1+C_2$ which contains all the ζ_k and contains all points of C_1 and C_2 at which $U(z)$ is positive also contains all critical points of $U(z)$.

Theorem 4 is proved by mapping the universal covering surface of R onto the interior R' of the unit circle Γ in the w-plane. The curves C_1 and C_2 each counted infinitely often map into complementary arcs Γ_1 and Γ_2 of Γ. Each function $\lambda_k g(z, \zeta_k, R)$ vanishes on C_1 and C_2, and is transformed into λ_k times a function of precisely the type considered in Theorem 3; each function of this type is in a suitable subregion of R' the limit of a sum of functions of the type which appears as the first term of the second member of (4), with positive coefficients. The function $U(z)-\sum_{k=1}^{m}\lambda_k g(z, \zeta_k, R)$ is transformed into a function bounded and harmonic except for removable singularities in R', which may have infinitely many discontinuities on Γ, but which can be represented as the limit of a sum of the type that appears as the last term of (4), with positive coefficients. Each locus $\omega(w, \Gamma_k, R')=\mu$ is the image of the locus $\omega(z, C_k, R)=\mu$ under the conformal map. Theorem 4 is a consequence of our generalization of Theorem 2.

Theorem 4 is the analog of Corollary 1 rather than of Theorem 2.

If in Theorem 4 we omit the requirements $\lambda_k > 0$ and $U(z) \geqq 0$ on C_1 and C_2, it is true that if Δ denotes the locus $\omega(z, C_1, R) = 1/2$, if all points ζ_k for which $\lambda_k > 0$ which do not lie on Δ lie between Δ and C_1, if $U(z)$ is non-negative on C_1, if all points ζ_k for which $\lambda_k < 0$ which do not lie on Δ lie between Δ and C_2, if $U(z)$ is nonpositive on C_2, and if either $U(z)$ is different from zero on an arc of C_1 or C_2 or there exist points ζ_k not on Δ, *then no critical points* of $U(z)$ *lie on* Δ *itself*. The proof here may be given, after conformal map of the universal covering surface of R onto the interior of the unit circle, following the second proof of Corollary 3.

The last part of Theorem 4 admits of immediate extension to an arbitrary region R bounded by a finite number of mutually disjoint Jordan curves, where non-euclidean geometry is defined in R by mapping the universal covering surface of R onto the interior of a circle. Details are left to the reader.

The location of critical points of a linear combination of Green's functions and harmonic measures with positive coefficients for a multiply connected region (compare equation (6)) can also be studied without the use of conformal mapping and non-euclidean geometry; results are obtained corresponding to and by the use of results on the zeros of the derivative of a rational function due to Bôcher and the present writer.[6]

In connection with the methods we are using, a remark due to Bôcher[7] is appropriate: "The proofs of the theorems which we have here deduced from mechanical intuition can readily be thrown, without essentially modifying their character, into purely algebraic form. The mechanical problem must nevertheless be regarded as valuable, for it suggests not only the theorems but also the method of proof."

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[6] Cf. Bull. Amer. Math. Soc. vol. 52 (1946) pp. 346–347.

[7] Proceedings of the American Academy of Arts and Sciences vol. 40 (1904) p. 479.

SOME PROPERTIES OF CONTINUED FRACTIONS
$1 + d_0 z + K(z/(1 + d_n z))$

W. J. THRON

1. Introduction. A continued fraction

$$b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \cdots, \tag{1.1}$$

with nth approximant $A_n(z)/B_n(z)$, is said to correspond to the power series

$$1 + c_1 z + c_2 z^2 + \cdots \tag{1.2}$$

if the power series expansion of $A_n(z)/B_n(z)$ agrees with (1.2) up to and including the term $c_{k(n)} z^{k(n)}$, where $k(n) \to \infty$ as $n \to \infty$.

Leighton and Scott [1][1] proved that there is one and only one continued fraction of the form

$$1 + \frac{a_1 z^{k_1}}{1} + \frac{a_2 z^{k_2}}{1} + \cdots, \tag{1.3}$$

where all k_n are positive integers, which corresponds to a given power series (1.2).

The class of all continued fractions of the form

$$1 + d_0 z + \frac{z}{1 + d_1 z} + \frac{z}{1 + d_2 z} + \cdots, \tag{1.4}$$

which is studied in this paper, has the same property. This is shown in §2. In §3 convergence, and in particular uniform convergence, of continued fractions (1.4) is investigated. §4, finally, is devoted to a study of necessary conditions for the uniform convergence of (1.4) in a neighborhood of the origin

2. Correspondence. A sufficient condition for the existence of a continued fraction (1.4) which corresponds to a given power series $P(z)$ of the form (1.2) is the existence of a solution for the system of formal identities

$$P_n(z) \equiv 1 + d_n z + \frac{z}{P_{n+1}(z)}, \qquad n = 0, 1, \cdots, \tag{2.1}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

where $P_0(z) = P(z)$ and $P_n(z)$ is a formal power series of the form (1.2). This is seen as follows. If such a solution exists one has for every $n \geq 1$

$$P(z) = 1 + d_0z + \frac{z}{1 + d_1z} + \cdots + \frac{z}{1 + d_{n-1}z} + \frac{z}{P_n(z)}.$$

Hence

$$P(z) = \frac{P_n(z)\cdot A_{n-1}(z) + z\cdot A_{n-2}(z)}{P_n(z)\cdot B_{n-1}(z) + z\cdot B_{n-2}(z)},$$

and therefore

$$P(z) - \frac{A_{n-1}(z)}{B_{n-1}(z)} = \frac{(-1)^{n-1}z^n}{B_{n-1}(z)\left[P_n(z)\cdot B_{n-1}(z) + z\ B_{n-2}(z)\right]}.$$

Since $B_n(0)$ and $P_n(0)$ are both different from zero for all n one now concludes that the power series expansion of the nth approximant agrees with $P(z)$ up to and including the term c_nz^n. The continued fraction therefore corresponds to $P(z)$.

Let $P_n'(z)$ be the formal derivative of $P_n(z)$. It is easily verified that $d_n = P_n'(0) - 1$ and

$$P_{n+1}(z) = \frac{z}{P_n(z) - 1 - d_nz}$$

provides a solution of the system (2.1). The following theorem has now been proved.

THEOREM 2.1. *For every power series of the form*

$$1 + c_1z + c_2z^2 + \cdots$$

there exists a continued fraction of the form

$$(1.4) \qquad 1 + d_0z + \frac{z}{1 + d_1z} + \frac{z}{1 + d_2z} + \cdots$$

which corresponds to the power series. The power series expansion of the nth approximant of the continued fraction agrees with the given power series up to and including the term c_nz^n.

The question whether more than one continued fraction of the type (1.4) can correspond to the same power series is answered in the next theorem.

THEOREM 2.2. *Two continued fractions*

$$1 + d_0 z + \overset{\infty}{\underset{r=1}{K}}\left(\frac{z}{1 + d_r z}\right), \tag{1}$$

$$1 + d_0' z + \overset{\infty}{\underset{r=1}{K}}\left(\frac{z}{1 + d_r' z}\right), \tag{2}$$

correspond to the same power series

$$1 + c_1 z + c_2 z^2 + \cdots$$

if and only if $d_n = d_n'$ *for all* $n \geqq 0$.

Let A_n/B_n and A_n'/B_n' be the nth approximants of the continued fractions (1) and (2) respectively. If both continued fractions correspond to the same power series then their $(n+1)$th approximants have power series expansions which certainly agree with each other up to and including the term involving z^{n+1}. Now

$$\frac{A_n}{B_n} - \frac{A_{n+1}}{B_{n+1}} = (-1)^{n+1} z^{n+1} + \cdots,$$

and similarly

$$\frac{A_n'}{B_n'} - \frac{A_{n+1}'}{B_{n+1}'} = (-1)^{n+1} z^{n+1} + \cdots.$$

It follows that the power series expansions of A_n'/B_n' and A_n/B_n agree up to and including the term involving z^{n+1}. Now $B_0 = B_0' = 1$ and hence $A_0 = A_0'$ and $d_0 = d_0'$.

The proof of the theorem is then complete if it is shown that the assumption

$$A_r = A_r', \quad B_r = B_r' \quad \text{and hence} \quad d_r = d_r',$$

for $r = 0, 1, \cdots, m-1$, leads to the conclusion $d_m = d_m'$ and $A_m = A_m'$, $B_m = B_m'$. Making those assumptions one obtains

$$\begin{aligned} \frac{A_m}{B_m} - \frac{A_m'}{B_m'} &= \frac{A_m B_m' - A_m' B_m}{B_m B_m'} \\ &= \frac{z^2(d_m - d_m')(A_{m-1}B_{m-2} - A_{m-2}B_{m-1})}{B_m B_m'} \\ &= \frac{(-1)^{m-2} z^{m+1}(d_m - d_m')}{B_m B_m'}. \end{aligned}$$

A combination of this result with the one concerning the power series expansion of this difference leads to the conclusion $d_n = d_n'$ and hence $A_n \equiv A_n'$, $B_n \equiv B_n'$.

The proof of the theorem below is omitted since it is analogous to the proof given for the corresponding theorem concerning continued fractions $1+K(a_n z/1)$ by Perron [2, p. 342].

THEOREM 2.3. *If the continued fraction* (1.4) *converges uniformly in a region* D, *containing the origin in its interior, then the corresponding power series converges to the same function in the largest open circular region, with center at the origin, completely contained in* D.

It is of course not implied that the power series converges only in that circle. It is not possible to deduce convergence of the continued fraction from the convergence of the power series. This is true for all types of "corresponding" continued fractions studied so far. (See [2, §65].) For the type studied here this phenomenon occurs for a very simple function. Contrary to the cases previously investigated there are no terminating expansions for this type of continued fraction. Even rational functions have nonterminating expansions. The expansion for $P(z) \equiv 1$ is easily seen to be

$$1 - z + \frac{z}{1 - z} + \frac{z}{1 - z} + \cdots .$$

This continued fraction converges to 1 only for $|z| < 1$, while it diverges for $|z| = 1, z \neq -1$ and converges to $-z$ for all other values of z.

The final two theorems of this section are analogues of two theorems of Leighton and Scott [1, pp. 600–601] for continued fractions (1.3). The proofs in our case are almost identical to the proofs given there and are therefore omitted.

THEOREM 2.4. *A necessary and sufficient condition that the continued fraction* (1.4) *converge uniformly to a holomorphic function in an open region* D, *containing the origin, is that the approximants* $A_n(z)/B_n(z)$ *be uniformly bounded in every closed set contained in* D *for all* $n > n_0$.

THEOREM 2.5. *Two infinite subsequences of approximants of the continued fraction* (1.4) *which converge uniformly in an open region* D, *containing the origin, converge to the same holomorphic function in* D.

3. **Convergence criteria.** It is well known [2, p. 196] that for $z \neq 0$ the continued fractions (1.4) and

$$x + d_0/x + \frac{1}{x + d_1/x} + \frac{1}{x + d_2/x} + \cdots , \tag{3.1}$$

where $x = z^{-1/2}$, $-\pi/2 < \arg x \leqq \pi/2$, converge and diverge together. The expression (3.1) is of the form $b_0 + K(1/b_n)$, where

$$b_n = x + d_n/x \tag{3.2}$$

and hence

$$d_n = x b_n - x^2. \tag{3.3}$$

Convergence criteria developed for continued fractions of the form $b_0 + K(1/b_n)$ (see [3]) are therefore applicable to (3.1) and consequently indirectly to (1.4).

Conditions on the sequence $\{d_n\}$ which insure convergence of (1.4) in a neighborhood of the point $z = 0$ can be obtained as follows. Let $|d_n| \leqq M$ for all n and let $|x| = r$. Then

$$|b_n| \geqq r - M/r.$$

Now for $r \geqq 1 + (M+1)^{1/2}$ one has $r - M/r \geqq 2$ and hence $|b_n| \geqq 2$. The Pringsheim criterion then insures the convergence of (3.1). It follows from the equivalence that the continued fraction (1.4) converges for all $z \neq 0$ for which

$$|z| \leqq 1/(1 + (M+1)^{1/2})^2.$$

The convergence of (1.4) for $z = 0$ is trivial. For all values of z under consideration the approximants of (1.4) are uniformly bounded. This is seen as follows:

$$\left| \mathop{K}_{\nu=1}^{n} \left(\frac{z}{1 + d_\nu z} \right) \right| = \left| \frac{1}{x} \right| \cdot \left| \mathop{K}_{\nu=1}^{n} \left(\frac{1}{x + d_\nu/x} \right) \right| \leqq \frac{1}{1 + (M+1)^{1/2}} \cdot 1,$$

since, for $|x + d_n/x| \geqq 2$, $|K(1/(x + d_\nu/x))| \leqq 1$ [3, Theorem 3.2]. An application of the Vitali theorem then leads to the conclusion that (1.4) converges uniformly in every closed set contained in the circular region

$$z < 1/(1 + (M+1)^{1/2})^2.$$

THEOREM 3.1. *If*

$$|d_n| \leqq M$$

for all n then the continued fraction (1.4) *converges for all z in the circular region*

$$|z| \leqq 1/(1 + (M+1)^{1/2})^2.$$

The convergence is uniform in every closed set contained in the interior of this circular region.

Observing that if $|d_n| \geqq m$ then

$$|b_n| \geqq m/r - r,$$

one easily shows that (1.4) converges if

$$|z| \geqq 1/(1 - (m + 1)^{1/2})^2.$$

To show that the convergence is uniform one makes use of the fact that in this case the approximants of (1.4) are uniformly bounded away from the origin but are finite. One thus proves the following result.

THEOREM 3.2. *If*

$$|d_n| \geqq m$$

for all n then the continued fraction (1.4) *converges for all z in the region*

$$|z| \geqq 1/(1 - (m + 1)^{1/2})^2.$$

The convergence is uniform in every closed bounded set in the interior of this region.

These theorems gain in importance if they are used in conjunction with the following results which give criteria for the convergence of (1.4) in certain angular openings.

THEOREM 3.3. *Let the region $D(\gamma, k, \epsilon)$ be defined by: $r \cdot e^{i\theta} \in D(\gamma, k, \epsilon)$ if*

$$r \geqq \frac{2 + \epsilon}{(c^2 - k^2)^{1/2} - c \cdot \cos(\gamma' - \theta)},$$

where $-\pi/2 < \gamma \leqq \pi/2$, $k < 1/2 \cos \gamma$, $\epsilon > 0$, and

$$\gamma = \arg(k \cdot e^{i\gamma} - e^{2i\gamma}),$$
$$c = |ke^{i\gamma} - e^{2i\gamma}|.$$

If there exist functions $\epsilon(\gamma)$ and $k(\gamma)$ and constants γ_1 and γ_2 such that all elements of the sequence $\{d_n\}$ lie in the intersections of the regions $D(\gamma, k(\gamma), \epsilon(\gamma))$ for $\gamma_1 \leqq \gamma \leqq \gamma_2$ then the continued fraction (1.4) *converges for all z in the angular opening*

$$-2\gamma_2 \leqq \arg z \leqq -2\gamma_1.$$

Moreover the convergence is uniform in every closed bounded set in the interior of this region.

Consider a fixed $x = \rho e^{i\gamma}$ and define as $C(\rho)$ the region formed by all points on or outside the circle with center at

$$\rho^2(ke^{i\gamma} - e^{2i\gamma})$$

and radius

$$\rho(4 + 2\epsilon + k^2\rho^2)^{1/2},$$

where ϵ is positive and k is real.

Assume that all numbers d_n of (3.1) lie in the region $C(\rho)$. Then the elements b_n of (3.1), connected with the numbers d_n by relation (3.2), satisfy the condition

$$|b_n - k\rho| \geq (4 + 2\epsilon + k^2\rho^2)^{1/2}. \tag{3.4}$$

For every fixed ρ therefore

$$|b_n - k\rho| \geq (4 + k^2\rho^2)^{1/2} + \eta(\rho),$$

where $\eta(\rho) > 0$. A known result [3, Corollary 6.2] then insures the convergence of (3.1) for for $x = \rho e^{i\gamma}$.

If all the elements of the sequence $\{d_n\}$ lie in a region $D(\gamma, k, \epsilon)$ which is the intersection of all sets $C(\rho)$, $0 < \rho < \infty$, then it follows from what was said above that the continued fraction (3.1) converges for every $x \neq 0$ on the ray $\arg x = \gamma$. The boundary of $D(\gamma, k, \epsilon)$ is the envelope of the boundaries of the regions $C(\rho)$.

Recalling the definition of γ', noting that

$$c = |ke^{i\gamma} - e^{2i\gamma}| = (k^2 + 1 - 2k\cdot\cos\gamma)^{1/2},$$

and denoting the boundary points of $C(\rho)$ by $r\cdot e^{i\theta}$ one easily arrives at the following equation for the boundary of $C(\rho)$

$$\begin{aligned} r^2 + \rho^4(k^2 + 1 - 2k\cdot\cos\gamma) - 2r\rho^2\cos(\gamma' - \theta)(k^2 + 1 - 2k\cos\gamma)^{1/2} \\ = (4 + 2\epsilon)\rho^2 + k^2\rho^4. \end{aligned}$$

Combination of like powers of ρ and introduction of c leads to

$$r^2 + \rho^4(c^2 - k^2) - 2\rho^2(rc\cdot\cos(\gamma' - \theta) + 2 + \epsilon) = 0. \tag{3.5}$$

Differentiating (3.5) with respect to ρ^2 one obtains

$$\rho^2(c^2 - k^2) = rc\cdot\cos(\gamma' - \theta) + 2 + \epsilon.$$

Elimination of ρ from (3.5) leads to

$$r^2(c^2 - k^2) - (rc\cos(\gamma' - \theta) + 2 + \epsilon)^2 = 0.$$

A part of the curve defined by this equation is the desired envelope. Thus one finds that $r\cdot e^{i\theta} \in D(\gamma, k, \epsilon)$ if and only if

$$r \geq \frac{2+\epsilon}{(c^2-k^2)^{1/2}-c\cdot\cos(\gamma'-\theta)}.$$

The proof of the first part of Theorem 3.3 is now easily completed with the remark that the convergence of (3.1) implies the convergence of (1.4) for the corresponding $z \neq 0$. That (1.4) converges for $z=0$ is obvious.

From the results concerning value regions of [3] it follows that the imaginary numbers on the line segment from $-i$ to i are not taken on by any of the approximants $\phi_n(x)$ of (3.1) for $\gamma_1 \leq \arg x \leq \gamma_2$, $x \neq 0$. Further, the functions $\phi_n(x)$ are rational functions of x which have no poles in the region under consideration [3, Lemma 2.2]. Thus the sequence $\{\phi_n(x)\}$ is a normal family of holomorphic functions in this region and since it converges it converges uniformly (Montel-Vitali theorem) in every closed bounded set in the interior of the region under consideration. Finally, if $\{f_n(z)\}$ is the sequence of approximants of (1.4) it follows from the relation

$$f_n(z) = \frac{1}{x}\phi_n(x)$$

that $\{f_n(z)\}$ converges uniformly wherever $\{\phi_n(x)\}$ does provided $|1/x|$ or $|z|$ is bounded. This completes the proof of the theorem.

For $k=0$ the number ϵ can be omitted in the statement of Theorem 3.3 since (3.1) is known to converge if $|b_n| \geq 2$. Further, if $k=0$ then $\gamma'=\pi-2\gamma$ and $c=1$. These remarks together with the substitution $\beta=-2\gamma$ suffice to establish the following corollary of Theorem 3.3.

COROLLARY 3.1. *Let $P(\beta)$ be the region, not containing the origin, bounded by the parabola*

$$r = \frac{2}{1+\cos(\theta-\beta)}.$$

If there exists an interval $[\beta_1, \beta_2]$ such that all elements of the sequence $\{d_n\}$ lie in the intersection of the regions $P(\beta)$ as β ranges over the interval $[\beta_1, \beta_2]$ then the continued fraction (1.4) *converges for all z in the angular opening*

$$\beta_1 \leq \arg z \leq \beta_2.$$

Moreover the convergence is uniform in every closed bounded set contained in the interior of this angular opening.

The result corresponding to the limiting case $k \to -\infty$ of Theorem 3.3 can be stated as follows.

THEOREM 3.4. *If there exist real numbers γ and δ, with $-\pi<\gamma<0<\delta<\pi$ and $\delta-\gamma<\pi$, such that for suitable choice of the argument*

$$\gamma \leqq \arg d_n \leqq \delta \qquad \textit{for all } n$$

then the continued fraction (1.4) *converges for all z in the angular opening*

$$-\pi - 2\gamma < \arg z < \pi - 2\delta.$$

The convergence is uniform in every closed bounded set in the interior of this angular opening.[1]

Van Vleck's convergence criterion insures the convergence of (3.1) provided

$$(3.6) \qquad -\pi/2 + \epsilon \leqq \arg b_n \leqq \pi/2 - \epsilon, \qquad \epsilon > 0,$$

and provided further $\sum|b_n|$ diverges. Consider a fixed $x\neq 0$, such that

$$-\pi/2 + \delta < \arg x < \pi/2 + \gamma.$$

The corresponding z then satisfies the conditions of the theorem. Now let

$$\epsilon(x) = \min(\arg x + \pi/2 - \delta; \pi/2 + \gamma - \arg x);$$

then $\epsilon(x)>0$ and

$$\begin{aligned}-\pi/2 + \epsilon(x) &\leqq -\pi/2 + \delta + \epsilon(x)\\ &\leqq \arg x \leqq \pi/2 + \gamma - \epsilon(x) \leqq \pi/2 - \epsilon(x).\end{aligned}$$

Further

$$\begin{aligned}-\pi/2 + \epsilon(x) &\leqq \gamma - \pi/2 - \gamma + \epsilon(x) \leqq \arg(d_n/x)\\ &\leqq \delta + \pi/2 - \delta - \epsilon(x) \leqq \pi/2 - \epsilon(x),\end{aligned}$$

provided the numbers d_n satisfy the conditions of the theorem. It is then clear that the elements b_n, defined by relation (3.2), satisfy condition (3.6) with $\epsilon=\epsilon(x)$. $\sum|b_n|$ converges only if $\lim d_n = d$ and if $z = -1/d$. This value however does not lie in the angular opening obtained for the variable z. This completes the proof of the convergence of (3.1) and hence of (1.4).

The uniform convergence is deduced in a manner analogous to that employed in the proof of Theorem 3.3. In this case the values of the approximants of (3.1) all lie in the half-plane $\Re(w)\geqq 0$.

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[1] The original statement of this theorem contained an error which was kindly pointed out by the referee.

All the preceding theorems of this section have the following common characteristics: For $\{d_n\}$ contained in a certain region D there exists a region $Z(D)$ such that for all $z \in Z(D)$ the continued fraction (1.4) converges. Moreover we saw that the convergence is uniform over every closed bounded set in the interior of $Z(D)$. Since the approximants of (1.4) are rational functions and have no poles for $z \in Z(D)$ they are holomorphic for $z \in Z(D)$. The uniform convergence then insures that the limit function is holomorphic in the interior of $Z(D)$.

THEOREM 3.5. *Let D be any one of the regions defined in the preceding theorems such that for $\{d_n\} \subset D$ the continued fraction* (1.4) *converges uniformly for all z contained in any arbitrary closed bounded set in the interior of the corresponding region $Z(D)$; then the function to which the continued fraction converges is holomorphic in the interior of $Z(D)$.*

THEOREM 3.6. *Let L be the set of limit points of the sequence $\{d_n\}$, and let S be an open set containing L. Let D and $Z(D)$ be regions of the type defined in Theorem* 3.5. *If S is contained in D the continued fraction* (1.4) *converges, at least in the wider sense, for all $z \in Z(D)$ and the function to which it converges is meromorphic in the interior of $Z(D)$.*

It is clear that there exists an integer n' such that, for all $n \geqq n'$, $d_n \in S$. Thus Theorem 3.5 applies to the continued fraction

$$1 + d_{n'}z + \frac{z}{1 + d_{n'+1}z +} \, \frac{z}{1 + d_{n'+2}z +} \cdots$$

and Theorem 3.6 follows.

4. **Behavior in a neighborhood of the origin.** In Theorem 3.1 it was shown that the condition $|d_n| \leqq M$ insures uniform convergence of the continued fraction (1.4) in some neighborhood of the origin. This raises the question: to what extent is boundedness of the sequence $\{d_n\}$ a necessary condition for the existence of a neighborhood of the origin in which the continued fraction converges uniformly. Three partial answers to this question are given below.

THEOREM 4.1. *If the sequence $\{d_n/n\}$ is unbounded then there does not exist a neighborhood of the origin in which the continued fraction* (1.4) *converges uniformly.*

The denominator $B_n(z)$ of the nth approximant of (1.4) is a polynomial of nth degree

$$B_n(z) = 1 + g_1^{(n)}z + \cdots + g_n^{(n)}z^n.$$

From the recursion formula for $B_n(z)$, that is,

$$B_n(z) = (1 + d_n z)B_{n-1}(z) + zB_{n-2}(z), \tag{4.1}$$

one easily deduces that

$$g_1^{(n)} = n - 1 + \sum_{\nu=1}^{n} d_\nu.$$

Let $\rho_\nu^{(n)}$ be the roots of $B_n(z)$; then it is clear that

$$g_1^{(n)} = -\sum_{\nu=1}^{n} \frac{1}{\rho_\nu^{(n)}}.$$

One now shows that the sequence $\{g_1^{(n)}/n\}$ is unbounded. The following relation holds:

$$\frac{g_1^{(n)}}{n} = \frac{d_n}{n} + \frac{1}{n} + \left(\frac{g_1^{(n-1)}}{n-1}\right)\left(\frac{n-1}{n}\right),$$

and hence

$$\left|\frac{g_1^{(n)}}{n}\right| + \left|\frac{g_1^{(n-1)}}{n-1}\right| \geq \left|\frac{d_n}{n}\right|.$$

The unboundedness of $g_1^{(n)}/n$ now follows immediately. Since $g_1^{(n)}/n$ is the arithmetic mean of the reciprocals of the roots one is now in a position to state that given an arbitrary positive δ and an arbitrary integer N there exists an integer $N' > N$ such that one of the roots of $B_{N'}(z)$ lies in the neighborhood $|z| < \delta$. In order that there be uniform convergence in some neighborhood of the origin it is necessary that there exist a neighborhood of the origin in which $B_n(z) \neq 0$ for all $n > n_0$, since for $z = 0$ (1.4) converges to 1. This completes the proof of the theorem.

THEOREM 4.2. *If $d_n > 0$ for all n and if $\{d_n\}$ is unbounded then there does not exist a neighborhood of the origin in which the continued fraction* (1.4) *converges uniformly.*

The proof of this theorem depends on three lemmas which are also of some interest for their own sake.

LEMMA 4.1. *If $d_\nu > 0$ for $\nu = 0, 1, \cdots, n$ then all the roots of $B_n(z)$ are real and negative and the roots of $B_n(z)$ and $B_{n-1}(z)$ separate each other, that is,*

$$\rho_1^{(n)} < \rho_1^{(n-1)} < \rho_2^{(n)} < \cdots < \rho_{n-1}^{(n-1)} < \rho_n^{(n)} < 0.$$

One notes first that all the coefficients of $B_n(z)$ are positive. That the root of $B_1(z)$ is negative is then clear. That no two successive $B_n(z)$ have common roots follows from the well known identity

$$A_n(z)B_{n-1}(z) - A_{n-1}(z)B_n(z) = (-1)^{n-1}z^n$$

and the fact that $B_n(0) = 1$ for all n. Now assume that the lemma holds for all ν up to and including $\nu = n-1$. Then

$$\begin{aligned} B_n(\rho_\nu^{(n-1)}) &= (1 + d_n\rho_\nu^{(n-1)})B_{n-1}(\rho_\nu^{(n-1)}) + \rho_\nu^{(n-1)}B_{n-2}(\rho_\nu^{(n-1)}) \\ &= \rho_\nu^{(n-1)}B_{n-2}(\rho_\nu^{(n-1)}). \end{aligned}$$

Thus for two successive roots of B_{n-1} the polynomial $B_n(z)$ takes on values with opposite signs. Hence there must be a root of B_n between any two roots of B_{n-1}. Further, for the first root (in order of size) $\rho_1^{(n-1)}$, B_n has a sign opposite to that of B_{n-2} while for $z<0$ and sufficiently small the signs of the two polynomials are the same. Hence, since by the induction hypothesis there is no root of B_{n-1} to the left of $\rho_1^{(n-1)}$, B_n must have a root to the left of $\rho_1^{(n-1)}$. In a similar way ($B_m(0)=1$, for all m) one shows that there must be a root of B_n between $\rho_{n-1}^{(n-1)}$ and 0. Thus all roots of B_n are accounted for and the lemma is proved.

LEMMA 4.2. *If $d_\nu > 0$ for $\nu = 0, 1, \cdots, n$ then either $-1/d_n$ is a root of B_n or there is at least one root of B_n to the left and at least one to the right of $-1/d_n$. (Note that Lemma 4.1 insures that all roots of B_n are real and negative.)*

For $z = -1/d_n$ one has

$$B_n(-1/d_n) = -1/d_nB_{n-2}(-1/d_n).$$

This expression may be equal to zero; if not one deduces from it that either B_n or B_{n-2} has a root to the left as well as to the right of $-1/d_n$. An application of Lemma 4.1 completes the proof of this lemma.

A combination of these two lemmas leads to the following result.

LEMMA 4.3. *If $d_\nu > 0$ for $\nu = 0, 1, \cdots, n$ then there is at least one root of B_n which is greater than or equal to every $-1/d_\nu$ and at least one root which is less than or equal to every $-1/d_\nu$.*

If $\{d_n\}$ is unbounded there then exist polynomials $B_n(z)$ with arbitrarily large n which have roots $\rho_n^{(n)}$ such that $|\rho_n^{(n)}|$ is less than any preassigned $\delta > 0$. This completes the proof of Theorem 4.2.

THEOREM 4.3. *If* $d_n \neq 0$ *for all n and if* $\lim d_n = \infty$ *then there does not exist a neighborhood of the origin in which the continued fraction* (1.4) *converges uniformly.*

One easily verifies that

$$g_n^{(n)} = \prod_{\nu=1}^{n} d_\nu = \prod_{\nu=1}^{n} 1/\rho_\nu^{(n)}.$$

If the conditions of the theorem are satisfied $(|\prod_{\nu=1}^{n} d_\nu|)^{1/n}$ is unbounded. Now at least one of the quantities $|1/\rho_\nu^{(n)}|$ is as large as the geometric mean of the set. The proof of the theorem is then completed by the same argument that was used in the two previous cases.

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SOME LIMIT THEOREMS

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1. **Introduction.** It is a classical result in the theory of trigonometric series that *if*

$$c_n \cos nx + d_n \sin nx \to 0 \qquad (n \to \infty) \tag{1.1}$$

for all (real) x on a set of positive measure, then

$$c_n \to 0, \qquad d_n \to 0. \tag{1.2}$$

Cantor proved this for the case that set $\{x\}$ is an interval, and Lebesgue established the result for a set of measure zero. A short proof is given by Hardy and Rogosinski.[1]

The following related result was proved and used by Szász.[2] *If*

$$a_n \sin nx + b_n \sin (n+1)x \to 0 \tag{1.3}$$

on a (real) set $\{x\}$ of positive measure, then

$$a_n \to 0, \qquad b_n \to 0. \tag{1.4}$$

Relations (1.1) and (1.3) can be put into complex form. For example, (1.1) becomes

$$a_n \exp \{nx\} + b_n \exp \{-nx\} \to 0, \tag{1.5}$$

with the conclusion that

$$a_n \to 0, \qquad b_n \to 0. \tag{1.6}$$

Here $\exp \{u\}$ is defined by

$$\exp \{u\} = e^{iu} \qquad (i = (-1)^{1/2}). \tag{1.7}$$

Our purpose in the present work is to extend the conclusions of the above-mentioned results to combinations more general than (1.3), (1.5). Thus in §2 we go from two terms to *k terms* and generalize the exponents; in §3 the coefficients of the exponentials are permitted

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[1] Hardy and Rogosinski, *Fourier series* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 38), Theorem 92, p. 84.

[2] Otto Szász, *On Lebesgue summability and its generalization to integrals*, Amer. J. Math. vol. 67 (1945) pp. 389–396, especially Lemma 2, p. 395. Dr. Szász has informed me that, with the intention of using it in work on trigonometric series, he has proved (but not published) a generalization of (1.3), namely where the left side of (1.3) is replaced by the expression $\sum_{p=n}^{n+k} a_p e^{ipx}$.

to be polynomials; and in §4 the multi-dimensional case is taken up.

2. One-dimensional case. In the following sections we suppose without further mention that all variables $x, y, \cdots$ are real.

LEMMA 2.1. *Let* $\{u_n\}$ *be a real sequence that does not have zero as its limit. The relation*

$$\lim_{n\to\infty} \exp \{u_n x\} = 1 \tag{2.1}$$

cannot hold on a set of positive measure.[3]

Suppose the lemma is false, so that there is a set $\mathcal{J}$ of positive measure on which (2.1) is satisfied. We may suppose that $\mathcal{J}$ is bounded. It is no restriction to assume that zero is not a limit point of $\{u_n\}$; for there exists an infinite subsequence of $\{u_n\}$ for which zero is not a limit point, and we may remove all u_n's not in this subsequence.

Suppose $\{u_n\}$ contains a bounded subsequence $\{u_{n_j}\}$; then from $\{u_{n_j}\}$ a further sequence can be chosen for which a limit exists. This limit, say L, cannot be zero, so from $\exp\{Lx\}=1$ for x in $\mathcal{J}$ we conclude that $\mathcal{J}$ is at most a denumerable set, contrary to the assumption that $\mathcal{J}$ is of positive measure.

Now suppose that $\{u_n\}$ contains no bounded subsequence, so that $|u_n|\to\infty$. By Egoroff's theorem there is a subset $\mathcal{J}_1$ of $\mathcal{J}$, of positive measure, on which (2.1), that is,

$$\cos u_n x + i \sin u_n x \to 1,$$

holds uniformly. Consequently, since $1 + \cos u_n x$ is uniformly bounded,

$$\cos^2 u_n x - 1 \to 0 \qquad \text{(uniformly on } \mathcal{J}_1\text{)}.$$

Integrating over $\mathcal{J}_1$:

$$\int_{\mathcal{J}_1} \cos^2 u_n x dx \to \int_{\mathcal{J}_1} dx = m(\mathcal{J}_1). \tag{2.2}$$

Now there exists an open set $\mathfrak{Q}$, consisting of a finite number of nonoverlapping intervals, say (a_j, b_j), $j=1, \cdots, p$, with the following two properties: (i) $\mathfrak{Q}$ contains $\mathcal{J}_1$; (ii) $m(\mathcal{J}_1)\leq m(\mathfrak{Q}) < 3m(\mathcal{J}_1)/2$. Hence,

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[3] As originally stated and proved, this and the other results of the paper used the condition *interval* everywhere in place of *set of positive measure*. We owe to Dr. Szász the suggestion to generalize to the case of positive measure, and also to extend the results to more than one variable (see §4).

$$\int_{\mathcal{J}_1} \cos^2 u_n x\,dx \leqq \int_{\mathfrak{Q}} \cos^2 u_n x\,dx = 2^{-1}\int_{\mathfrak{Q}} (1+\cos 2u_n x)\,dx$$

$$= 2^{-1} m(\mathfrak{Q}) + 2^{-1}\sum_{j=1}^{p}\left[\frac{\sin 2u_n x}{2u_n}\right]_{a_j}^{b_j};$$

so for all n sufficiently large,

$$\int_{\mathcal{J}_1} \cos^2 u_n x\,dx < \frac{3}{4} m(\mathcal{J}_1). \tag{2.3}$$

But this contradicts (2.2), so the lemma cannot be false.[4]

LEMMA 2.2. *Let* $\{t_{s,n}\}$, $s=1, \cdots, k$, *be real sequences with the property that none of the sequences* $\{t_{s,n}\}$, $\{t_{s,n}-t_{p,n}\}$ $(s\neq p)$ *has zero as a limit point. If real or complex constants* $\{A_{s,n}\}$ *exist such that*

$$\sum_{s=1}^{k} A_{s,n}[\exp\{t_{s,n}x\} - 1] \to 0 \tag{2.4}$$

for all x *on a set* $\mathcal{E}$ *of positive measure, then*

$$A_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{2.5}$$

If $k=1$ the lemma is true in virtue of Lemma 2.1. Assume it true for the case $k-1$; we shall then prove it for k by an induction argument, and this will establish the truth of Lemma 2.2.

It is no restriction to suppose that $\mathcal{E}$ is a bounded set. $\mathcal{E}$ contains a point x_1 with the property that every interval containing x_1 in its interior meets $\mathcal{E}$ in a set of positive measure.[5] For suppose not. Then about each x in $\mathcal{E}$ exists an interval I_x, with x in its interior, such that $\mathcal{E}\cdot I_x$ is of measure zero. Let x_1 be in $\mathcal{E}$, and let I_{x_1} be the largest associated interval. It is clear that there *is* a largest interval. If x_2 in $\mathcal{E}$ is not in I_{x_1}, then it too has a largest associated interval I_{x_2}, and I_{x_1}, I_{x_2} do not meet. It is now a straightforward argument to show that $\mathcal{E}$ is covered by at most a denumerable number of such intervals I_x, thus establishing $\mathcal{E}$ as a set of zero measure. This contradiction shows that a point such as the aforementioned x_1 exists.

Let x in (2.4) take on such a value x_1 and subtract from (2.4). There results the relation

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[4] This contradiction does not preclude the possibility that the set of points for which (2.1) holds is non-measurable, but in this case the set cannot contain a subset of positive measure.

[5] A much stronger conclusion as to the number of such points x_1 is of course possible, but we require only the above mild assertion.

$$\sum_{s=1}^{k} A_{s,n} \exp\{t_{s,n}x_1\}[\exp\{t_{s,n}y\} - 1] \to 0 \tag{2.6}$$

for all y in $\mathcal{E}_1 = \{y = x - x_1$, as x ranges over $\mathcal{E}\}$. $\mathcal{E}_1$ and $\mathcal{E}$ have the same (positive) measure, and the replacement of (2.4) and $\mathcal{E}$ by (2.6) and $\mathcal{E}_1$ insures that the origin $(y = 0)$ is a point of $\mathcal{E}_1$ every neighborhood of which contains a subset of $\mathcal{E}_1$ of positive measure.

Let

$$B_{s,n} = A_{s,n} \exp\{t_{s,n}x_1\}. \tag{2.7}$$

Then

$$\sum_{s=1}^{k} B_{s,n}[\exp\{t_{s,n}y\} - 1] \to 0 \qquad (y \text{ in } \mathcal{E}_1), \tag{2.8}$$

and (2.5) is equivalent to

$$B_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{2.9}$$

Suppose the lemma is false for case k. Then a value s, say $s = k$, exists such that[6] $B \to 0$ is false. There therefore is a subsequence $\{n_j = n(j)\}$ of $\{n\}$, and a number $M > 0$, such that

$$|B_{k,n(j)}| > M;$$

so on replacing $\{n\}$ by $\{n(j)\}$ in (2.8) and dividing by $B_{k,n(j)}$, we have

$$[\exp\{t_{k,n(j)}y\} - 1] + \sum_{s=1}^{k-1} C_{s,n(j)}[\exp\{t_{s,n(j)}y\} - 1] \to 0 \qquad (y \text{ in } \mathcal{E}_1). \tag{2.10}$$

Here

$$C_{s,n(j)} = \frac{B_{s,n(j)}}{B_{k,n(j)}}. \tag{2.11}$$

Let y_1 be an arbitrary point of $\mathcal{E}_1$. The set of points $\{u\}$ defined by $u = y - y_1$ as y ranges over $\mathcal{E}_1$ will be denoted by $\mathcal{E}_{y_1}$, and will be termed a *translation set* (relative to $\mathcal{E}_1$). Clearly, $m(\mathcal{E}_{y_1}) = m(\mathcal{E}_1)$.

Let $\mathfrak{Q}$ be an open set containing $\mathcal{E}_1$, with $m(\mathcal{E}_1) < m(\mathfrak{Q}) < 3m(\mathcal{E}_1)/2$. There exists a set of nonoverlapping open intervals $\mathfrak{Q}_1 = I_1 + \cdots + I_r$ contained in $\mathfrak{Q}$, for which

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[6] Actually, for *every* $s = 1, \cdots, k$ the quantity $B_{s,n}$ does not approach zero; for otherwise we can drop from sum (2.8) all terms for which $B_{s,n} \to 0$, and thus reduce (2.8) to $k - 1$ or less terms, in which case the lemma is true by our induction assumption.

$$m(\mathcal{E}_1) < m(\mathcal{Q}_1) < 3m(\mathcal{E}_1)/2,$$

and also such that

$$m(\mathcal{Q}_1 \cdot \mathcal{E}_1) > 3m(\mathcal{E}_1)/4.$$

To each end of I_p ($p=1, \cdots, r$) add an interval of length Δ, forming a new interval J_p, where Δ is chosen small enough so that on setting $\mathcal{Q}_2 = J_1 + \cdots + J_q$ ($q \leqq r$ since some intervals may overlap and thus be combined) then $m(\mathcal{Q}_2) < 3m(\mathcal{E}_1)/2$.

Let $\mathcal{H}$ be the subset of numbers y of $\mathcal{E}_1$ for which $|y| < \Delta$. We know that $\mathcal{H}$ is of positive measure. Moreover, for an arbitrary y in $\mathcal{H}$,

$$m(\mathcal{Q}_2 \cdot \mathcal{E}_y) > 3m(\mathcal{E}_1)/4, \qquad m(\mathcal{Q}_2 \cdot \mathcal{E}_1) > 3m(\mathcal{E}_1)/4.$$

Since $m(\mathcal{Q}_2) < 3m(\mathcal{E}_1)/2$, it follows that

$$m(\mathcal{E}_1 \cdot \mathcal{E}_y) > 0 \qquad \text{(all } y \text{ in } \mathcal{H}\text{)}.$$

We see from (2.10) and Lemma 2.1 that we cannot have $C_{s,n(j)} \to 0$ for all $s=1, \cdots, k-1$. Hence there is an s, say $s=1$, for which $C_{1,n(j)} \to 0$ is false; and a subsequence $\{m(j)\}$ of $\{n(j)\}$, and a positive number K, such that

$$|C_{1,m(j)}| > K. \tag{2.12}$$

Now the relation

$$\exp\{(t_{1,m(j)} - t_{k,m(j)})z\} \to 1 \qquad (j \to \infty) \tag{2.13}$$

cannot hold on a set of positive measure (Lemma 2.1); consequently, there is a point y_1 in $\mathcal{H}$ such that (2.13) is false for $z = y_1$. Choose $y = y_1$ in (2.10) and subtract from (2.10):

$$\begin{aligned} &\exp\{t_{k,n(j)}y_1\}[\exp\{t_{k,n(j)}u\} - 1] \\ &\quad + \sum_{s=1}^{k-1} C_{s,n(j)} \exp\{t_{s,n(j)}y_1\}[\exp\{t_{s,n(j)}u\} - 1] \to 0, \end{aligned} \tag{2.14}$$

where $u = y - y_1$ (y in $\mathcal{E}_1$), so that u ranges over the translation set $\mathcal{E}_{y_1}$. In (2.14), replace $\{n(j)\}$ by $\{m(j)\}$ and divide by $\exp\{t_{k,m(j)}\, y_1\}$. This gives us

$$\begin{aligned} &[\exp\{t_{k,m(j)}u\} - 1] \\ &+ \sum_{s=1}^{k-1} C_{s,m(j)} \exp\{(t_{s,m(j)} - t_{k,m(j)})y_1\}[\exp\{t_{s,m(j)}u\} - 1] \to 0 \\ &\qquad\qquad (u \text{ in } \mathcal{E}_{y_1}). \end{aligned} \tag{2.15}$$

Let $\mathcal{L} = \mathcal{E}_1 \cdot \mathcal{E}_{y_1}$. We know that $m(\mathcal{L}) > 0$. If we restrict u in (2.15) and y in (2.10) to lie in $\mathcal{L}$, then u and y may be identified; so on subtracting (2.15) from (2.10) (with $n(j)$ replaced by $m(j)$) we obtain

$$(2.16)\quad \sum_{s=1}^{k-1} C_{s,m(j)}[1-\exp\{(t_{s,m(j)}-t_{k,m(j)})y_1\}][\exp\{t_{s,m(j)}y\}-1]\to 0 \qquad (y \text{ in } \mathcal{L}).$$

This relation has only $k-1$ terms, and for it the hypotheses of Lemma 2.2 hold. By our induction assumption, therefore, each coefficient approaches zero. Since (2.12) holds, we must have

$$\exp\{(t_{1,m(j)}-t_{k,m(j)})y_1\}\to 1,$$

which is contrary to the choice of y_1.

Thus the induction chain is complete, and the lemma is established.

THEOREM 2.1. *Let* $\{a_{s,n}\}$, $s=1, \cdots, k$, *be real or complex number sequences, and let the real sequences* $\{r_{s,n}\}$ *have the property that none of the sequences* $\{r_{s,n}-r_{p,n}\}(s\neq p)$ *has zero as a limit point. If*

$$(2.17)\qquad \sum_{s=1}^{k} a_{s,n}\exp\{r_{s,n}x\}\to 0$$

for all x *on a set* $\mathcal{E}$ *of positive measure, then*

$$(2.18)\qquad a_{s,n}\to 0 \qquad (s=1, \cdots, k).$$

REMARK. For sequences $\{r_{s,n}\}$ satisfying the above hypothesis, Theorem 2.1 asserts what may be termed the *asymptotic linear independence* of the functions $\exp\{r_{s,n}x\}$, $s=1, \cdots, k$.

If the theorem is false, there is an index s, say $s=1$, for which $a_{1,n}\to 0$ is false; so a subsequence $\{n(j)\}$ of $\{n\}$ exists, and a positive number M, such that $|a_{1,n(j)}|>M$. Replace $\{n\}$ by $\{n(j)\}$ in (2.17) and divide by $a_{1,n(j)}\exp\{r_{1,n(j)}x\}$:

$$(2.19)\qquad 1+\sum_{s=2}^{k} b_{s,n(j)}\exp\{(r_{s,n(j)}-r_{1,n(j)})x\}\to 0,$$

where

$$(2.20)\qquad b_{s,n(j)}=\frac{a_{s,n(j)}}{a_{1,n(j)}}.$$

Take $x=x_1$ in (2.19) and subtract from (2.19):

$$(2.21)\qquad \sum_{s=2}^{k} b_{s,n(j)}\exp\{t_{s,n(j)}x_1\}[\exp\{t_{s,n(j)}y\}-1]\to 0.$$

Here

$$t_{s,n(j)} = r_{s,n(j)} - r_{1,n(j)}; \quad y = x - x_1 \qquad (x \text{ in } \mathcal{E}),$$

so y ranges over a set $\mathcal{E}_1$ of positive measure.

The hypotheses of Lemma 2.2 are satisfied, so

$$b_{s,n(j)} \to 0 \qquad (s = 2, \cdots, k).$$

But this contradicts (2.19). Thus Theorem 2.1 is established.

COROLLARY 2.1. *Let the sequences* $\{t_{s,n}\}$ *satisfy the hypothesis of Lemma* 2.2. *If a constant* A *and constants* $\{a_{s,n}\}$ *exist such that*

$$\sum_{s=1}^{k} a_{s,n} \exp\{t_{s,n}x\} \to A \tag{2.22}$$

for all x *on a set of positive measure, then*

$$A = 0; \qquad a_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{2.23}$$

For, (2.22) can be written

$$\sum_{s=1}^{k+1} a_{s,n} \exp\{t_{s,n}x\} \to 0, \tag{2.24}$$

where

$$a_{k+1,n} \equiv -A, \qquad t_{k+1,n} \equiv 0. \tag{2.25}$$

The hypothesis of Theorem 2.1 is fulfilled in (2.24), so $a_{s,n}\to 0$, $s=1, \cdots, k+1$.

In Theorem 2.1 the condition on the sequences $\{r_{s,n}\}$ cannot be weakened. This is shown by the following theorem.

THEOREM 2.2. *Let the real sequences* $\{r_{s,n}\}$, $s=1, \cdots, k$, *be such that at least one of the sequences* $\{r_{s,n}-r_{p,n}\}(s\neq p)$ *has zero as a limit point. There exist sequences* $\{a_{s,n}\}$, *at least one of which does not approach zero, such that* (2.17) *holds for all* x.

We may suppose that $t_{n(j)}=r_{1,n(j)}-r_{2,n(j)}\to 0$ as $j\to\infty$. Choose $a_{s,n}=0$, $s=3, \cdots, k$, and $a_{1,n}=a_{2,n}=0$ for $n\neq n_1, n_2, \cdots$. The left side of (2.17) becomes

$$a_{1,n(j)} \exp\{r_{1,n(j)}x\} + a_{2,n(j)} \exp\{r_{2,n(j)}x\}, \tag{2.26}$$

and this approaches zero if and only if

$$a_{1,n(j)} \exp\{t_{n(j)}x\} + a_{2,n(j)} \to 0. \tag{2.27}$$

It is clear that if we *define* $a_{1,n(j)}=1$, $a_{2,n(j)}=-1$, then (2.27) does hold for every x. Actually, these coefficients can be chosen to be *unbounded*. For let $R>0$ be given. There exists an $M=M(R)$ such that

$$\left|\exp\{t_{n(j)}x\}-1\right| \leqq M\left|t_{n(j)}\right|$$

for all $|x|\leqq R$. On writing the left side of (2.27) as

$$a_{1,n(j)}[\exp\{t_{n(j)}x\}-1]+[a_{1,n(j)}+a_{2,n(j)}],$$

we see that (2.27) follows if we choose $a_{1,n(j)}$, $a_{2,n(j)}$ so that

$$a_{1,n(j)}t_{n(j)}\rightarrow 0, \qquad a_{1,n(j)}+a_{2,n(j)}\rightarrow 0. \tag{2.28}$$

Since $t_{n(j)}\rightarrow 0$, conditions (2.28) can be satisfied by sequences $a_{1,n(j)}$, $a_{2,n(j)}$ that are unbounded.

3. **Polynomial coefficients.** The result of Theorem 2.1 can be extended to the case of polynomial coefficients of bounded degree:

THEOREM 3.1. *Let* $\{r_{s,n}\}$, $s=1,\cdots,k$, *be real sequences such that none of the sequences* $\{r_{s,n}-r_{p,n}\}$ $(s\neq p)$ *has zero as a limit point. Let* $\{P_{s,n}(x)\}$ *be real or complex polynomial sequences*:

$$P_{s,n}(x)=a_{s,0,n}+a_{s,1,n}x+\cdots+a_{s,q_s,n}x^{q_s} \qquad (s=1,\cdots,k) \tag{3.1}$$

in which q_s *is independent of* n. *If*

$$\sum_{s=1}^{k}P_{s,n}(x)\exp\{r_{s,n}x\}\rightarrow 0 \tag{3.2}$$

for all x *on a set* $\mathcal{E}$ *of positive measure, then*

$$a_{s,p,n}\rightarrow 0 \qquad (p=0,1,\cdots,q_s;\ s=1,\cdots,k). \tag{3.3}$$

Let

$$q=\max\{q_1,\cdots,q_k\}. \tag{3.4}$$

If $q=0$ the result follows from Theorem 2.1. Suppose the theorem is false. Then there is an integer $Q>0$ such that whenever $q<Q$ the result is true, but for at least one case with $q=Q$ the theorem is untrue. In each case of failure, with $q=Q$, at least one polynomial coefficient is of degree Q. Let λ be the number of such polynomials; then there is a positive integer Λ with the property that whenever $\lambda<\Lambda$ (and $q=Q$), the theorem is true, but there is a case $\lambda=\Lambda$, $q=Q$ for which it is false.

Let (3.2) be such a case, so that exactly Λ polynomials, that we may take to be $P_{s,n}(x)$, $s=1,\cdots,\Lambda$, are of degree Q while all other polynomial coefficients (if any) are of lower degree. Since the theorem is false for this case, not all the coefficients approach zero as $n\to\infty$. For each $n=1, 2, \cdots$ let

$$\mu_n = \max \{ | a_{s,p,n} | \} \qquad (p = 0, 1, \cdots, q_s;\ s = 1, \cdots, k). \tag{3.5}$$

Then μ_n does not approach 0. There therefore exist values $s=\sigma$, $p=p_\sigma$, a positive number M, and a subsequence $\{n(j)\}$ of $\{n\}$, such that

$$| a_{\sigma,p_\sigma,n(j)} | = \mu_{n(j)} > M \qquad (j = 1, 2, \cdots). \tag{3.6}$$

Replace $\{n\}$ by $\{n(j)\}$ in (3.2) and divide by $a_{\sigma,p_\sigma,n(j)} \exp \{r_{1,n(j)}x\}$:

$$R_{1,n(j)}(x) + \sum_{s=2}^{k} R_{s,n(j)}(x) \exp \{t_{s,n(j)}x\} \to 0 \qquad (x \text{ in } \mathcal{E}), \tag{3.7}$$

where

$$t_{s,n(j)} = r_{s,n(j)} - r_{1,n(j)}$$

and

$$R_{s,n(j)}(x) = \sum_{p=0}^{q_s} b_{s,p,n(j)}x^p \equiv \frac{1}{a_{\sigma,p_\sigma,n(j)}}\cdot P_{s,n(j)}(x).$$

The b-coefficients are bounded, and $b_{\sigma,p_\sigma,n(j)} \equiv 1$ for all j. Consequently, there exists a subsequence $\{m(j)\}$ of $\{n(j)\}$ for which the following limits exist:

$$\lim_{j\to\infty} b_{s,p,m(j)} = b_{s,p} \qquad (p = 0, 1, \cdots, q_s;\ s = 1, \cdots, k); \tag{3.8}$$

and not all of $b_{s,p}$ are zero, since $b_{\sigma,p_\sigma} = 1$.

From (3.7) it follows that

$$R_1(x) + \sum_{s=2}^{k} R_s(x) \exp \{t_{s,m(j)}x\} \to 0 \qquad (x \text{ in } \mathcal{E}), \tag{3.9}$$

where

$$R_s(x) = \sum_{p=0}^{q_s} b_{s,p}x^p \qquad (s = 1, \cdots, k). \tag{3.10}$$

Moreover, $R_1(x)$ is of degree Q; for if it is of lower degree, then (3.9) presents a case in which fewer than Λ polynomials are of degeee Q, so from the definition of Λ it will follow that the theorem is true for (3.9). Thus all coefficients in all the polynomials approach zero as $j\to\infty$. But this is contrary to the fact that $b_{\sigma,p_\sigma} = 1$. Hence the degree of $R_1(x)$ must be Q.

We know from the proof of Lemma 2.2 that set $\mathcal{E}$ contains a point

x_1 every neighborhood of which contains a subset of $\mathcal{E}$ of positive measure; and using this fact, we may conclude (as was similarly argued in establishing Lemma 2.2) that there exist distinct numbers h_1, h_2 such that on defining $\mathcal{E}_1, \mathcal{E}_2$ by

$$\mathcal{E}_p \equiv \{y_p = x + h_p,\ x \text{ ranging over } \mathcal{E}\} \qquad (p = 1, 2),$$

then $\mathcal{E}_3 = \mathcal{E}_1 \cdot \mathcal{E}_2$ is a set of positive measure.

Relation (3.9) may then be written in each of the forms

$$\begin{aligned} &R_1(y_p - h_p) \\ (3.11) \qquad &+ \sum_{s=2}^{k} R_s(y_p - h_p) \exp\{-t_{s,m(j)}h_p\} \exp\{t_{s,m(j)}y_p\} \to 0 \\ &\qquad\qquad (y_p \text{ in } \mathcal{E}_p,\ p = 1, 2). \end{aligned}$$

If we consider only points in $\mathcal{E}_3$, then y_1 and y_2 may be identified:

$$\begin{aligned} &R_1(y - h_p) \\ (3.12) \qquad &+ \sum_{s=2}^{k} R_s(y - h_p) \exp\{-t_{s,m(j)}h_p\} \exp\{t_{s,m(j)}y\} \to 0 \\ &\qquad\qquad (y \text{ in } \mathcal{E}_3,\ p = 1, 2). \end{aligned}$$

On subtracting we have

$$\begin{aligned} (3.13) \quad &[R_1(y - h_1) - R_1(y - h_2)] + \sum_{s=2}^{k} [R_s(y - h_1) \exp\{-t_{s,m(j)}h_1\} \\ &\qquad - R_s(y - h_2) \exp\{-t_{s,m(j)}h_2\}] \exp\{t_{s,m(j)}y\} \to 0 \\ &\qquad\qquad (y \text{ in } \mathcal{E}_3). \end{aligned}$$

Since $R_1(x)$ is of actual degree Q, and $Q > 0$, we see that

$$(3.14) \qquad H(y) \equiv [R_1(y - h_1) - R_1(y - h_2)]$$

is a polynomial of degree exactly $Q-1$, and is therefore not identically zero. But $H(y)$ being of degree less than Q, this places (3.13) in the category of cases for which the theorem is true, since now fewer than A polynomials are of degree Q. Hence all coefficients approach zero. This is however contrary to the condition that $H(y) \not\equiv 0$.

We have thus arrived at a contradiction, so the assumption that Theorem 3.1 is false is untenable.

4. **Higher dimensions.** We shall now show that the foregoing results extend to the general case of p dimensions. Throughout this section the term *measure* refers to *p-dimensional measure*. Proofs for

the general case usually follow those of the preceding sections, and are accordingly given briefly or not at all.

LEMMA 4.1. *Let* $\{u_{s,n}\}$, $s=1, \cdots, p$, *be real sequences such that for at least one value of* s, $\{u_{s,n}\}$ *does not have zero as its limit. The relation*

$$\lim_{n\to\infty} \exp\left\{\sum_{s=1}^{p} u_{s,n}x_s\right\} = 1 \tag{4.1}$$

cannot hold on a set of points $(x)\equiv(x_1, \cdots, x_p)$ *of positive measure.*

Assume that the lemma is false, so there is a set $\mathcal{J}$ of positive measure for which (4.1) holds. Suppose $s=q$ is the value for which $u_{q,n}$ does not approach 0. We may then assume that $\{u_{q,n}\}$ does not have zero as limit point. If a subsequence $\{n(j)\}$ of $\{n\}$ exists for which all the sequences $\{u_{s,n(j)}\}$, $s=1, \cdots, p$, are bounded, then there will be a further subsequence $\{m(j)\}$ for which the following limits exist:

$$\lim_{j\to\infty} u_{s,m(j)} = l_s \qquad (s = 1, \cdots, p),$$

with $l_q \neq 0$. Hence if (x) is in $\mathcal{J}$, then (x) must satisfy one of the equations

$$\frac{1}{2\pi}\sum_{s=1}^{p} l_s x_s = 0, \ \pm 1, \ \pm 2, \cdots . \tag{4.2}$$

For each choice of the right side, (4.2) is a hyperplane, and is of measure zero. The totality of planes (4.2) is likewise of zero measure, and so, therefore, is $\mathcal{J}$, which is contrary to assumption.

There remains to consider the case where for at least one value of s, say $s=1$, and a subsequence $\{n(j)\}$,

$$|u_{1,n(j)}| \to \infty.$$

The remainder of the argument now follows that of Lemma 2.1, with obvious p-dimensional modifications.

LEMMA 4.2. *Let* $\{t_{s,r,n}\}$, $s=1, \cdots, k$; $r=1, \cdots, p$ *be real sequences with the following property: a value* $r=q$ *exists such that none of the sequences* $\{t_{s,q,n}\}$, $\{t_{s,q,n}-t_{\sigma,q,n}\}$ $(s\neq\sigma)$ *has zero as a limit point. If real or complex constants* $\{A_{s,n}\}$ *exist such that*

$$\sum_{s=1}^{k} A_{s,n}\left[\exp\left\{\sum_{r=1}^{p} t_{s,r,n}x_r\right\} - 1\right] \to 0 \tag{4.3}$$

for all $(x)\equiv(x_1, \cdots, x_p)$ *on a set* $\mathcal{E}$ *of positive measure, then*

$$A_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{4.4}$$

The proof is like that of Lemma 2.2 with simple modifications that need not be detailed here.

Lemma 4.2 leads directly to the following theorem.

THEOREM 4.1. *Let* $\{a_{s,n}\}$, $s=1, \cdots, k$, *be real or complex sequences, and let the real sequences* $\{q_{s,r,n}\}$, $s=1, \cdots, k$; $r=1, \cdots, p$, *be such that for some value* $r=\omega$, *none of the sequences* $\{q_{s,\omega,n}-q_{\sigma,\omega,n}\}$ $(s\neq\sigma)$ *has zero as a limit point. If*

$$\sum_{s=1}^{k} a_{s,n} \exp \left\{ \sum_{r=1}^{p} q_{s,r,n} x_r \right\} \to 0 \tag{4.5}$$

for all $(x)\equiv(x_1, \cdots, x_p)$ *on a set* $\mathcal{E}$ *of positive measure, then*

$$a_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{4.6}$$

The proof follows an earlier one (Theorem 2.1), as does the next result:

COROLLARY 4.1. *Let the sequences* $\{t_{s,r,n}\}$ *satisfy the hypothesis of Lemma* 4.2. *If a constant* A *and constants* $\{a_{s,n}\}$ *exist such that*

$$\sum_{s=1}^{k} a_{s,n} \exp \left\{ \sum_{r=1}^{p} t_{s,r,n} x_r \right\} \to A \tag{4.7}$$

for all (x) *on a set of positive measure, then*

$$A = 0; \qquad a_{s,n} \to 0 \qquad (s = 1, \cdots, k). \tag{4.8}$$

Finally, we have

THEOREM 4.2. *Let* $\{q_{s,r,n}\}$, $s=1, \cdots, k$; $r=1, \cdots, p$, *be real sequences satisfying the hypothesis of Theorem* 4.1. *Let* $\{P_{s,n}(x_1, \cdots, x_p)\}$ *be real or complex polynomial sequences*:

$$P_{s,n}(x_1, \cdots, x_p) = \sum_{k_1+\cdots+k_p=0}^{e_s} a_{s,n;k_1,\cdots,k_p} x_1^{k_1} \cdots x_p^{k_p} \tag{4.9}$$

$$(s = 1, \cdots, k),$$

in which e_s *is independent of* n. *If*

$$\sum_{s=1}^{k} P_{s,n}(x_1, \cdots, x_p) \exp \left\{ \sum_{r=1}^{p} q_{s,r,n} x_r \right\} \to 0 \tag{4.10}$$

for all (x) *on a set* $\mathcal{E}$ *of positive measure, then*

$$(4.11) \qquad a_{s,n;k_1,\cdots,k_p} \to 0 \qquad (0 \leq k_1 + \cdots + k_p \leq e_s;\ s = 1, \cdots, k).$$

Up to a point the proof is patterned after that of Theorem 3.1. When the equivalent of (3.13) is obtained, however, we can no longer assert that $H(y_1, \cdots, y_p)$ is not identically zero simply from the fact that two distinct sets $(h)_1 = (h_{11}, \cdots, h_{1p})$, $(h)_2 = (h_{21}, \cdots, h_{2p})$ exist such that

$$(4.12) \qquad \begin{aligned} H(y_1, \cdots, y_p) \equiv {} & R_1(y_1 - h_{11}, \cdots; y_p - h_{1p}) \\ & - R_1(y_1 - h_{21}, \cdots, y_p - h_{2p}). \end{aligned}$$

In fact, nonconstant polynomials in more than one variable exist that are "periodic." We avoid this difficulty by observing that for a fixed point $(h)_1$, the point $(h)_2$ can be chosen arbitrarily on a set of positive measure. Examination of the proof of Lemma 2.2 shows this. Now if a polynomial $L(x_1, \cdots, x_p)$ has the property that

$$L(x_1 + c_1, \cdots, x_p + c_p) \equiv L(x_1, \cdots, x_p)$$

for all sets $(c) \equiv (c_1, \cdots, c_p)$ on a set of positive measure, then surely $L \equiv$ constant.

In our case, therefore, if $H(y_1, \cdots, y_p) \equiv 0$ for all possible choices of $(h)_2$, then R_1 is a constant, contrary to the fact that its degree is $Q > 0$ (cf. Theorem 3.1). The remainder of the proof offers no difficulty.

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NICHOLSON'S INTEGRAL FOR $J_n^2(z)+Y_n^2(z)$

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The integral in question is

$$J_n^2(z)+Y_n^2(z) = (8/\pi^2)\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt, \tag{1}$$

and its validity for arbitrary complex n when the real part of z is positive is proved in [1, pp. 441–444][1] with the help of Hardy's theory of generalized integrals and integrations over contours in the complex plane. It is the purpose of this paper to give a much more elementary proof of (1).

We begin by observing [1, p. 146] that if $D=z(d/dz)$, then three linearly independent solutions of the equation

$$[D(D^2-4n^2)+4z^2(D+1)]y = 0 \tag{2}$$

are $J_n^2(z)$, $Y_n^2(z)$ and $J_n(z)Y_n(z)$. Equation (2) may be written as

$$z^2y'''+3zy''+(1-4n^2+4z^2)y'+4zy = 0. \tag{3}$$

We shall now show that $y(z)=\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt$ is a solution of (3). When the real part of z is positive it is clear that $K_0(2z\sinh t)$ is sufficiently small at ∞ to permit us to differentiate under the integral sign as many times as we please. Therefore,

$$y'(z) = 2\int_0^\infty \sinh t K_0'(2z\sinh t)\cosh 2nt\,dt. \tag{4}$$

If we make use of the differential equation

$$xK_0''(x)+K_0'(x)-xK_0(x) = 0 \tag{5}$$

satisfied by $K_0(x)$, then we find that

$$y'' = \int_0^\infty \{4\sinh^2 tK_0(2z\sinh t)-2z^{-1}\sinh tK_0'(2z\sinh t)\}\cosh 2nt\,dt,$$

$$y''' = \int_0^\infty \{(8\sinh^3 t+4z^{-2}\sinh t)K_0'(2z\sinh t) - 4z^{-1}\sinh^2 tK_0(2z\sinh t)\}\cosh 2nt\,dt.$$

It follows that

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[1] Numbers in brackets refer to the reference cited at the end of the paper.

$$z^2y''' + 3zy'' + (1 + 4z^2)y' + 4zy$$

$$= \int_0^\infty \{4z^2 \sinh 2t \cosh t K_0' (2z \sinh t) + 4z \cosh 2t K_0(2z \sinh t)\} \cosh 2nt\,dt. \tag{6}$$

If now (4) is integrated by parts and use is made of (5) we find that

$$4n^2y' = -4nz \int_0^\infty \sinh 2tK_0(2z \sinh t) \sinh 2nt\,dt,$$

whence another integration by parts shows that $4n^2y'$ is equal to the right-hand side of (6). Therefore $y(z)$ is a solution of (3). Consequently, there exist constants A, B, C such that

$$y(z) = AJ_n^2(z) + BY_n^2(z) + CJ_n(z)Y_n(z). \tag{7}$$

We shall now show that

$$\lim_{z\to\infty} zy(z) = \lim_{z\to\infty} \int_0^\infty zK_0(2z \sinh t) \cosh t\,dt = \frac{\pi}{4}, \tag{8}$$

the last equality being a consequence of the result [1, p. 388]

$$\int_0^\infty K_0(u)du = \frac{\pi}{2}\cdot$$

In (8), z is restricted to real values. In fact, the difference of the integrands in the limitands in (8) is

$$F(z, t) = zK_0(2z \sinh t)(\cosh 2nt - \cosh t).$$

Now $x^{1/2}e^xK_0(x)$ is bounded on $(0, \infty)$, so that

$$|F(z, t)| \leq A_0(z \operatorname{csch} t)^{1/2}e^{-2z\sinh t}|\cosh 2nt - \cosh t|.$$

Moreover, $\operatorname{csch} t \leqq 1/t$ and the mean value theorem shows that

$$|\cosh 2nt - \cosh t| \leq (2|n| + 1)t(\sinh 2|n|t + \sinh t),$$

whence we see that

$$|F(z, t)| \leq A_1(zt)^{1/2}e^{-2z\sinh t}(\sinh 2|n|t + \sinh t).$$

We can suppose that $z \geqq 1$. Since $\sinh t \geqq t$ and $(zt)^{1/2}e^{-zt}$ is bounded, we find that

$$\begin{aligned} |F(z, t)| &\leq A_2(\sinh 2|n|t + \sinh t)e^{-z\sinh t} \\ &\leq A_2(\sinh 2|n|t + \sinh t)e^{-\sinh t}. \end{aligned}$$

Therefore, $F(z, t)$ converges dominatedly to zero as z approaches ∞, and this suffices to prove (8).

It is known [1, p. 199] that

$$J_n(z) = (2/\pi z)^{1/2} \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2}),$$

$$Y_n(z) = (2/\pi z)^{1/2} \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2}).$$

From (7) we conclude that

$$\frac{\pi z y(z)}{2} = A + (B - A) \sin^2\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$
$$+ \frac{C}{2} \sin\left(2z - n\pi - \frac{\pi}{2}\right) + O(z^{-1}).$$

This result is incompatible with (8) unless $A = \pi^2/8$, $B = A$, $C = 0$, and in this case $y(z) = (\pi^2/8)\{J_n^2(z) + Y_n^2(z)\}$. This completes the proof of (1).

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THE KOLMOGOROFF PRINCIPLE FOR THE LEBESGUE AREA

R. G. HELSEL AND E. J. MICKLE

1. **Introduction.** Kolmogoroff [3][1] has considered an axiomatic development of measure theory for analytic sets in a metric space. One of his axioms is the following: If the set E^* is the image of a set E under a Lipschitzian transformation with Lipschitz constant equal to or less than one, then the measure of E^* should not exceed the measure of E. As stated, the principle has meaning only for point sets, but we shall be concerned with the area (or two-dimensional measure) of Fréchet surfaces in Euclidean 3-space; so we first reformulate the Kolmogoroff principle as follows: If T^* and T are two continuous transformations from the unit disc D: $u^2+v^2\leqq 1$ into Euclidean 3-space which satisfy the distance inequality $|T^*(p_1)-T^*(p_2)|\leqq|T(p_1)-T(p_2)|$ for every pair of points $p_1, p_2\in D$, then the area of the Fréchet surface determined by T^* should not exceed the area of the one determined by T. The purpose of this paper is to show that the Kolmogoroff principle, as reformulated above for Fréchet surfaces, is satisfied by the Lebesgue area.

We first discuss the case in which both the Fréchet surfaces are represented by Lipschitzian transformations. Since the Lebesgue area is given by the usual double integral formula in this case (see Radó [5]), an elementary inequality suffices to establish the desired relation between the areas. In an independent study of a related question, Reichelderfer [6] has obtained a result which can be shown to be equivalent. The general case, in which the transformations representing the surfaces are merely assumed to be continuous, depends upon the extension of a Lipschitzian vector function, defined originally on a bounded and closed set, to an open neighborhood of the set. We are indebted to Professor S. Eilenberg for calling our attention to the fact that such an extension has already been established by Kirszbraun [2].

2. **The Lipschitz case.** To begin, let us establish the elementary inequality mentioned in the introduction.

LEMMA 1. *If E, F, G and E^*, F^*, G^* are constants such that*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$E\cos^2\theta+2F\cos\theta\sin\theta+G\sin^2\theta\geqq E^*\cos^2\theta+2F^*\cos\theta\sin\theta+G^*\sin^2\theta\geqq 0$ for $0\leqq\theta\leqq 2\pi$, then $EG-F^2\geqq E^*G^*-F^{*2}$.

PROOF. Since the quadratic form $E^*\cos^2\theta+2F^*\cos\theta\sin\theta+G^*\sin^2\theta$ is positive, we have

$$E^*G^*-F^{*2}\geqq 0. \tag{1}$$

The quadratic form $(E-E^*)\cos^2\theta+2(F-F^*)\cos\theta\sin\theta+(G-G^*)\sin^2\theta$ is also positive; so

$$(E-E^*)(G-G^*)\geqq (F-F^*)^2, \tag{2}$$

which may be written in the form

$$\begin{aligned} EG-F^2\geqq E^*G^*-F^{*2}&+G^*(E-E^*)\\ &-2F^*(F-F^*)+E^*(G-G^*). \end{aligned} \tag{3}$$

Clearly the desired inequality will follow from (3) if we can show that

$$G^*(E-E^*)-2F^*(F-F^*)+E^*(G-G^*)\geqq 0. \tag{4}$$

The numbers $G^*(E-E^*)$ and $E^*(G-G^*)$ are non-negative; hence $G^*(E-E^*)+E^*(G-G^*)\geqq 2[G^*(E-E^*)E^*(G-G^*)]^{1/2}$. But multiplication of (1) and (2) yields the inequality $[G^*(E-E^*)E^*(G-G^*)]^{1/2}\geqq|F^*(F-F^*)|$ and (4) now follows.

In the sequel it will be convenient to use $\mathfrak{x}$ and $\mathfrak{y}$ to represent vectors in Euclidean 3-space with components (x_1, x_2, x_3) and (y_1, y_2, y_3) respectively. The $x_1x_2x_3$ and $y_1y_2y_3$ spaces may or may not coincide. If the components of $\mathfrak{x}$ are functions of the parameters u and v, we shall indicate the fact by the notation $\mathfrak{x}(u, v)$. Then $\mathfrak{x}_u(u, v)$ and $\mathfrak{x}_v(u, v)$ will denote vector functions whose components are obtained from those of $\mathfrak{x}(u, v)$ by taking the indicated partial derivative. We shall also use the conventional notations:

$$E=\mathfrak{x}_u^2,\qquad F=\mathfrak{x}_u\cdot\mathfrak{x}_v,\qquad G=\mathfrak{x}_v^2,\qquad W=(EG-F^2)^{1/2}.$$

LEMMA 2. *If $\mathfrak{x}(u, v)$ is a Lipschitzian vector function on $D: u^2+v^2\leqq 1$, then, for a.e. $(u, v)\in D$,*

$$\left|\frac{\mathfrak{x}(u+r\cos\theta, v+r\sin\theta)-\mathfrak{x}(u, v)}{r}\right| \xrightarrow[r\to+0]{} |\mathfrak{x}_u(u, v)\cos\theta+\mathfrak{x}_v(u, v)\sin\theta|$$

for $0\leqq\theta\leqq 2\pi$.

PROOF. See Rademacher [4] for the case of a single-valued function.

The extension to a vector function is immediate.

THEOREM 1. *If* $T^*:\mathfrak{x}=\mathfrak{x}^*(u,v)$ *and* $T:\mathfrak{x}=\mathfrak{x}(u,v)$ *are Lipschitzian transformations defined on* $D:u^2+v^2\leq 1$ *such that* $|\mathfrak{x}^*(u_1,v_1)-\mathfrak{x}^*(u_2,v_2)|\leq|\mathfrak{x}(u_1,v_1)-\mathfrak{x}(u_2,v_2)|$ *for every pair of points* $(u_1,v_1),(u_2,v_2)\in D$, *then* $A(T^*)\leq A(T)$· *(where* $A(T^*)$ *and* $A(T)$ *are the Lebesgue areas of the Fréchet surfaces determined by* T^* *and* T).

PROOF. Since $|\mathfrak{x}^*(u+r\cos\theta,v+r\sin\theta)-\mathfrak{x}^*(u,v)|\leq|\mathfrak{x}(u+r\cos\theta,v+r\sin\theta)-\mathfrak{x}(u,v)|$, providing (u,v), $(u+r\cos\theta,v+r\sin\theta)\in D$, it follows from Lemma 2 that, a.e. in D,

$$|\mathfrak{x}^*_u(u,v)\cos\theta+\mathfrak{x}^*_v(u,v)\sin\theta|\leq|\mathfrak{x}_u(u,v)\cos\theta+\mathfrak{x}_v(u,v)\sin\theta|$$

for $0\leq\theta\leq 2\pi$. Squaring both sides of this inequality, we have, a.e. in D, $0\leq E^*\cos^2\theta+2F^*\cos\theta\sin\theta+G^*\sin^2\theta\leq E\cos^2\theta+2F\cos\theta\sin\theta+G\sin^2\theta$ for $0\leq\theta\leq 2\pi$. Thus, by Lemma 1, a.e. in D, $W^{*2}=E^*G^*-F^{*2}\leq EG-F^2=W^2$. Since W^* and W are non-negative, it follows that $W^*\leq W$ a.e. in D, and integration yields the desired inequality $A(T^*)=\iint_D W^*dudv\leq\iint_D Wdudv=A(T)$.

3. **The general case.** To establish the Kolmogoroff principle for the Lebesgue area in the general case, we shall need the following result concerning the extension of the range of definition of a Lipschitzian vector function.

LEMMA 3. *If* $L:\mathfrak{y}=\mathfrak{y}(x_1,x_2,x_3)$ *is a Lipschitzian transformation defined on a bounded set* B *in* $x_1x_2x_3$*-space, then* L *can be defined on the complement of* B *in such a way that the extended transformation is Lipschitzian on the whole space with the same Lipschitz constant.*

PROOF. See Kirszbraun [2].

LEMMA 4. *If* $T^*:\mathfrak{y}=\mathfrak{y}^*(u,v)$ *and* $T:\mathfrak{x}=\mathfrak{x}(u,v)$ *are any two bounded transformations defined on* $D:u^2+v^2\leq 1$ *such that* $|\mathfrak{y}^*(u_1,v_1)-\mathfrak{y}^*(u_2,v_2)|\leq|\mathfrak{x}(u_1,v_1)-\mathfrak{x}(u_2,v_2)|$ *for every pair of points* $(u_1,v_1),(u_2,v_2)\in D$, *then there exists a Lipschitzian transformation* $L:\mathfrak{y}=\mathfrak{y}(x_1,x_2,x_3)$, *with Lipschitz constant equal to one, which is defined on all of* $x_1x_2x_3$*-space and is such that* $T^*=LT$ *on* D.

PROOF. Let p be any point in $T(D)$. Consider the inverse set $T^{-1}(p)$. It is clear, since $|\mathfrak{y}^*(u_1,v_1)-\mathfrak{y}^*(u_2,v_2)|\leq|\mathfrak{x}(u_1,v_1)-\mathfrak{x}(u_2,v_2)|$ for every pair of points $(u_1,v_1),(u_2,v_2)\in D$, that under T^* every point in $T^{-1}(p)$ must have one and the same image p^*. Thus T^*T^{-1} is a Lipschitzian transformation from $T(D)$ to $T^*(D)$, with Lipschitz constant equal to one, such that $(T^*T^{-1})T=T^*$ on D. By Lemma 3, there exists a transformation L (the extension of T^*T^{-1}) satisfying the requirements of the lemma.

THEOREM 2. *If* $T^*: \mathfrak{y} = \mathfrak{y}^*(u, v)$ *and* $T: \mathfrak{x} = \mathfrak{x}(u, v)$ *are continuous transformations defined on* $D: u^2+v^2 \leq 1$ *such that* $|\mathfrak{y}^*(u_1, v_1) - \mathfrak{y}^*(u_2, v_2)| \leq |\mathfrak{x}(u_1, v_1) - \mathfrak{x}(u_2, v_2)|$ *for every pair of points* (u_1, v_1), $(u_2, v_2) \in D$, *then* $A(T^*) \leq A(T)$.

PROOF. Let L be the Lipschitzian transformation satisfying the requirements of Lemma 4 and let T_n be a sequence of Lipschitzian transformations defined on D and such that $T_n \rightarrow T$ uniformly on D and $A(T_n) \rightarrow A(T)$ (see Youngs [7], Helsel [1]). Define $T_n^* = LT_n$ on D. Then $T_n^* \rightarrow T^*$ uniformly on D and the Lipschitzian transformations T_n^* and T_n satisfy the hypotheses of Theorem 1. Therefore, $A(T_n^*) \leq A(T_n)$ and, by the lower semi-continuity of the Lebesgue area, $A(T^*) \leq \lim\inf A(T_n^*) \leq \lim A(T_n) = A(T)$.

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ANALYTIC CONTINUATION AND INFINITELY DIFFERENTIABLE FUNCTIONS

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It is our purpose to show that several branches of the theory of functions: detection of singularities, study of the behaviour of an analytic function in a strip or in an angle, a much more general theory than that of quasi-analyticity, convergence theorems, and so on, can all be based on a single principle, which may be regarded, in some ways, as a basic generalization of Cauchy's inequalities on Taylor coefficients.

This principle gives estimates of the d_n's in the series $\sum d_n e^{-\lambda_n s}$ by means of the maximum of the modulus of the function $F(s)$ it "represents." But the novelty of these evaluations lies in two essential facts: (1) It is supposed that the series represents the function in a certain horizontal strip-like region containing points with arbitrarily large abscissae, the series being, however, *not supposed a priori* to converge to the function anywhere. It has merely to "represent" the function with a certain "great precision," or, what amounts to the same thing, with a small error. (2) The evaluation is made by means of the maximum of the function in a circle which may lie as far to the left of the original region as the analytical continuation permits. The magnitude of the radius of this circle depends only on the distribution of the λ_n. The radius may be small if the λ_n are sparse.

We have given some results of the same nature in papers published a few years ago, but our latest results—the general inequality and its applications to infinitely differentiable functions—seem to be of the "best possible" kind, since "nearly" converse theorems are also proved. At any rate, they are much more general than those we published in 1944.

Before we try to state the results clearly it is necessary to introduce some definitions.

Let $\{\lambda_n\}$ be an increasing sequence of positive numbers. We introduce the function $N(\lambda)=\sum_{\lambda_n<\lambda}1$, which is the number of λ_n smaller than λ, the function $D(\lambda)=N(\lambda)/\lambda$—the density function, and $D^*(\lambda)=\text{l.u.b.}_{x\geq\lambda}D(x)$—the upper density function. The upper density function is obviously the smallest decreasing function larger than the density function.

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The quantity $D^* = \limsup_{\lambda\to\infty} D(\lambda) = \lim_{\lambda\to\infty} D^*(\lambda)$ is called the upper density of $\{\lambda_n\}$. We shall suppose that $D^* < \infty$.

It will be convenient to introduce also the mean density function: $\overline{D}(\lambda) = \lambda^{-1}\int_0^\lambda D(x)dx$, the mean upper density function $\overline{D}^*(\lambda) = \text{l.u.b.}_{x\geq\lambda}\overline{D}(x)$, and the mean upper density of $\{\lambda_n\}$: $\overline{D}^* = \limsup \overline{D}(\lambda) = \lim \overline{D}^*(\lambda)$. The inequality $\overline{D}^* \leqq D^*$ always holds, but examples show that the inequality $\overline{D}^* < D^*$ may actually hold.

Let Δ be a region in the $s = \sigma + it$ plane containing points with σ positive and arbitrarily large. Let $F(s)$ be a function holomorphic in Δ, and let $\{d_n\}$ be a sequence of real or complex numbers. We shall write $R_m(s) = F(s) - \sum_1^m d_k e^{-\lambda_k s}$. If n is a given positive integer we say that $F(s)$ is represented in Δ by the set of Dirichlet polynomials $\sum_1^m d_k e^{-\lambda_k s}$, with $m \geqq n$, with logarithmic precision $p(\sigma)$, if for s belonging to Δ, and x sufficiently large the inequality holds:

$$\underset{m\geq n}{\text{g.l.b.}}\ (\underset{\sigma\geq x}{\text{l.u.b.}}\ |R_m(s)|) \leqq e^{-p(x)}.$$

We shall always suppose that the logarithmic precision is a continuous function increasing to $+\infty$ (or eventually identically equal to $+\infty$ for σ large).

If x is sufficiently large, the difference between $\sum_1^m d_k e^{-\lambda_k s}$ and $F(s)$ can be made, in absolute value, smaller, in the part of Δ in which $\sigma \geqq x$, than any quantity $\delta > e^{-p(x)}$, provided m (larger than n) is suitably chosen.

If the series converges to the function, the logarithmic precision can be taken equal to $+\infty$ for each $n > 0$.

The converse is not generally true. But, if the logarithmic precision is equal to $+\infty$ (for σ large) the series overconverges to $F(s)$ in Δ.

A channel of width $2R$ is the union of circles $C(s', R)$ ($|s - s'| < R$), the radius R being fixed, the centers s' taking all the values on a Jordan arc. The Jordan arc itself is the central line of the channel.

If a function $F(s)$ is holomorphic in the region composed of two regions Δ_1 and Δ_2, and a channel C such that one of the circles of C lies in Δ_1, another in Δ_2, we say that $F(s)$ can be continued analytically from Δ_1 to Δ_2 through the channel C.

The idea of the fundamental theorem is the following one:

If the sums $\sum_1^m d_k e^{-\lambda_k s}$ ($m \geqq n$) represent $F(s)$, in a region sufficiently expansive (when measured vertically, for σ large), with a sufficiently great logarithmic precision, and if the function can be continued analytically through a channel sufficiently wide to a circle $C(s_0, r)$, then $|d_n|$ is smaller than a certain quantity which depends on the following elements: (1) the abscissa of s_0, (2) the radius r, (3) the maximum of $|F(s)|$ in $C(s_0, r)$, (4) the sequence $\{\lambda_n\}$ by means of the quantity

$$\Lambda_n = \lambda_n \prod_{m \neq n} \frac{\lambda_m^2}{|\lambda_n^2 - \lambda_m^2|}.$$

Let us now state the theorem rigorously:

FUNDAMENTAL THEOREM. *Let* $\{\lambda_n\}$ *be a positive increasing sequence with upper density* $D^* < \infty$. *Let* Δ *be a region in the* $s = \sigma + it$ *plane defined by*: $\sigma > a$, $|t| < \pi g(\sigma)$ *where* $g(\sigma)$ *is a function of bounded variation (in* $[a, \infty)$), *such that* $g(\sigma) > D^*$, $\lim g(\sigma) > D^*$.

Let $F(s)$ *be holomorphic in* Δ, *and suppose that* $F(s)$ *can be continued analytically through a channel wider than* $2\pi D^*$ *to a circle* $C(s_0, \pi R)$ $(R > D^*)$.

Suppose that the sequence $\{d_k\}$ *is such that, for a given integer* $n > 0$, *the Dirichlet polynomials* $\sum_1^m d_k e^{-\lambda_k s}$ *with* $m \geqq n$ *represent* $F(s)$ *in* Δ *with a logarithmic precision* $p(\sigma)$ *satisfying the relationship*:

$$\text{(P)} \qquad \int^{\infty} p(\sigma) \exp\left[-2^{-1}\int^{\sigma} (g(u) - \overline{D}^*(p(u))^{-1}du\right] d\sigma = \infty.$$

Then

$$\text{(I)} \qquad |d_n| \leqq A(R)\Lambda_n M(s_0, R) \exp [\lambda_n \Re(s_0)]$$

where $A(R)$ *depends only on* R,[1] *and where* $M(s_0, R) = \max |F(s)|$ $(s \in C(s_0, \pi R))$.

In this theorem one may replace D^* by $\overline{D}^*$ and $D^*(\lambda)$ by $\overline{D}^*(\lambda)$. If the sequence $\{\lambda_n\}$ is such that $\overline{D}^* < D^*$, then with these replacements the hypotheses become less restrictive.

The condition (P) seems to be the most involved but as matter of fact, from a certain point of view it is, if not the best possible condition, at least an "almost best possible" one. We shall see the meaning of this assertion later on. But from now on we may notice that the theorem ceases to be true if, in (P), $D^*(p(\sigma))$ is replaced by its limit D^* (such a condition would be less restrictive than (P)).

If the series $\sum d_k e^{-\lambda_k s}$ converges to $F(s)$ for σ sufficiently large, then $p(\sigma) \equiv \infty$ (for σ large) and the relationship (P) is satisfied automatically. In this case $g(\sigma)$ can be taken as large as possible, and the theorem is then reduced to the following statement:

If $\sum d_k e^{-\lambda_k s}$, *with* $D^* < \infty$, *converges for* σ *large to a function* $F(s)$ *which can be continued analytically, through a channel wider than* $2\pi D^*$,

---

[1] For a given sequence $\{\lambda_n\}$.

to a circle $C(s_0, \pi R)$ $(R>\bar{D}^*)$, *then the evaluation* (I) *holds for each* $n\geqq 1$.

Let us, for a moment, limit ourselves to the case in which the series converges to the function for large values of σ $(\sigma>\sigma_0)$. It follows then from the theorem that if $F(s)$ can be continued analytically from the half-plane of convergence, through channels wider than $2\pi D^*$, to circles $C(s_i, \pi R)$ with $\lim_{i\to\infty}\Re(s_i)=-\infty$, $F(s)$ being bounded in the set of these circles, then $F(s)$ is identically zero.

If, for instance, $F(s)$ *is holomorphic and bounded in a strip* $|t|<\pi C$ *with* $C>\bar{D}^*$, *then* $F(s)$ *is identically zero.*

On introducing, moreover, the theory of normal families, we get the following result:

If the series $\sum d_l e^{-\lambda_l s}$ *with* $D^*<\infty$ *converges, for* σ *large, to a function* $F(s)$ *non-identically zero, then in each strip* $\Delta: |t|<\pi C$, *with* $C>\bar{D}^*$, $F(s)$ *satisfies one of the three conditions*:

(*a*) $F(s)$ *admits there at least one singularity*,

(*b*) $F(s)$ *tends to infinity as* $\lim \sigma=-\infty$ *uniformly with respect to* t, s *belonging to the strip* $|t|<\pi(C-\epsilon)$, *with* ϵ *arbitrarily such that* $0<\epsilon<C$.

(*c*) $F(s)$ *takes in* Δ *each value, except at most one, infinitely many times.*

Many years ago, Ritt defined the order of a function given by a Dirichlet series convergent everywhere by:

$$\rho_R=\limsup_{\sigma=-\infty}\frac{\log_2 M(\sigma)}{-\sigma}$$

where

$$M(\sigma)=\underset{-\infty<t<\infty}{\text{l.u.b.}}\ |F(\sigma+it)|.$$

$F(s)$ being an integral function, let us set

$$M_C(\sigma)=\underset{|t|<\pi C}{\text{l.u.b.}}\ |F(\sigma+it)|,$$

$$\rho_R(C)=\limsup_{\sigma=-\infty}\frac{\log_2 M_C(\sigma)}{-\sigma}.$$

We shall call ρ_R the order (*R*) of $F(s)$ in the plane, and $\rho_R(C)$ the order (*R*) in the strip $|t|<\pi C$.

If $\liminf(\lambda_{n+1}-\lambda_n)=h>0$, then it is proved that

$$\limsup_{n=\infty}\frac{\log A_n}{\lambda_n}\leq A(h, D^*),$$

where $A(h, D^*)$ is a quantity tending to zero, when h is constant, D^* tending to zero.

It is then easy to see from our general inequalities that if $C > \overline{D}^*$, $h > 0$, then $\rho_B(C) = \rho_B$.

In other words: *the order* (R) *of a function given by a Dirichlet series convergent everywhere is, in each horizontal strip larger than* $2\pi\overline{D}^*$, *equal to its order* (R) *in the whole plane.*

This, together with a theorem proved by Bieberbach and Valiron (concerning the growth of functions in an angle), allows us to prove that *if* $D^* < \infty$, $h > 0$, *and if* $\sum d_n e^{-\lambda_n s}$ *represents an integral function* $F(s)$ *which is not a Dirichlet polynomial, then in each strip wider than* $2\pi \max\,(\overline{D}^*, (2\rho)^{-1})$ *there exists a straight line* $t = t_0$ *such that in each horizontal strip containing this line* $F(s)$ *takes each possible value, except at most one, infinitely many times.*

With λ_n integers and $\rho = \infty$, this result contains a theorem of Pólya. The line $\arg z = t_0$ is then a "line of Julia" of the function $\sum d_n z^{\lambda_n}$.

In a more restrictive form, these particular results concerning convergent Dirichlet series were already obtained by Gergen and myself many years ago.

On supposing still that the series admits an axis of convergence, let us see how the fundamental theorem gives results on singularities of functions represented by Dirichlet series. It follows from the general inequalities that if σ_C is the abscissa of convergence of the series then

$$\sigma_C = \limsup_{n \to \infty} \frac{\log |d_n|}{\lambda_n} \leqq \limsup_{n \to \infty} \frac{\log \Delta_n}{\lambda_n} + \Re(s_0).$$

In other words, if $h > 0$, and if $F(s)$ can be continued analytically through a channel wider than $2\pi\overline{D}^*$ to a circle $|s - s_0| < \pi R$ $(R > \overline{D}^*)$, then

$$\Re(s_0) \geqq \sigma_C - A(h, D^*).$$

This leads immediately to a theorem of Ostrowski by which:

For each $h > 0$ *there exists a function* $B_h(D^*)$, *with* $\lim_{D^* \to 0} B_h(D^*) = 0$, *such that in each circle with center at* $\sigma = \sigma_C$ *and radius* $B_h(D^*)$ *the function* $F(s)$ *admits a singularity.*

If, in particular, $D^* = 0$, we must have $\Re(s_0) \geqq \sigma_C$, the width of the channel being arbitrarily small. *This shows that if* $\lim n/\lambda_n = 0$ $(h > 0)$, *then each point on* $\sigma = \sigma_C$ *is a singular point.* For Taylor series (λ_n integers) this is the famous Fabry's "gap theorem." For Dirichlet series it was proved by Szász. Other theorems on singularities can be obtained.

Of course, the theorems on bounded functions in a strip or in a set of circles (theorems of Liouville's type), theorems of Weierstrass and Picard's type, still hold, if, instead of supposing that the series converges, we suppose merely that it represents the function in a region Δ (as in the fundamental theorem) with a sufficiently large logarithmic precision (the condition (P) to be satisfied).

It may seem unnecessary to generalize results on convergent series to asymptotic ones. But we shall now show that our general theorem, in which it is not supposed that the series converges (instead, the condition (P) is satisfied), gives results on infinitely differentiable functions, results which can be obtained *only* because the series is not supposed a priori to converge.

The link between the fundamental theorem and the theory of infinitely differentiable functions is given by a lemma which we shall state next. We must, however, first introduce a notion connected with infinitely differentiable functions.

If $f(x)$ is an infinitely differentiable function on the interval I $(x \geqq 0)$, such that $\text{l.u.b.}_{x\geqq 0}|f^{(n)}(x)| = m_n < \infty$, we shall call the function $C_f(\sigma) = \text{l.u.b.}_{n\geqq 1}(n\sigma - \log m_n)$ the character of the function $f(x)$.

We prove the following lemma:

Let $f(x)$ be an infinitely differentiable function in I $(x \geqq 0)$ such that $|f^{(n)}(x)| \leqq M_n < \infty$ $(n \geqq 0)$. Let $\{\nu_n\}$ be a sequence of non-negative integers such that

$$f^{(\nu_n)}(0) = 0 \qquad (n \geqq 1).$$

Let $\{q_n\}$ be the sequence of non-negative integers complementary to $\{\nu_n\}$.

For each real a the function

$$F_a(s) = \int_0^\infty \exp\left[-xe^{s-a}\right] f(x)dx$$

is holomorphic in the region $\bar{\Delta}$ defined by $e^\sigma \cos t \geqq 1$. In this region the inequality holds: $|F_a(s)| < M_0 e^a$; and for each $n \geqq 1$; $F_a(s)$ is represented in $\bar{\Delta}$ by $\sum_1^m d_k^{(a)} e^{-\lambda_k s}$ with $m \geqq n$, where $d_k^{(a)} = e^{\lambda_k a} f^{(q_k)}(0)$, $\lambda_k = q_k + 1$, with the logarithmic precision equal to $C_f(\sigma - a) - a$, where $C_f(\sigma)$ is the character of $f(x)$.

It is clear that this lemma together with our fundamental theorem allows us to evaluate the derivatives of infinitely differentiable functions at the origin, if sufficiently many of these derivatives are supposed a priori to be zero (at the origin), and if the character of the function is sufficiently large. As a matter of fact, the evaluations so obtained depend on the quantity a in such a way that they enable us to prove, by making a tend to infinity, that all the derivatives are zero at the origin (if many of them are supposed to be zero, and if the character is large).

Here is the principal theorem concerning infinitely differentiable functions, obtained on combining the fundamental theorem with the lemma.

Let $f(x)$ *be an infinitely differentiable function on* I $(x \geqq 0)$, *bounded with all its derivatives on* I. *Let* $\{\nu_n\}$ *be a sequence of non-negative integers with* $\nu_1 = 0$, *such that the upper density* D^* *of its complementary sequence* $\{q_n\}$ *satisfies the inequality* $D^* < 1/2$. *Let* $C_f(\sigma)$ *denote the character of* $f(x)$, *and let* $D^*(q)$ *be the upper density function of* $\{q_n\}$. *If*

$$(Q) \qquad \int^{\infty} C_f(\sigma) \exp\left[-\int^{\sigma} (1 - 2D^*(C_f(u)))^{-1} du\right] d\sigma = \infty,$$

and if $f^{(\nu_k)}(0) = 0$ $(k \geqq 1)$, *then* $f(x)$ *is identically zero.*

Of course, in this theorem, too, D^* and $D^*(q)$ can be respectively replaced by $\bar{D}^*$ and $\bar{D}^*(q)$ (the mean upper density and the mean upper density function of $\{q_n\}$).

It is obvious that this theorem contains, as a very particular case, the classical theorem on quasi-analyticity. If, indeed, we suppose that all the derivatives are zero at the origin, that is to say, that the sequence $\{\nu_n\}$ is composed of all the non-negative integers, then $D^*(q) \equiv 0$, and the theorem becomes:

If

$$f^{(n)}(0) = 0 \quad (n \geqq 0), \qquad |f^{(n)}(x)| \leqq M_n < \infty \quad (n \geqq 0,\ x \geqq 0)$$

and if

$$\int^{\infty} C_f(\sigma) e^{-\sigma} d\sigma = \infty,$$

then $f(x) \equiv 0$.

In particular if $f^{(n)}(0) = 0$, $|f^{(n)}(x)| \leqq M_n$ $(n \geqq 0)$, and if, on setting

$$A(\sigma) = \underset{n \geqq 1}{\text{l.u.b.}} (n\sigma - \log M_n),$$

we have:

$$(Q') \qquad \int^{\infty} A(\sigma) e^{-\sigma} d\sigma = \infty,$$

then $f(x) \equiv 0$. This is exactly another wording of the classical theorem (Denjoy-Carleman) on quasi-analyticity (Ostrowski's form).

However, not only does our theorem require merely $D^* < 1/2$ in order to enable us to affirm that $f(x) \equiv 0$, if a suitable condition on $C_f(\sigma)$ is satisfied (condition (Q)), but, moreover, even if the classical condition is satisfied (condition (Q')) we do not need to suppose that *all* the derivatives are zero at the origin in order to conclude that $f(x) \equiv 0$.

For instance, the functions f such that

$$|f^{(n)}(x)| < K^n (n \log n)^n \qquad (n \geqq 2)$$

constitute a quasi-analytic class. By the classical theorem one can affirm that $f(x) \equiv 0$ only if it is supposed that $f^{(n)}(0) = 0$ $(n \geqq 0)$. But, since $C_f(\sigma) > Ce^{\sigma}/\sigma$ we see, by our theorem, that if we suppose $f^{(\nu_n)}(0) = 0$ $(n \geqq 1,\ \nu_1 = 0)$, the complementary sequence $\{q_n\}$ being such that $q_n > an \log n \log_2 n$, we still may conclude that $f(x) \equiv 0$.

That the theorem on generalized quasi-analyticity cannot be much improved is shown by the following example: If in the just quoted example we replace the inequality $q_n > an \log n \log_2 n$ by $q_n > an \log n$ then the assertion that $f(x) \equiv 0$ is not any more true.

As a matter of fact, the following theorem can be proved:

If the sequence of positive integers $\{q_n\}$ *complementary to the sequence of non-negative integers* $\{\nu_n\}$ $(\nu_1 = 0)$ *is such that* $\lim \log q_n / n = 0$, *then there exists a function* $f(x)$ *not identically zero on* I $(x \geqq 0)$, *infinitely differentiable and bounded on* I *such that* $f^{(\nu_n)}(0) = 0$ *and such that*

$$|f^{(n)}(x)| \leqq q_1 q_2 \cdots q_n \qquad (n \geqq 1,\ x \geqq 0).$$

The proof of the fundamental theorem is difficult and technical and it would be impossible to indicate here even its main outlines. But it may be interesting to mention a theorem which plays an important role in the proof of the most general case.

The theorem generalizes a classical one which solves the Watson problem in a half-plane. As a matter of fact, this classical theorem served us as a tool when our statement of the fundamental theorem was less general.

Let $\Phi(s)$ *be holomorphic in a region* Δ *given by* $\sigma > a$, $|t| < G(\sigma)$, *where* $G(\sigma)$ *is a function of bounded variation with* $\lim G(\sigma) > 0$, *and let* $\Phi(s)$ *be continuous and bounded in* $\bar{\Delta}$. *Let* $N(\sigma)$ *be an increasing function.*

If

$$(\alpha) \qquad \log |\Phi(\sigma + iG(\sigma))| < -N(\sigma),$$

and if on setting, for $d>a$,

$$S(\sigma) = \frac{\pi}{2}\int_d^\sigma \frac{du}{G(u)},$$

we have

$$(\beta) \qquad \int^\infty N(\sigma)e^{-S(\sigma)}d\sigma = \infty,$$

then the function $\Phi(s)$ *is identically zero.*

This theorem has its converse.

If, $N(\sigma)$ *being an increasing function,* (β) *does not hold, then there exists a function* $\Phi(s)$, *not identically zero, holomorphic and bounded in* $\bar{\Delta}$ *and such that* (α) *holds.*

These two theorems were proved in a paper by G. R. MacLane and myself. If $G(\sigma)=\pi/2$, $a=-\infty$, the theorem gives the classical solution of a classical problem (generally stated for a half-plane $z=e^s$).

Let us mention that very recently we have proved jointly with Norbert Wiener results concerning infinitely differentiable functions of a character similar, if not easily comparable, to the theorem on the same subject stated above. The methods we used with Wiener are quite different from those I used to prove the theorem mentioned here. We employ an auxiliary function, holomorphic in the half-plane, with an argument corresponding to an order of differentiation, real or complex, with non-negative real part, and a value given by the value of the derivative of that order at the origin of a given infinitely differentiable function.

Let us notice that the fundamental theorem gives also conditions for convergence of Dirichlet series.

Suppose that $F(s)$ is holomorphic in a region described in the statement of the fundamental theorem, and let in this region

$$(\mathrm{C}) \qquad \left| F(s) - \sum_1^n d_k e^{-\lambda_k s} \right| \leqq M_n e^{-\lambda_n \sigma}.$$

It is then readily seen that for each integer n, $\sum_1^m d_k e^{-\lambda_k s}$, with $m \geqq n$, represents $F(s)$ in Δ with the logarithmic precision equal to:

$$(\mathrm{M}) \qquad p(\sigma) = \underset{n\geq 1}{\mathrm{l.u.b.}}\ (\lambda_n\sigma - \log M_n).$$

Therefore, if, with the notations just introduced, (P) holds, then

(I) holds for each $n>0$. It can then be easily shown that *if* (C) *and* (P) *hold* (with $p(\sigma)$ defined by (M)), *and if* $h>0$, *then* $\sum d_n e^{-\lambda_n s}$ *converges to* $F(s)$ *for* σ *sufficiently large.*

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COLLÈGE DE FRANCE AND RICE INSTITUTE

GLOBAL THEOREMS IN RIEMANNIAN GEOMETRY

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1. **Introduction.** I come to you today disguised as a missionary in an effort to interest you in the present day activities of differential geometers. The chief stimulus to differential geometry in the present century has been the General Theory of Relativity which is written in the language of Riemannian Geometry. Much of the effort of differential geometers in the twenties and thirties was directed toward generalizations of Riemannian Geometry which might lead to the elusive Unified Field Theory. With the virtual exhaustion of efforts in this direction several leading American differential geometers turned their attention to other fields of mathematics, others retired from active research, and the study of differential geometry in this country dropped to its lowest point in many years.

As this situation was deteriorating at home, a new trend in differential geometry was taking root abroad, namely the study of differential geometry "in the large." The general problem proposed was to develop relationships between the local differential properties of a space and its topologic structure as a whole. Leaders in this movement were H. Hopf and his pupils in Switzerland, de Rham in France, and Hodge in England. Contributions were made by various writers in the United States including Myers, Chern, Weil, and Allendoerfer; and the field has now matured to a point at which its general scope can be outlined and its major problems enumerated. It is now an active frontier in mathematics, ripe for development, and full of interest to mathematicians in numerous branches of our subject. Furthermore its investigation raises new questions in ordinary differential geometry, and these give promise of providing the stimulus needed to revive activity in differential geometry throughout this country.

Time is too short today to give a systematic survey of this field;[1] so I shall sketch the status of some of the major problems and give details of several topics which should serve as illustrations of the methods employed and the type of results that can be obtained.

2. **The main problems.** The raw material of the subject is a dif-

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[1] Such an article will appear in the author's detailed report on the session on differential geometry of the Princeton Bicentennial Conference on Mathematics.

ferential manifold, namely a topological manifold defined in terms of neighborhoods each supplied with a coordinate system such that the coordinates of overlapping neighborhoods are differentiable functions of each other (with nonvanishing Jacobian) within the common region. The order of differentiability involved (usually called the class C^s) varies with the problem to be considered; but except for a few cases which require analyticity, fourth order partial derivatives are usually sufficient. On this manifold is imposed a Riemann metric

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

where $g_{\alpha\beta}$ is usually required to be positive definite. Such a manifold is called "complete" if it is complete in the usual sense; that is, if every Cauchy sequence converges; and it is proved that complete manifolds are also locally compact. Five main problems have arisen concerning such manifolds:

A. *The problem of continuation.* Here one is given an n-cell provided with an analytic Riemann metric, and the problem is *to determine those complete manifolds (if any) to which this element can be continued.* The chief contributions to this problem have been made by Myers [8].[1] He has shown that not every analytic Riemann element can be continued to a complete manifold and has given a number of necessary conditions and some sufficient conditions. A definitive set of necessary and sufficient conditions has yet to be found. Myers has also proved the following uniqueness theorem:

THEOREM. *Every n-dimensional Riemann element can be continued to at most one complete, simply-connected n-dimensional manifold M; that is, if two such continuations exist, they are isometric.*

B. *The problem of metrisation.* In a sense this is the inverse of the problem of continuation, namely: *Given a manifold, what Riemann metrics (analytic or otherwise) can be used to metrise the entire manifold?* The final solution of this question must await, among other things, the development of definitive topologic criteria for identifying homeomorphic manifolds. But even in the two-dimensional case where the topologic problem is solved, the problem of metrization is still open. Developments so far have been concerned (except in trivial cases) with necessary conditions. These seek to establish relationships between the differential invariants of the Riemann metric and the topology of the space. Connections of this sort have been made with the Euler-Poincaré characteristic, cohomology characteristic classes,

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Betti numbers, and the fundamental group. Since I shall return to certain aspects of this problem, for the moment I shall pass on to a statement of the other problems.

C. *The problem of imbedding.* Here we are given a Riemann manifold, M^n, and ask the question: *Is M^n isometrically imbeddable in a Euclidean space? If so, what is the least dimensional Euclidean space for which the imbedding is possible? When the imbedding is possible, is it unique?* Results in this field are most fragmentary, and consist chiefly of negative information and uniqueness theorems. For example, Hilbert has shown the impossibility of imbedding the hyperbolic plane in Euclidean three-space; but the fundamental questions stated above are unanswered for this manifold.

The uniqueness theorems state that an imbedding, if possible, is unique to within rigid motions or reflections. For two-dimensional surfaces in Euclidean three-space, Weyl [13] has shown that every convex surface is uniquely imbedded. For higher dimensions Beez [3] has shown that unicity of imbedding of M^n in a Euclidean space of $n+1$ dimensions follows from the hypothesis that the second fundamental form of M^n is of rank $\geqq 3$. His results were extended by Allendoerfer [1] to cover imbeddings in Euclidean spaces of arbitrary dimensions. These theorems, however, hold in the small as well as in the large, and the question of the unicity of imbedding in the large under more general hypotheses is still unanswered.

D. *The problem of geodesics.* In general terms this problem may be stated: *What are the properties in the large of geodesics lying in a complete manifold?* To be more specific consider the following question: If we have a geodesic, C, passing through a point P of a complete manifold, how far can C be extended from P before it intersects itself or another geodesic through P, and how far along its path does C measure a minimum distance from P? Considerable attention has been directed to the description of the loci of minimum and conjugate points of P. The results of these investigations are intended (among other things) to answer the question as to the maximum size of a normal coordinate system centered at a point P. Full results for $n=2$ have been obtained by Myers [9, 10], but a number of questions remain open for general values of n.

E. *Metric theorems in the large.* A number of theorems of differential geometry deal with properties of the whole of a curve or higher-dimensional subspace. Many of these were developed initially for plane curves, but when these are generalized to curves in Riemann manifolds it becomes essential to study the topology of these manifolds. Indeed without such study it is not possible to give an efficient or properly general statement of the theorem. As an example, I cite the well known theorem of Steiner on parallel curves. Let C be a closed, convex plane curve and let C_ρ be a curve outside C parallel to it at a distance ρ; that is, C_ρ is obtained by displacing the points of C a distance ρ along outward drawn normals to C. If L is the length of C and L_ρ the length of C_ρ, Steiner showed that

$$L_\rho = L + 2\pi\rho.$$

The convexity hypothesis may be replaced by the equivalent requirement that no two normals to C intersect outside C. However, when we turn to a right circular cylinder (which has the same local differential geometry as a plane), the theorem is no longer true. For a circular cross section of the cylinder and its parallels all have equal lengths. The difference is that the topology of a cylinder is not that of a plane. The theorem, however, is true on a cylinder if we restrict C to curves which bound a finite area. Other theorems of this nature are the isoperimetric theorem, and the four vertex theorem.

3. **The Kronecker index.** Returning now to the problem of metrization, I wish to discuss certain aspects of this problem and to describe several tools which have proved useful in attacking it. The first of these is the Kronecker index of a vector field on a closed hypersurface S^n of a manifold M^{n+1}. When M^{n+1} is Euclidean this index takes the form:

$$I = \frac{1}{\omega^n}\int_{S^n} D\,dx^1 \cdots dx^n \tag{1}$$

where

$$D = \begin{vmatrix} V^1 & \cdots & V^{n+1} \\ \dfrac{\partial V^1}{\partial x^1} & \cdots & \dfrac{\partial V^{n+1}}{\partial x^1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \dfrac{\partial V^1}{\partial x^n} & \cdots & \dfrac{\partial V^{n+1}}{\partial x^n} \end{vmatrix},$$

$x^1, \cdots, x^n$ are coordinates on S^n, and V^i is a unit vector field in the Euclidean space having no singularities on S^n. ω^n is the area of an n-dimensional sphere. The useful properties of I follow from the fact that it can be proved to be an integer; and this integer is shown to equal the algebraic sum of the orders of the singularities of V^i within S^n.

For present purposes the most useful result is obtained when n is even and V^i is the unit normal vector field to S^n in the Euclidean space. Then $I=\chi/2$ where χ is the Euler-Poincaré characteristic of S^n. Moreover in this case the integrand in (1) is expressible in terms of the determinants of the first and second fundamental forms of S^n, namely $g_{\alpha\beta}$ and $b_{\alpha\beta}$. Hence

$$\frac{\chi}{2}=\frac{1}{\omega^n}\int_{S^n}\frac{|b_{\alpha\beta}|}{|g_{\alpha\beta}|}(|g_{\alpha\beta}|)^{1/2}dx^1\cdots dx^n. \tag{2}$$

The ratio $|b_{\alpha\beta}|/|g_{\alpha\beta}|$ is called the "total curvature," K_T, of S^n since it represents the product of the n principal curvatures of the hypersurface. Since we have assumed that n is even, K_T can be expressed solely in terms of the intrinsic metric of S^n, namely in terms of $g_{\alpha\beta}$ and their derivatives. This generalization of Gauss' "Theorema Egregium" follows from the equations of Gauss:

$$R_{\alpha\beta\gamma\delta}=b_{\alpha\gamma}b_{\beta\delta}-b_{\alpha\delta}b_{\beta\gamma} \tag{3}$$

where $R_{\alpha\beta\gamma\epsilon}$ is the Riemann tensor, an algebraic expression in $g_{\alpha\beta}$ and their derivatives. The result of this substitution is:

$$K_T=\frac{R_{\alpha_1\alpha_2\beta_1\beta_2}R_{\alpha_3\alpha_4\beta_3\beta_4}\cdots R_{\alpha_{n-1}\alpha_n\beta_{n-1}\beta_n}e^{\alpha_1\cdots\alpha_n}e^{\beta_1\cdots\beta_n}}{n!2^{n/2}|g_{\alpha\beta}|} \tag{4}$$

and hereafter K_T will stand for this expression, even when we do not assume S^n to be a subspace of any Euclidean space. Thus we have an integral relationship

$$\int_{S^n}K_Tg^{1/2}dx=\omega^n\cdot\frac{\chi}{2} \qquad (n \text{ even}) \tag{5}$$

between the intrinsic differential invariant K_T and the topologic invariant χ. Regarding S^n as a given topologic manifold, this becomes a necessary condition upon the Riemann metrics which may be used to metrize it, and hence this is a contribution to the problem of metrization.

The validity of (5) so far has been established only under the assumption that S^n is a hypersurface of a Euclidean space; but its intrinsic form suggests that it may be true for any closed differentiable manifold. This is indeed the case, and it should be remarked at this point that an excellent way of discovering relations like (5) is first to consider an imbedded manifold and then later to invent a new proof applicable to a general abstract manifold.

In order to establish the general validity of (5) and to illustrate further the Kronecker index, we shall follow a method due to Chern [5] in obtaining this result, which was first proved by Allendoerfer-Weil [2] in a more complicated fashion. It is well known that the Euler-Poincaré characteristic of a closed manifold M^n is equal to the algebraic sum of the orders of the singularities of an otherwise continuous vector field on M^n. By contrast to the normal vector field used above, these vectors are intrinsic, or tangent, vectors. It therefore appears desirable to find a generalization of (1) valid on Riemann manifolds. Following the usual procedure for making such generalizations, we replace the partial derivatives in (1) by covariant derivatives and introduce components of the Riemann tensor at appropriate places. However, there are so many ways in which the Riemann tensor can be introduced, that it is not easy to guess the correct form of the final answer. In order to do this systematically we consider for the moment only manifolds M^n which are hypersurfaces of an $(n+1)$-dimensional Euclidean space. Using the method of tubes as developed by Allendoerfer-Weil we arrive at the following expression for I:

$$I = \lim_{S^{n-1}\to P} \int_{S^{n-1}} \phi\, dx^1 \cdots dx^{n-1} \tag{6}$$

where

$$\phi = \frac{1}{\omega^n} \int_{-\pi/2}^{\pi/2} \Delta(\theta)\, d\theta \tag{7}$$

where

$$\Delta(\theta) = \begin{array}{|c|l} \multicolumn{1}{c}{\longleftarrow n \text{ columns} \longrightarrow} & \\ V^i & 1 \text{ row} \\ [\cos\theta V^i_{,j} - \sin\theta g^{ik} b_{kj}] \dfrac{\partial y^j}{\partial x^\alpha} & n-1 \text{ rows.} \end{array}$$

In (6) and (7) P is an isolated singularity of the unit vector field V^i; S^{n-1} is an $(n-1)$-dimensional topologic sphere enclosing P; y^i are coordinates on M^n; x^α are coordinates on S^{n-1}; g_{ij} and b_{ij} are the first and second fundamental forms of M^n relative to the Euclidean space in which it lies; $V^i_{,j}$ is the covariant derivative of V^i relative to the metric g_{ij}; and for convenience in the sequel the dimensions have been taken to be one lower than those of the corresponding formula (1).

At first sight ϕ does not appear to be intrinsic relative to M^n. However, after the integration the elements b_{ij} appear only in second order minors of the matrix $\|b_{ij}\|$, and hence they can be expressed in terms of the Riemann tensor of M^n by equations (3). The result of this substitution is a very complex expression which is omitted here (see Chern [5]). This expression has been derived on the assumption that M^n is imbedded in a Euclidean space; but now that it has been invented, we use it to define ϕ for all manifolds M^n whether imbedded or not. In all cases I can be proved to be an integer which represents the order of the singularity of V^i at P.

To find χ we then take the sum of the indices at each singularity, and the resulting sum can be expressed as an integral over M^n by virtue of the generalized form of Stokes' Theorem which states that:

$$\int_B \phi = \int_R d\phi \tag{8}$$

where R is a region of M^n and B is its boundary. It turns out that

$$d\phi = \frac{2K_T}{\omega^n} \quad (n \text{ even}) \tag{9}$$

$$= 0 \quad (n \text{ odd})$$

independently of the vector field V^i from which we started. This establishes (5) in general. In this form (5) is a restricted case of the generalized Gauss-Bonnet Formula for Riemannian Polyhedra developed by Allendoerfer-Weil [2].

4. **Problems growing out of the Gauss-Bonnet formula.** As I stated at the outset, the development of such a result leads to new problems both in differential geometry and topology. The first of these concerns the analogous results for open manifolds. Cohn-Vossen [6] has proved that

$$\int_{M^n} K_T g^{1/2} dx \leqq \omega^n \cdot \frac{\chi}{2} \tag{10}$$

for two-dimensional open manifolds without boundary provided that the left-hand integral exists as an improper integral. The problem for n-dimensional open manifolds is unsolved. Its solution appears to require on the one hand a study of the topologic properties of vector fields defined on open manifolds, and on the other hand the additional study of the differential geometry of the expression ϕ. To illustrate one of the topologic difficulties I might mention that Cohn-Vossen makes essential use of the fact that any finitely connected two-dimensional open manifold (without boundary) is homeomorphic to a closed manifold from which a finite number of points have been removed. No suitable generalization of this result for n dimensions appears to be known.

The expression ϕ appears in the work of Allendoerfer-Weil and that of Chern, but its full meaning is yet to be uncovered. As a partial step in this direction I have noted that:

$$A^{n-1} = \int_{R^{n-1}} \phi dx^1 \cdots dx^{n-1} \tag{11}$$

is a proper definition of the solid angle of the vector field V^i relative to a hypersurface R^{n-1} of M^n. R^{n-1} may be the whole of a closed hypersurface or may be any $(n-1)$-dimensional subset of a hypersurface. Because of the role which solid angle plays in physical field theories in ordinary space, it is likely that this expression may have applications to similar theories in general Riemann spaces.

Let the hypersurface R^{n-1} have a boundary B^{n-2} and let $\bar{R}^{n-1}$ be another hypersurface with the same boundary such that the vector field V^i has no singularity in the volume enclosed by these two hypersurfaces. If the manifold M^n is Euclidean, it is known that the solid angle of V^i relative to R^{n-1} is equal to that of V^i relative to $\bar{R}^{n-1}$ and hence depends solely on the boundary B^{n-2}. In a curved Riemann space of even dimension this conclusion follows in general only if $K_T = 0$. For the difference between these two solid angles is equal to the integral of K_T over the enclosed volume. This means that an absolute notion of $(n-1)$-dimensional solid angle exists in an even-dimensional Riemann space when and only when $K_T = 0$. When n is odd, the two solid angles are equal without restriction. This situation is analogous to the known result that the angle between two vectors at different points in a Riemann space is independent of the path joining these points only when the Riemann space is flat; that is, when $R_{ijkl} = 0$.

Levi-Civita has called a vector field "parallel" with respect to a curve, C, if $V^i_{,j} dy^j/ds = 0$ along C (whose equations are $y^i = f^i(s)$). When $n = 2$,

$$\phi = \frac{1}{2\pi} \begin{vmatrix} V^1 & V^2 \\ V^1_{,j} \dfrac{dy^j}{ds} & V^2_{,j} \dfrac{dy^j}{ds} \end{vmatrix}. \tag{12}$$

Since V^i is assumed to be a unit vector, it follows that V^i is "parallel" along C if and only if $\phi=0$ along C. By way of generalization we may therefore say that a unit vector field on M^n is parallel (of order $n-1$) relative to a hypersurface R^{n-1} whenever $\phi=0$ on R^{n-1}. Further, just as absolute parallelism of Levi-Civita exists only when $R_{ijkl}=0$, absolute parallelism of order $n-1$ exists only when n is odd, or (when n is even) when $K_T=0$. A full discussion of this generalized type of parallelism together with the obvious intermediate cases must await another occasion. I mention it here chiefly as an illustration of the type of stimulus to ordinary Riemannian geometry which results from the study of problems in the large.

5. **The nature of curvature.** Let us now turn to the total curvature, K_T, itself. Although this quantity appears here and there in the literature, its study has been sadly neglected. Practically all the work on the "curvature" of Riemann spaces is concerned with the Riemann tensor R_{ijkl} itself or with its contraction R_{ij}. Let me remind you that the common interpretation of R_{ijkl} is in terms of the Gaussian two-dimensional curvature of a geodesic two-dimensional surface associated with an arbitrary bi-vector. We have thus been limiting ourselves to an interpretation of the curvature of n-space in terms of our knowledge of two-space. It is time that we really set out to find some truly n-dimensional theorems not bound by this limitation. For example we can discuss the total curvature K_T of the geodesic surface of $2k$ dimensions associated with an arbitrary set of $2k$ independent vectors $\lambda_1^i, \cdots, \lambda_{2k}^i$. Then we have the expression:

$$K_{2k}=\frac{R_{r_1\cdots r_{2k}s_1\cdots s_{2k}}\lambda_1^{r_1}\cdots\lambda_{2k}^{r_{2k}}\lambda_1^{s_1}\cdots\lambda_{2k}^{s_{2k}}}{g_{r_1\cdots r_{2k}s_1\cdots s_{2k}}\lambda_1^{r}\cdots\lambda_{2k}^{r_{2k}}\lambda_1^{s_1}\cdots\lambda_{2k}^{s_{2k}}} \tag{13}$$

where

$$R_{r_1\cdots r_{2k}s_1\cdots s_{2k}}=R_{i_1i_2j_1j_2}\cdots R_{i_{2k-1}i_{2k}j_{2k-1}j_{2k}}\epsilon^{i_1\cdots i_{2k}}_{r_1\cdots r_{2k}}\epsilon^{j_1\cdots j_{2k}}_{s_1\cdots s_{2k}}$$

$$g_{r_1\cdots r_{2k}s_1\cdots s_{2k}}=g_{i_1j_1}\cdots g_{i_{2k}j_{2k}}\epsilon^{i_1\cdots i_{2k}}_{r_1\cdots r_{2k}}\epsilon^{j_1\cdots j_{2k}}_{s_1\cdots s_{2k}}$$

There are thus $[n/2]$ such curvatures available to describe the nature of an n-space, the highest of which for n even is K_T.

Very little is known about these curvatures; in particular it would be interesting to know the properties of a space which are implied by the vanishing of one or more of the curvatures K_{2k}. For example, a four-dimensional space with $K_2\neq 0$ and with $K_4=0$ is still unexplored for the purposes of relativity theory. The difficulties involved in inventing such theorems arise partly from our lack of intuition when the dimension exceeds three and partly from the nonlinearity of the differential equations which these conditions imply.

6. The Hodge theory.

Much more modern than Kronecker is the the work of Hodge [7] which has just begun to be exploited towards the ends with which I have been concerned today. Hodge has defined for skew-symmetric tensors two derived tensors which generalize the curl and divergence of a vector. For the tensor $T_{\alpha_1 \cdots \alpha_r}$ (skew-symmetric in each pair of indices) these are:

$$dT = T_{\alpha_1 \cdots \alpha_r, \alpha_{r+1}} \epsilon^{\alpha_1 \cdots \alpha_{r+1} \alpha_{r+2} \cdots \alpha_n} \tag{14}$$

and

$$\delta T = T_{\alpha_1 \cdots \alpha_r, \alpha_{r+1}} g^{\alpha_r \alpha_{r+1}}. \tag{15}$$

If both $dT = 0$ and $\delta T = 0$, T is called a harmonic tensor of order r. His main theorem is that the rth Betti number of a closed orientable manifold M^n is equal to the number of linearly independent harmonic tensors of order r on M^n. This should lead to a method of computing the Betti number of such a manifold in terms of its local differential geometry. The only result so far is due to Bochner [4]: *If the Ricci tensor $R_{jk} = R_{ijkl}g^{il}$ of a closed Riemann manifold is positive definite, the first Betti number of the manifold is zero.* This result, however, was first proved in an entirely different fashion and in a somewhat stronger form by Myers [11].

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HAVERFORD COLLEGE

THE ANNUAL MEETING OF THE SOCIETY

The fifty-fourth Annual Meeting of the American Mathematical Society was held at the University of Georgia, Athens, Georgia, Monday to Wednesday, December 29–31, 1947, in conjunction with the Annual Meeting of the Mathematical Association of America. Over two hundred persons registered, including the following one hundred eighty-four members of the Society:

Louise Adams, V. W. Adkisson, R. P. Agnew, G. E. Albert, C. B. Allendoerfer, W. F. Atchison, E. A. Bailey, D. F. Barrow, E. F. Beckenbach, W. S. Beckwith, E. G. Begle, R. G. Blake, L. M. Blumenthal, M. G. Boyce, H. J. Bradley, J. W. Bradshaw, A. T. Brauer, H. E. Bray, Foster Brooks, N. R. Bryan, R. S. Burington, L. P. Burton, L. E. Bush, S. S. Cairns, Iris Callaway, E. A. Cameron, C. C. Camp, W. B. Carver, Alonzo Church, Randolph Church, R. V. Churchill, C. E. Clark, F. E. Clark, J. M. Clarkson, A. C. Cohen, Eckford Cohen, L. W. Cohen, E. C. Coker, J. B. Coleman, L. P. Copeland, N. A. Court, R. R. Coveyou, W. H. H. Cowles, H. B. Curry, J. H. Curtiss, C. H. Denbow, A. H. Diamond, J. M. Dobbie, H. H. Downing, W. L. Duren, L. A. Dye, E. D. Eaves, F. A. Ficken, L. R. Ford, Tomlinson Fort, J. S. Frame, H. K. Fulmer, A. S. Galbraith, H. M. Gehman, M. E. Gillis, Wallace Givens, Casper Goffman, H. E. Goheen, V. D. Gokhale, J. S. Gold, Michael Goldberg, M. O. González, W. H. Gottschalk, S. H. Gould, Fletcher Gray, D. E. Green, R. E. Greenwood, W. C. Griffith, William Gustin, D. W. Hall, E. A. Hedberg, G. A. Hedlund, G. W. Hess, E. H. C. Hildebrandt, T. H. Hildebrandt, Einar Hille, G. B. Huff, W. R. Hutcherson, L. C. Hutchinson, M. P. Jarnagin, Jr., E. D. Jenkins, F. B. Jones, H. T. Karnes, A. J. Kempner, D. E. Kibbey, J. R. Kline, F. W. Kokomoor, H. L. Krall, S. H. Lachenbruch, A. E. Landry, G. B. Lang, G. A. Larew, E. H. Larguier, C. G. Latimer, H. L. Lee, T. H. Lee, Solomon Lefschetz, R. J. Levit, D. C. Lewis, F. A. Lewis, R. A. Lytle, Dorothy McCoy, E. J. McShane, C. C. MacDuffee, E. L. Mackie, J. D. Mancill, Szolem Mandelbrojt, W. A. Martin, W. T. Martin, L. E. Mehlenbacher, B. E. Meserve, H. A. Meyer, A. N. Milgram, D. D. Miller, R. A. Miller, T. W. Moore, W. B. Moye, S. L. Nelson, P. F. Neményi, F. H. Murray, C. V. Newsom, E. B. Ogden, Morris Ostrofsky, E. R. Ott, F. W. Owens, H. B. Owens, W. V. Parker, I. E. Perlin, B. J. Pettis, H. B. Phillips, Harry Polachek, G. B. Price, R. C. Prim, F. M. Pulliam, J. F. Randolph, A. S. Rayl, L. M. Reagan, Mina Rees, P. K. Rees, P. R. Rider, R. F. Rinehart, L. A. Ringenberg, E. K. Ritter, J. H. Roberts, H. A. Robinson, L. V. Robinson, W. J. Robinson, A. C. Schaeffer, E. B. Shanks, T. M. Simpson, C. B. Smith, H. L. Smith, P. A. Smith, W. S. Snyder, E. P. Starke, R. P. Stephens, R. W. Stokes, J. L. Synge, Otto Szász, Olga Taussky-Todd, J. S. Taylor, H. P. Thielman, J. M. Thomas, H. S. Thurston, E. W. Titt, H. C. Trimble, G. R. Trott, C. A. Truesdell, Henry Van Engen, A. H. Van Tuyl, T. L. Wade, A. D. Wallace, J. A. Ward, F. P. Welch, W. F. Whitmore, G. T. Whyburn, W. L. Williams, R. L. Wilson, L. C. Young.

Sessions for reading contributed papers were held at 2:00 P.M., Monday, at 10:00 A.M. and 3:15 P.M., Tuesday, and at 3:15 P.M., Wednesday. The presiding officers for these sessions were Professors

Tomlinson Fort, L. R. Ford, S. S. Cairns, T. H. Hildebrandt, G. T. Whyburn, and G. A. Hedlund.

The twenty-first Josiah Willard Gibbs Lecture was given on Monday evening by Professor P. M. Morse of the Massachusetts Institute of Technology and the Brookhaven National Laboratories on *Mathematical problems in operations research.* President Einar Hille presided.

On Tuesday afternoon with Vice President L. R. Ford presiding, Professor E. F. Beckenbach of the University of California at Los Angeles delivered an address entitled *Convex functions.*

The annual business meeting and election of officers was held on Wednesday morning, President Einar Hille presiding. The proceedings are included later in this report. At the close of the business session Professor T. H. Hildebrandt of the University of Michigan delivered his retiring presidential address on *Integration in abstract spaces.* Following Professor Hildebrandt's address those members of the Committee on Reorganization who were present at the meeting outlined some of the problems which the committee faces and asked for suggestions from the members present.

Sessions of the Mathematical Association of America were held on Thursday morning and afternoon.

On Monday morning and afternoon tours were conducted for mathematicians to some of the antebellum homes in Athens. On Tuesday afternoon a tea and reception for the guests of the University was held in Lyndon Hall. On Tuesday evening a concert was given in the University Chapel by members of the Department of Music. A special art exhibit for those attending the meetings was held on Wednesday afternoon in the Fine Arts Building.

At the dinner on Wednesday evening for mathematicians and their guests, Professor Tomlinson Fort of the University of Georgia was toastmaster. President H. W. Caldwell of the University of Georgia gave an address of greeting. Music was provided by Messrs. James Griffith and Byron Warner. The following addresses were presented: *What is applied mathematics?*, Professor J. L. Synge of Carnegie Institute of Technology; *Applied mathematics in government service,* Dr. J. H. Curtiss of the National Bureau of Standards; *Applied mathematics in the college and university,* Professor Emeritus H. B. Phillips of the Massachusetts Institute of Technology. These were followed by a dramatic skit entitled *Applied mathematics,* presented by members of the Department of Mathematics of the University of Georgia. Professor R. P. Agnew of Cornell University presented resolutions expressing the thanks and appreciation of the members of the Society and of the Association to the various groups at the University of Georgia for the excellent arrangements made for the meeting.

At the meeting of the Board of Trustees at 6:00 P.M. on December 29, 1947, in Snelling Hall, there was no quorum present and the Board adjourned to January 17, 1948.

At this meeting of the Board of Trustees the resignations of Professor G. W. Mullins and Dr. Warren Weaver as members of the Board of Trustees were presented and accepted with regret. The following resolutions were adopted by the Trustees regarding the significant contributions of Professor Mullins and Dr. Weaver to the work of the Society:

Professor George Walker Mullins of Columbia University has resigned from the Board of Trustees of the American Mathematical Society after nearly twenty years of signal service as Treasurer and Trustee. This period coincided with one of distinguished usefulness to the University where he was notably influential in formulation of policies. Happy relations between Columbia University and the Society began with its founding by Professor Fiske sixty year ago. Through the good offices of Professor Mullins these relations became ever more cordial and the University ever more generous in its attitudes.

The present favorable status of the financial affairs of our Society is in no small measure due to our colleague's devotion and skill. The Board is profoundly appreciative of all these services, and tenders to him its heartiest good wishes.

It is not possible to record in a few paragraphs the qualities of mind and spirit that have made the work of Dr. Warren Weaver, as a Trustee of the American Mathematical Society, so successful and outstanding and so satisfying to his associates on the Board. During his seven years of service as a Trustee, he has given generously of his time and effort in trying to solve the many perplexing problems that constantly arose during this period of expansion. Equipped with a liberal mind of dignity and real distinction, he brought to the discussion of any problem that degree of human tolerance and understanding so necessary in the guidance of the affairs of the Society.

Beginning with 1933 Dr. Weaver, realizing that science was being misdirected in some regions of Europe, was a powerful factor in recruiting for American mathematics a large number of scholars from the various European countries, thus enriching on a widespread and national scale the several branches of our subject. His statesmanlike assistance in founding Mathematical Reviews is only one of the more obvious of the outcomes of his services. His initiative in the establishment of pre-doctoral fellowships for young men who had been diverted from their studies through scientific work for the war effort will result in a lessening of the alarming gap in personnel which threatens education and research.

As Chief of the Applied Mathematics Panel of the National Defense Research Committee during World War II, Dr. Weaver rendered signal service. Through his insight and diplomatic skill, he was able to convince the armed forces that mathematics is vital in the solution of many urgent problems of defense and offense. In the various events which resulted in mathematics emerging from the war with greatly enhanced prestige, he was a notable leader.

Dr. Weaver's vision of the future concerning the part that mathematics may play in American civilization will be greatly missed in the deliberations of the Trustees, but it is the good fortune of the Board that his advice and help will still be available.

The Council met at 8:00 P.M. on December 30 at the home of Professor Tomlinson Fort.

The Secretary announced the election of the following seventy-one persons to ordinary membership in the Sociey:

Professor Maria Yolanda de Mello Nogueria Abdelhay, University of Brazil;
Mr. E. Leonard Arnoff, Case Institute of Technology;
Mr. Leon Auerbach, Gallaudet College, Washington, D. C.;
Mr. Edwin Ward Banhagel, Physics Department, Wayne University;
Professor Otakar Borůvka, Masaryk University, Brno, Czechoslovakia;
Mr. George Bott, Operations Evaluation Group, Navy Department, Washington, D. C.;
Miss Thelma Elizabeth Bradford, Claflin University, Orangeburg, S. C.;
Mr. João Augusto Breves Filho, University of São Paulo;
Mr. Rudolph Richard Caputo, Naval Shipyard, Brooklyn, N. Y.;
Mr. Phillip George Carlson, Jr., Rensselaer Polytechnic Institute;
Mr. Eric R. I. Carson, St. Francis College, Brooklyn, N. Y.;
Professor Alonzo Clifford Cohen, Jr., University of Georgia;
Mr. Louis M. Court, Rutgers University;
Mr. William Francis Dice, St. Francis College, Brooklyn, N. Y.;
Mr. Alexander Weaver Ebin, American Institute of Man, Chicago, Ill.;
Mr. Jason Arundel Ellis, State University of Iowa;
Mr. John Fall, Acquisition Division, New York Public Library;
Mr. Athanasius Floris, Pacific Electric Railway Co., Los Angeles, Calif.;
Professor Luiz Barros Freire, Department of Physics, University of Recife, Recife, Pernambuco, Brazil;
Mr. Marcos Galper, Rio de Janeiro, Brazil;
Mr. Harold Grad, New York University;
Mr. Ross Edwin Graves, University of Minnesota;
Mr. Fletcher Gray, Birmingham, Ala.;
Miss Dorothy Eunice Green, Huntingdon College, Montgomery, Ala.;
Dr. Edwin Leland Harder, Westinghouse Electric Corporation, East Pittsburgh, Pa.;
Mr. Keith Ortlepp Househam, Princeton University;
Mr. Albert Charles Howell, Actuarial Department, John Hancock Mutual Life Insurance Co., Boston, Mass.;
Mr. Roy Sherman Kingsbury, Operations Evaluation Group, Navy Department, Washington, D. C.;
Professor Walter Eddie Koss, Agricultural and Mechanical College of Texas;
Professor Julien Kravtchenko, University of Grenoble, Grenoble, France;
Mr. Maurice Albert L'Abbé, Princeton University;
Professor Friedrich Wilhelm Levi, Tata Institute of Fundamental Research, Bombay, India;
Mr. Samuel Levy, National Bureau of Standards, Washington, D. C.;
Brother George Lewis, La Salle College, Philadelghia, Pa.;
Sister Mary Teresine Lewis, Fontbonne College, St. Louis, Mo.;
Mr. Geraldo dos Santos Lima Filho, University of São Paulo;
Mr. Clair George Maple, Carnegie Institute of Technology;
Mr. Gerald W. L. Van Douglas Martynes, Rio de Janeiro, Brazil;
Professor Ross Raymond Middlemiss, Washington University;
Mr. Warren Keith Moore, University of Kansas;

Professor Abrahão de Moraes, University of São Paulo;
Sister Charles Anne Mulligan, Albertus Magnus College, New Haven, Conn.;
Professor Francisco Antonio Lacaz Netto, University of São Paulo;
Mr. William Alfred Nichols, Canadian Broadcasting Corporation, Montreal, Canada;
Dr. Yves Nubar, New York University;
Miss Thelma Eve Peacock, Wells College;
Mr. Chester Clement Piersol, Naval Gun Factory, Washington, D. C.;
Mr. Johnny Webster Ponds, West Virginia State College;
Mr. Robert Clay Prim, 3rd, Naval Ordnance Laboratory, White Oak, Md.;
Professor Ary Norton de Murat Quintella, Military Academy, Rio de Janeiro, Brazil;
Mr. Robert Abram Roberts, West Virginia University;
Professor Plinio Sussekind Rocha, University of Brazil;
Mr. Milton Edward Rose, Irvington, N. J.;
Miss Rose Lee Rowen, Operations Evaluation Group, Navy Department, Washington, D. C.;
Miss Leila Raines Rubashkin, Cornell University;
Mr. Bernard Sachs, Polytechnic Institute of Brooklyn;
Mr. Anthony Henry Sarno, St. John's University;
Mr. Jacob T. Schwartz, New York, N. Y.;
Mr. George Shapiro, Brooklyn College;
Dr. Jack Sherman, Research Laboratory, The Texas Company, Beacon, N. Y.;
Sister Mary Carmel Sullivan, Albertus Magnus College, New Haven, Conn.;
Mr. John Joseph Taylor, Bendix Aviation Corporation, Teterboro, N. J.;
Mr. Affonso Penteado de Toledo Piza, State Department of Statistics of São Paulo, São Paulo, Brazil;
Dr. Francis Eligius Van Bergen, Atheneum St. Nicolas-Waas, Belgium;
Visiting Professor Bartel Leendert van der Waerden, The Johns Hopkins University;
Professor Rin Wang, Changbai Normal College, Babaerun, Kirin, China;
Mr. Gerard Washnitzer, Brooklyn, N. Y.;
Mr. Daniel Robson Waterman, Brooklyn College;
Dean Cecil Lamborn Woods, Pacific Union College, Angwin, Calif;
Mr. John Leonard Yarnell, Topeka, Kan.;
Mr. Philipp Wolfgang Zettler-Seidel, Naval Ordnance Laboratory, White Oak, Md.

It was reported that the following one hundred forty-six persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Messrs. Richard Earl Allen and Henry Conrad Miller;
Brooklyn College: Mr. Leon H. Herbach;
Brown University: Messrs. Peter Chiarulli and Morton Finston, Dr. James Arthur Krumhansl, Miss Jacqueline Lydie Penez, Dr. Rohn Truell;
Bryn Mawr College: Miss Joan Elizabeth Robinson;
University of California: Messrs. John Gurland, Terry Allen Jeeves and Edward Silverman;
University of California at Los Angeles: Messrs. John David Newburgh and Lawrence Theodore Ratner;
The Case Institute of Technology: Mr. Francis Dunbar Parker;
University of Cincinnati: Mr. Nelson Paul Yeardley;
City College of the College of the City of New York: Messrs. Jerome Berkowitz, Irwin Fischer, Milton Martin Gutterman, Samuel Linial, John Jay Stachel and Herman Zabronsky;

Columbia University: Messrs. Edward Kenneth Blum, Kuo-Tsai Chen and Arnold Lionel Fass, Miss Alice Osterberg, Messrs. Jack Maynard Patterson and Sherman Kopald Stein;

Cornell University: Messrs. George Laush and Walter Lynn Murdock, Miss Maria Alice Weber;

Duke University: Messrs. Samuel Wilfred Hahn and Thomas Davies Reynolds;

Equitable Life Insurance Company: Mr. Kermit Lang;

Harvard University: Messrs. Richard Eliot Chamberlin, Frederic Cunningham, Horace Chandler Davis, Paul Roesel Garabedian, Thomas Andrew Lehre, Peter MacNair, Jerome Joseph Newman, Henry Otto Pollak and Morton Lincoln Slater, Miss Dorothy Browne Shaffer, Mr. (William) Forrest Stinespring;

Illinois Institute of Technology: Messrs. Harold Greenspan, Louis Joseph, Donald Williamson Pounder and David Rubinfien, Dr. Herbert A. Simon, Miss Barbara Reese Vogel;

University of Illinois: Mr. James Bercos, Miss Marie Louise Boddy, Mr. George Truett Cates, Miss Lois Vivian Cline, Messrs. Cevdet A. Erzen, Albert Ansell Gopaul and Clemens Bernard Hanneken, Miss Ruth Heinsheimer, Messrs. Richard Edwin Janssen, Richard Lynn Mentzer, Leland Latham Scott, Joe Willie Wray;

Indiana University: Mr. Israel Nathan Herstein;

Institute for Advanced Study: Drs. Alfons Maria Frans Borgers and Hel Braun, Professor Marie Radegoude Jeanne Charpentier, Dr. Kien-Kwong Chen, Professor Hubert Delange, Drs. Erling Følner, Harish-Chandra, Nicolaas Hendrik Kuiper, Jean Mariane, Szu-Hoa Min and Paolo Nesbeda, Professor Ake Vilhelm Carl Pleijel, Dr. Atle Selberg;

Iowa State College of Agriculture and Mechanic Arts: Mr. Carl Eric Langenhop;

The Johns Hopkins University: Messrs. Arthur Simpson Littell, Henry Llewellyn Merritt, Arthur Stein, Mitchell Trauring, Mathews C. Waddell, and Arthur Wouk;

Lehigh University: Messrs. Wilbur Hibbard and Ralph Waldo Young;

University of Maryland: Mr. Arnold George Rawling;

Massachusetts Institute of Technology: Messrs. John Bertram Giever, John Beckwith Kelly, Warren Paul Manger, Oliver Gordon Selfridge;

University of Michigan: Miss Marjorie Lee Browne, Messrs. Bertram Joseph Eisenstadt, Kenneth Arthur Fowler, Meyer Jerison, Benjamin Lapidus, Clay Lamont Perry, Jr., and David Van Vranken Wend;

Michigan State College: Mr. Jean Joseph LeJunter;

University of Minnesota: Messrs. Watson Bryn Fulks, Jack Indritz and Chih-yi Wang;

New York University: Miss Anneli Leopold;

Northwestern University: Mr. Lester Gothard Riggs;

University of Notre Dame: Mr. John Charles Burke;

Ohio State University: Messrs. Joseph Elliott Adnew, Jr., Byron Brown Dressler and George Marsaglia;

Oklahoma Agricultural and Mechanical College: Mr. Crosman Jay Clark;

University of Oregon: Mr. Walter Mossman Gilbert;

Pennsylvania State College: Messrs. Stephen Jerome Bilo and Robert Taylor Foote;

University of Pennsylvania: Misses Leila Ann Dragonette and Anne Marie Whitney;

Rice Institute: Mr. Neville Carter Hunsaker;

University of Rochester: Mr. William Anderson;

The College of St. Thomas: Mr. Ferdinand Roy Ohnsorg;

Smith College: Miss Evelyn Boyd;

Swarthmore College: Mr. Robert Garlin Kuller;

Syracuse University: Miss Elizabeth Harriet Wetherell;

University of Tennessee: Mr. William Austin Rutledge;

University of Texas, Department of Pure Mathematics: Mr. Mark Hall Clarkson, Miss Mary Ellen Estill, Mr. John Sheridan MacNerney;

University of Toronto: Messrs. Kenneth Dudley Fryer and Atholl Livingstone Wilson;

University of Washington: Messrs. Fred C. Andrews and James L. Hildebrandt;

Wellesley College: Mrs. Elaine Van Aken Cowen;

University of Wisconsin: Messrs. Howard Ernest Campbell, Kermit Howard Carlson, Marion Preston Emerson, Gerald Paul Dinneen, Delbert Ray Fulkerson, Leonard Eugene Fuller, Rodney Taber Hood, and John E. McAdam, Miss Ida Merle Mitchell, Messrs. Vernon Hugh Ross and Herbert Ruderfer;

Yale University: Messrs. Robert Goodell Brown, Jack Clark Gibson, John McGaughey Heath, William Gordon Lister, John Edward Lancelot Peck, Donald Aubry Quarles, Jr., and John Newton Williams.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Dr. Hermann Kober, University of Birmingham, Birmingham, England; Dr. Robert Alexander Rankin, Cambridge University, England; Société Mathématique de France: Mr. Paul Belgodère, Centre National de la Recherche Scientifique, Paris; Dr. Gustave Choquet, Université de Grenoble; Professor Jean Alexandre Dieudonné, Faculté des Sciences de Nancy; Mr. Victor Dana, Faculty of Science, Cairo, Egypt; Professor Paul Jacques Dubreil, Faculté des Sciences, L'Université de Paris; Professor Luc Edmond Nicolas Gauthier, University of Nancy; Professor Gaston Maurice Julia, Ecole Polytechnique, Paris; Dr. Marc Krasner, Centre National de la Recherche Scientifique, Paris; Professor Paul Métral, Lycée de Marseille; Professor André Metté, Lycée Pasteur, Neuilly; Professor Victor Michel Jean-Marie Thebault, Ecole Normale, Tennie; Swiss Mathematical Society: Professor Walter Nef, University of Fribourg, Fribourg, Switzerland; Unione Matematica Italiana: Professor Enrico Bompiani, University of Rome; Professor Mauro Picone, University of Rome.

It was reported that the Council had voted by mail: to accept the recommendation of the Committee on Applied Mathematics that the Second Symposium on Applied Mathematics be held at Massachusetts Institute of Technology during the summer of 1948; to authorize and request the President to appoint a committee to nominate two candidates for the vacancy in the Policy Committee for Mathematics, to occur on December 31, 1947; to accept the recommendation of the Emergency Committee for the International Congress of Mathematicians that steps be taken at once to nominate and elect officers and committees for the Congress; to authorize and request the President to appoint a committee to nominate officers and committees for the International Congress. It was also reported that the Council, by mail vote, had elected Professor Einar Hille as a representative of the Society on the Policy Committee for Mathematics, to succeed Professor T. H. Hildebrandt and to serve for a period of four years beginning January 1, 1948.

The Secretary is pleased to report at this time that the ordinary membership of the Society is now 3494, including 321 nominees of institutional members and 58 life members. There are also 103 institutional members. The total attendance at all meetings in 1947 was 1950; the number of papers read was 511; there were 12 hour addresses, 2 symposium addresses, 4 addresses and 21 papers at the Applied Mathematics Symposium, 1 Gibbs Lecture, 4 Colloquium Lectures, and 1 Retiring Presidential Address; the number of members attending at least one meeting was 1191.

The following appointments by President Einar Hille of representatives of the Society were reported: Professor R. P. Agnew at the inauguration of Katherine Gillette Blyley as President of Keuka College on October 4, 1947; Professor Einar Hille at the Convocation of the Sheffield Scientific School on October 17, 1947; Professor L. M. Kells at the inauguration of John Spangler Kieffer as President of St. John's College on October 25, 1947; Professor W. L. Miser at the inauguration of Charles Spurgeon Johnson as President of Fisk University on November 6–8, 1947; Professor M. H. Martin at the inauguration of Lowell Skinner Ensor as President of Western Maryland College on November 8, 1947; Professor C. B. Wright at the inauguration of James Gilliam Gee as President of East Texas State Teachers College on November 15, 1947; Professor G. T. Whyburn at the inauguration of Frank Bell Lewis as President of Mary Baldwin College on November 18, 1947.

The following additional appointments by the President were reported: Professors M. H. Stone (Chairman), R. P. Agnew and John von Neumann as a Committee on the Award of the Bôcher Prize, to be awarded at the 1948 Annual Meeting for the period 1943–1947; Drs. R. P. Boas (Chairman), Arthur Erdélyi, Professors George Pólya and Antoni Zygmund as a Committee on Known Results; Professors W. T. Martin (Chairman), T. R. Hollcroft, J. A. Stratton, and J. L. Synge as a Committee on Arrangements for the Second Annual Symposium in Applied Mathematics, to be held at Massachusetts Institute of Technology in 1948; Professors H. P. Evans

(Chairman), R. H. Bruck, R. V. Churchill, Dr. B. H. Colvin, Professor H. M. Gehman, Dr. Elizabeth S. Sokolnikoff, and Professor R. D. Specht as a Committee on Arrangements for the 1948 Summer Meeting, to be held at the University of Wisconsin; Dr. E. R. Kolchin, Professors R. J. Levit and Walter Strodt as tellers for the 1947 election of officers and members of the Council; Professor T. H. Hildebrandt as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for the period 1948–1949 (committee now consists of Professors J. R. Kline, Chairman, T. H. Hildebrandt, and Oscar Zariski); Professor Deane Montgomery as a member of the Committee to Select Hour Speakers for Eastern Sectional Meetings for the period 1948–1949 (committee now consists of Professors T. R. Hollcroft, Chairman, W. T. Martin, and Deane Montgomery); Professor Marshall Hall as a member of the Committee to Select Hour Speakers for Western Sectional Meetings for the period 1948–1949 and Professor C. J. Nesbitt as a member of the same committee for the year 1948 (committee now consists of Professors R. H. Bruck, Chairman, Marshall Hall, and C. J. Nesbitt); Professor Ralph Hull as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings for the period 1948–1949 (committee now consists of Professors J. W. Green, Chairman, Ralph Hull, and C. B. Morrey); Professors J. C. Oxtoby and A. C. Schaeffer as members of the Committee on Publicity for the period 1948–1950 (committee now consists of Professors C. O. Oakley, Chairman, Garrett Birkhoff, J. C. Oxtoby, A. C. Schaeffer, R. M. Thrall, and R. J. Walker); Professor J. W. Tukey as a member and Chairman of the Committee on Reorganization of the Society's Programs; Professor J. W. T. Youngs as Acting Chairman of the Birkhoff Finance Committee, to serve until July 1, 1948; Professors S. B. Myers (Chairman), R. H. Bruck, Ben Dushnik, J. S. Frame, T. H. Hildebrandt, and R. M. Thrall as a Committee on Arrangements for the April, 1948, meeting at the University of Michigan; Professors W. L. Ayres (Chairman), S. S. Cairns, B. P. Gill, J. R. Kline, P. A. Smith, and M. H. Stone as a Committee on Reorganization; Professor Hassler Whitney as Chairman of the Committee on Visiting Lectureships for the period 1948–1950 and Dean R. G. D. Richardson as a member of the same committee for the period 1948–1949 (committee now consists of Professor Hassler Whitney, Chairman, Dean R. G. D. Richardson, and Professor Norbert Wiener); Professor C. J. Rees as Chairman of the Committee on Printing Contracts for the period 1948–1950 (committee now consists of Professor C. J. Rees, Chairman, Dean M. H. Ingraham, and Professor J. M. Thomas); Professor

G. B. Price as Chairman of the Committee on Places of Meetings for 1948 and Professor R. L. Wilder as a member of the committee for 1948–1950 (committee now consists of Professors G. B. Price, Chairman, C. B. Allendoerfer, and R. L. Wilder); Professors G. C. Evans (Chairman), J. R. Kline and G. T. Whyburn as a committee to nominate representatives of the Society on the Policy Committee for Mathematics; Dean Walter Bartky and Professor G. C. Evans as members of the Committee on Applied Mathematics for the period 1948–1950 (committee now consists of Professors J. L. Synge, Chairman, Walter Bartky, Richard Courant, G. C. Evans, John von Neumann, and William Prager); Professors D. V. Widder (Chairman), A. B. Coble, Arnold Dresden, T. H. Hildebrandt, and Deane Montgomery as a committee to nominate officers and committees for the International Congress of Mathematicians.

The following were reported as the Chairmen and voting members of Editorial Committees for the year 1948: Professor R. E. Langer, Bulletin; Professor A. A. Albert, Transactions; Professor A. W. Tucker, Mathematical Surveys; Professor J. F. Ritt, Colloquium.

It was reported that Professor G. A. Hedlund of the University of Virginia had accepted the invitation to deliver a series of Colloquium Lectures at the 1949 Summer Meeting.

The Secretary reported that the biennial membership campaign had been conducted during November, 1947, at which time 4,524 letters of invitation to join the Society were mailed to teachers of mathematics in the colleges and universities of the United States and Canada (including 515 invitations to members of the Association).

It was reported that a reciprocity agreement had been established between the Society and the Société Mathématique de France. The details of this agreement and of other reciprocity agreements are given elsewhere in this issue of the Bulletin.

At the annual election which closed on December 31, and at which 656 votes were cast, the following officers were elected:

Vice Presidents, Professors John von Neumann and Hassler Whitney.

Associate Secretaries, Professors R. H. Bruck, J. W. Green, and G. B. Price.

Librarian, Professor Arnold Dresden.

Members of the Editorial Committee of the Bulletin, Professor Nathan Jacobson and Dean E. B. Stouffer.

Member of the Editorial Committee of the Transactions, Professor G. T. Whyburn.

Member of the Editorial Committee of the Colloquium Publications, Professor C. C. MacDuffee.

Member of the Editorial Committee of Mathematical Reviews, Professor William Feller.

Member of the Editorial Committee of Mathematical Surveys, Professor J. L. Walsh.

Representative on the Editorial Board of the American Journal of Mathematics, Professor Richard Brauer.

Members at large of the Council, Professors J. L. Doob, R. L. Jeffrey, J. B. Rosser, N. E. Steenrod, and Alfred Tarski.

The Secretary reported the resignation in November of Dr. Warren Weaver as a Trustee of the Society and the resignation, to be effective after January 17, 1948, of Professor G. W. Mullins as a Trustee. He further reported that Professor G. T. Whyburn had been elected by the Trustees to fill the unexpired term (to December 31, 1948) of Dr. Weaver.

The Librarian reported the following additions to the Library: 219 volumes of periodicals, 42 books, 82 pamphlets (including 72 dissertations). He also reported that exchanges with foreign periodicals are being reestablished and that he is assisting in the revival of mathematical activity abroad in so far as the policy of liberal exchange conditions can contribute to this end.

In an appendix to this report are excerpts from the report of the Treasurer for the fiscal year 1947 as verified by the auditors. A copy of the complete report will be sent, on request, to any member of the Society.

The American Journal of Mathematics reported that it had published 871 pages in 1947. The Society now contributes an annual subvention of $2,000 to the American Journal.

Times and places for meetings of the Society in 1948 were set as follows: June 19 at the University of British Columbia; September 7–10 at the University of Wisconsin; October 30 in New York City; November 26–27 in Chicago, Illinois; November 27 at the University of California at Los Angeles. The Council voted in favor of holding regular meetings of the Society in the Pacific Northwest. On recommendation of the Committee on Places of Meetings, the Council voted to hold the 1948 Annual Meeting at The Ohio State University and the 1949 Annual Meeting in New York City, the latter to be held in conjunction with the meetings of the American Association for the Advancement of Science. (It has proved necessary to postpone the New York meeting from December, 1948, to December, 1949, because of the change in plans of the A.A.A.S.)

Certain invitations to give addresses in 1948 were announced: Professor Eric Reissner for the February meeting in New York City; Professor André Weil for the February meeting in Chicago; Professors Charles Loewner and O. E. Neugebauer for the April meeting in New York; Professors Irving Kaplansky and H. S. M. Coxeter for the April meeting in Ann Arbor; Professor H. F. Bohnenblust for the April meeting in Berkeley; Professor J. W. T. Youngs for the Summer Meeting at the University of Wisconsin; Professor Mark Kac for the Annual Meeting at The Ohio State University.

Associate Secretary G. B. Price reported on the Society's plan for institutional contributing memberships. At the present time the Society has one hundred seven institutional contributing members which contribute over $9,000 annually to the Society's publication program. Dues for institutional contributing members are computed every six years and are based on the number of pages published by the member's staff. The per page rate used at present is $2.75; this rate has been used since institutional memberships were inaugurated in 1934 and does not reflect present printing costs. Professor Price pointed out that the Society's income from industry is very small and that the Society might well hope for an increase in income from this source, since an increasingly large number of persons employed by industry are becoming affiliated with the Society.

The Bulletin Editorial Committee reported that the high standard of acceptability for articles is being maintained, that the number of papers received from abroad has increased and that the flow of papers from this country has continued. Professors A. H. Clifford, M. M. Day, F. G. Dressel and G. W. Whitehead were reported as new Assistant Editors for the Bulletin.

The Transactions Editorial Committee reported that a total of 1,100 pages have been published in the 1947 volumes and that a backlog of papers accepted for publication is accumulating, so that a year will elapse between submission of a manuscript and publication. Professors Samuel Eilenberg and Harry Levy were reported as new Associated Editors and Professor A. A. Albert as Managing Editor for the year 1948.

The Mathematical Reviews Editorial Committee reported an increase in the amount of material to be reviewed, due to the availability of foreign journals and to the increased production in most countries. During 1947 the Matematisk Forening i København and the Polish Mathematical Society were added to the list of sponsors of Mathematical Reviews. The subscription list, as of December 1, 1947, was 1,765. A contract with the Office of Naval Research has recently been negotiated which will yield over $23,000 during 1948 for the support of Mathematical Reviews.

On recommendation of the Colloquium Editorial Committee, the Council voted to recommend to the Trustees the acceptance of a revised edition of *Lattice theory* by Professor Garrett Birkhoff for publication in the Colloquium Series. On recommendation of the Mathematical Surveys Editorial Committee, a manuscript by Dr. Stefan Bergman entitled *Methods in complex orthogonal functions* was accepted for publication, subject to the approval of the Trustees.

It was reported that an Editorial Committee had started work on the compilation of material to be published as the *Collected mathematical papers* of G. D. Birkhoff and that the Finance Committee for the project had secured, as of December 12, 1947, a total of $7,102.80 in contributions and subscriptions.

The Council approved July 29–31, 1948, as the dates for the Second Symposium on Applied Mathematics, to be held at the Massachusetts Institute of Technology on the subject of *Electro-magnetic theory*. The Council also approved the recommendation that the following groups be invited to act as co-sponsors of the Symposium: American Institute of Physics, American Institute of Radio Engineers, American Institute of Electrical Engineers.

The Committee on Aid to Devastated Libraries reported that a second shipment of books and journals is on its way to twenty-four different libraries in ten different countries and that the committee hopes to find a satisfactory way for forwarding other material now at its disposal.

The Council voted to donate to the new journal, *Applied Mechanics Reviews*, two subscriptions to the Bulletin and Transactions.

The President was authorized to appoint a committee to consider the question of holding regular meetings of the Society in the southeastern section of the country. This action was taken at the request of a group of members of the Society representing various institutions in the southeast.

Abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 110 was presented by Dr. Cohen, number 122 by Professor Gustin, number 129 by Professor Cohen, number 149 by Dr. Neményi, number 169 by Professor Gottschalk, and number 176 by Professor Young. Dr. Dvoretzky was introduced by Professor Szolem Mandelbrojt and Professor Jacob Wolfowitz, Dr. Hu by Professor Oscar Zariski, and Mr. Seth by Professor Jacob Wolfowitz.

ALGEBRA AND THEORY OF NUMBERS

106*t*. Grace E. Bates and F. L. Kiokemeister: *An example in the theory of quasigroups.*

A loop is constructed in such a way that its image under a given homomorphism is not a quasigroup. The loop may be commutative. It is shown that if under a given homomorphism of a loop the kernel is finite or the image is finite or the image is associative, then the image is a loop. (Received November 12, 1947.)

107*t*. Garrett Birkhoff and Orrin Frink: *Representations of lattices by sets.*

This deals with methods of representing a general lattice L by sets. The first part concerns lattices of subalgebras of abstract algebras. Necessary and sufficient conditions are given for L to be isomorphic with the lattice of all subalgebras of a suitable abstract algebra. In a lattice satisfying these conditions, it is shown that every element is a meet of completely meet-irreducible elements. This generalizes the main result of Garrett Birkhoff on subdirect unions in universal algebra. Necessary and sufficient conditions are given for L to be isomorphic with the lattice of all ideals of another lattice A. The second part deals with representations by sets of dual ideals. It is shown that isomorphic meet-representations are furnished by the use of all meet-irreducible, of all completely meet-irreducible, and of all principal dual ideals. In distributive lattices, meet-irreducible and prime dual ideals are shown to be the same. They provide the representation by Stone; while the more economical representation by sets of completely meet-irreducible dual ideals is equivalent to one given previously by Garrett Birkhoff. It is shown that in lattices with the disjunction property of Wallman, the representation by sets of maximal dual ideals is in general more economical still. In lattices with the descending chain condition, some of the representations considered reduce to that introduced by A. D. Campbell. (Received October 22, 1947.)

108*t*. Bailey Brown and N. H. McCoy: *The radical of a ring.*

In a previous paper (Amer. J. Math. vol. 69 (1947) pp. 46–58), a radical N has been defined for an arbitrary ring R, and various theorems on the radical established by use of Zorn's Lemma. Certain of these theorems basic in the theory, for example, that N is a two-sided ideal in R and that R/N has zero radical, are now proved by finite methods, as are also the following new results. The radical N consists of the elements b of R such that the ideal RbR is H_1-regular. If A is a two-sided ideal in R, the radical of the ring A is the intersection of A and N. (Received November 18, 1947.)

109*t*. Leonard Carlitz: *A problem of Dickson's.*

Let the homogeneous polynomial $Q(x, y)$ with coefficients in $GF(p^n)$, p odd, be such that $Q(a, b)$ is a nonzero square in $GF(p^n)$ for all (a, b) except $(0, 0)$ in the field. Dickson (Trans. Amer. Math. Soc. vol. 10 (1909) pp. 109–122) calls such forms ***definite***. He conjectured that every definite binary form of degree $2r$, $r>1$, is the square of a polynomial with coefficients in $GF(p^n)$ provided p^n exceeds a certain bound N_r; he proved this in the case $r=2, 3$. In the present paper this conjecture is shown to be correct in the slightly more general case of polynomials $F(x)$ all of whose values are nonzero squares in $GF(p^n)$: If $p^n>(k-1)^2$, where $k=\deg F$, then F is a square.

This follows easily from the Riemann hypothesis for the Artin zeta-function; the truth of this hypothesis has been proved by A. Weil (Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 345–347). (Received October 16, 1947.)

110. Leonard Carlitz and Eckford Cohen: *Representations of a polynomial in $GF[p^n, x]$ in certain special quadratic forms.*

The authors are concerned with the number of representations of a polynomial in $GF[p^n, x]$ in the form $\sum \alpha_i X_i^2 + \sum L_{ij} Y_i Y_j$, where $\alpha_i \in GF(p^n)$ and deg $L_{ij} \leq 1$. The method is an application of the singular series method developed in a recent paper (Duke Math. J. vol. 14 (1947)). (Received October 16, 1947.)

111*t*. M. J. Herzberger and Robert Morris: *Normal equations with nearly vanishing determinants.*

In least square problems it is frequently necessary to compute the coefficients of the normal equations with greater accuracy than the solution requires because of the fact that the determinant has fewer significant figures than the coefficients. In some cases, in which the number of digits is limited by the use of a calculating machine, a direct result is unobtainable. In the case of a small determinant, an approximate solution of the problem may be found, which is not an approximation to the exact solution of the normal equations but may have more significance than the exact solution, because of the fact that the data might be slightly in error. The paper shows how to obtain such approximations, using only a limited number of digits. By translating the algebraic problem into a problem of geometry, some help is given to the intuition of the computer in arranging the successive operations in the most effective order. (Received November 24, 1947.)

112*t*. Saunders MacLane: *A non-associative method for associative algebras.*

In a forthcoming issue of Trans. Amer. Math. Soc. Eilenberg and the author have established the following theorem for a crossed product algebra A. Let $K \supset N \supset P$ be fields, K and N finite, normal and separable over P, and let σ, τ be automorphisms of K/N, α, β, γ automorphisms of K over P. Let $f(\sigma, \tau)$ be a factor set in K for the crossed product algebra $A = (K, f)$, and $h(\alpha, \beta)$ any extension of the function f to all α, β. If h can be so chosen that $[\alpha \cdot h(\beta, \gamma)] h(\alpha\beta, \gamma)^{-1} h(\alpha, \beta\gamma) h(\alpha, \beta)^{-1}$ is a function $t(\alpha', \beta', \gamma')$ only of the automorphisms α', β', γ' induced on N by α, β, γ, respectively, then every automorphism of N/P can be extended to an automorphism of A, and t is the "Teichmüller cocycle" of A. In the present paper the author gives a simple non-associative proof of this theorem. (Received November 7, 1947.)

113*t*. G. D. Mostow: *On closed subgroups of a Lie group.* Preliminary report.

It is proved that a semi-simple Lie subgroup S of a simply connected Lie group G is closed in G. If G is not simply connected, S need not be closed. A corollary of this theorem is that there is no proper representation of a noncompact connected simple Lie group into a compact Lie group. (Received November 6, 1947.)

114*t*. Gordon Pall: *Composition of binary quadratic forms.*

A compound of binary quadratic forms is defined, and the uniqueness of the product class is proved, without the use of bilinear transformations. There is included also an extension to forms in n variables of a criterion due to Gauss for the equivalence of binary quadratic forms. (Received November 24, 1947.)

115. W. V. Parker: *Characteristic roots and the field of values of a matrix.*

Let $A=(a_{ij})$ be an n-rowed square matrix of complex numbers and write $P_k=\sum_{j=1}^{n}|a_{kj}|\,(j\neq k)$, $Q_k=\sum_{i=1}^{n}|a_{ik}|\,(i\neq k)$, $R_k=P_k+|a_{kk}-m|$, $T_k=Q_k+|a_{kk}-m|$, where $m=n^{-1}\sum_{i=1}^{n}a_{ii}$, $2S_k=R_k+T_k$ and denote by P, Q, R, S and T the greatest of the P_k, Q_k, R_k, S_k and T_k respectively. In this paper it is shown that: 1. The *field of values of* A (the set of complex numbers $xA\bar{x}'$ where $x\bar{x}'=\sum_{i=1}^{n}x_i\bar{x}_i=1$) lies in the circle of radius S with center at m. 2. Each characteristic root, λ, of the matrix A lies in (or on) at least one of the n ellipses $|\lambda-m|+|\lambda-a_{kk}|=P_k+R_k$ and in (or on) at least one of the n ellipses $|\lambda-m|+|\lambda-a_{kk}|=Q_k+T_k$. 3. Each characteristic root, λ, of the matrix A lies in (or on) at least one of the n Cassini ovals $|\lambda-a_{kk}|\cdot|\lambda-m|=P_kR_k$ and in (or on) at least one of the n Cassini ovals $|\lambda-a_{kk}|\cdot|\lambda-m|=Q_kT_k$. (Received November 10, 1947.)

116t. Edward Rosenthall: *Multiplicative diophantine equations in quaternions.*

The diophantine equation $XY=ZW$ in Lipschitz integral quaternions with proper parts of odd norm is solved completely in integral parameters, namely: $X=abS\bar{S}FPABC$, $Y=cdT\bar{T}\bar{C}\bar{B}\bar{A}(\bar{P}K+\bar{L}M)E$, $Z=adC\bar{C}F(K\bar{M}+P\bar{L})RST$, $W=cbB\bar{B}\bar{T}\bar{S}\bar{R}ME$, where the small letters represent rational integers and the others represent odd integral quaternions except that K or L must be even, and also $A\bar{A}=R\bar{R}$, $P\bar{P}=M\bar{M}$, which can be solved explicitly in rational integral parameters (E. T. Bell, Trans. Amer. Math. Soc. vol. 57 (1945) p. 95). Through properties of these quaternions developed by G. Pall (Trans. Amer. Math. Soc. vol. 47 (1940) pp. 487–500) the resolution of this equation is made to depend on the system $PN=QM$, $P\bar{P}=M\bar{M}$, the complete solution for which is $N=\bar{P}K+\bar{L}M$, $Q=K\bar{M}+P\bar{L}$. Other multiplicative equations are considered. (Received November 18, 1947.)

117t. R. D. Schafer: *Structure of genetic algebras.*

Etherington (in Proc. Roy. Soc. Edinburgh vol. 59 (1939) pp. 242–258, and other papers) has studied the non-associative algebras which arise in the symbolism of genetics. The author introduces a concept of genetic algebra which is intermediate between his (commutative) train algebra and special train algebra: a commutative baric algebra A over F is a genetic algebra in case the coefficients of the characteristic function $|\lambda I-T|$ of $T=\alpha I+f(R_{x_1}, R_{x_2}, \cdots)$ in the transformation algebra $T(A)$ are polynomials which, insofar as they depend on the x_i, depend only on the weights of the x_i. This definition is more satisfactory than that of special train algebra on two counts: the structure of the algebra is not postulated, and the duplicate of a genetic algebra is itself a genetic algebra (so that all of the fundamental algebras of symmetrical inheritance—zygotic, copular, and so on, as well as gametic—are genetic algebras). It is proved that the radical of A is the kernel of the weight function, and that it is nilpotent. Some special cases are treated in detail. (Received November 25, 1947.)

118*t*. A. R. Schweitzer: *Remarks on fields of ordered dyads.*

Postulates are constructed for a field of ordered dyads which permit interpretation of the latter as rational numbers. The postulates are based on multiplicative and additive groups of dyads and are analogous to postulates for a field previously reported (Bull. Amer. Math. Soc. Abstract 52-11-353). The postulates for the multiplicative group are the same as those given in the author's paper, *Remarks on groups of ordered dyads* (Bull. Amer. Math. Soc. Abstract 54-1-11). The postulates for the additive group are first stated in terms of undefined addition of dyads with the same posterior elements; it is then assumed: 1. α/β, γ/δ in T imply the existence of ξ, η, ζ in S such that $\alpha/\beta=\xi/\zeta$, $\gamma/\delta=\eta/\zeta$; α/β, γ/δ, λ/μ in T imply the existence of ξ, η, ζ, τ in S such that $\alpha/\beta=\xi/\tau$, $\gamma/\delta=\eta/\tau$, $\lambda/\mu=\zeta/\tau$. 2. $\xi/\tau+\eta/\tau=\zeta/\tau$, $\xi'/\tau+\eta'/\tau=\zeta'/\tau$, $\xi/\tau=\xi'/\tau$, $\eta/\tau=\eta'/\tau$ imply $\zeta/\tau=\zeta'/\tau$. Definition. $\alpha/\beta\dotplus\gamma/\delta=\lambda/\mu$ means: There exist ξ, η, ζ, τ in S such that $\alpha/\beta=\xi/\tau$, $\gamma/\delta=\eta/\tau$, $\lambda/\mu=\zeta/\tau$, $\xi/\tau+\eta/\tau=\zeta/\tau$. (Received November 24, 1947.)

119*t*. Olga Taussky and John Todd: *Covering theorems for groups.*

The following two problems have been studied. (1) G being an abelian group with n generators each of order p, S being the set of the $s=n(p-1)+1$ distinct powers of each of the elements of a fixed base, determine the smallest integer σ for which there exists at lest one set Σ of σ elements of G such that $G=\Sigma S$. (2) Determine the cases when Σ can be a subgroup or a coset in G as in the cases $p=2$, $n=7$; $p=3$, $n=4$. (Received October 31, 1947.)

ANALYSIS

120. R. P. Agnew: *Abel transforms and partial sums of Tauberian series.*

Let $u_0+u_1+\cdots$ be a series of complex terms for which $L=\limsup |nu_n|<\infty$. Let $\sigma(t)=\sum t^n u_n$ $(0<t<1)$ and let $s(x)=u_0+u_1+\cdots+s_n$ where n is the greatest integer not greater than x. Let $q>0$. Then, as $t\to 1-$, the relation (*) $\limsup |\sigma(t)-s(q/\log t^{-1})|\leq A(q)L$ holds when $A(q)$ is the constant defined by $A(q)=\gamma+\log q+2\int_q^\infty x^{-1}e^{-x}dx$. Moreover $A(q)$ is the best constant in the following sense. There is a real series $\sum u_n$ such that $0<L<\infty$ and the members of (*) are equal. The same is true when the relation (*) is replaced by the relation (**) $\limsup_{n\to\infty} |\sigma(e^{-q/n})-s(n)|\leq A(q)L$. This generalizes and unifies special results obtained in six recent papers by Hadwiger, Agnew, Hadwiger, Wintner, Hartman, and Hadwiger for special cases in which $q=\log 2$ and $q=1$. Relations among the six papers are set forth in recent and forthcoming issues of Mathematical Reviews. (Received November 17, 1947.)

121. D. F. Barrow: *On expressing a function of three variables in nomograph form.*

A function of three variables (x, y, z) is in nomograph form if it is expressed as a three-rowed determinant whose first, second, and third rows are functions of x alone, of y alone, and of z alone respectively. Nine necessary and sufficient conditions that a given function may be so expressed are found. These conditions are in the form of determinants which must vanish identically, the elements of the determinants being the given function and certain of its derivatives. Also, a formula is furnished for expressing the function in nomograph form if this is possible. (Received October 24, 1947.)

122. E. F. Beckenbach and William Gustin: *On the mean modulus of an analytic function.*

Let f denote a function of a complex variable analytic in the open unit circle. In this paper the convexity properties of the circumferential and areal mean moduli of order t $(0 \leqq t \leqq \infty)$ of the analytic function f over a circle of radius r $(0 \leqq r < 1)$ are investigated as functions of r. It is known that the logarithm of such a mean modulus is a convex function of $\log r$ for any f and any t; for $t = \infty$ this is the Hadamard three circle theorem. The following results are obtained. The set of t-values for which the circumferential mean modulus is a convex function of r for any f is a closed bounded set which includes the t-values $0, 1/2, 2/3, 1, 2$; and the set of t-values for which the areal mean modulus is a convex function of r for any f is a closed bounded set which includes the t-values $0, 1, 2$. (Received November 10, 1947.)

123*t*. Stefan Bergman and M. M. Schiffer: *Kernel functions in the theory of partial differential equations of elliptic type.*

Let B be a plane domain with smooth boundary C. $P(Z)$ is a continuous positive function of $Z = (x, y)$ in B. Let $a(Z)$ and $b(s)$ be continuous differentiable in B and in C and suppose $a(Z) \cos (n, x) + b(Z) \cos (n, y) = -\lambda(Z)$ on C continuous and negative. The scalar product $D(\phi, \psi) = \iint_B [\phi_x\psi_x + \phi_y\psi_y + (a\phi\psi)_x + (b\phi\psi)_y + P\phi\psi]dxdy$, $\phi_x = \partial\phi/\partial x, \cdots$, is defined for each pair ϕ, ψ in the function space Ω, consisting of all continuously differentiable functions in B for which $D\{\phi, \phi\} < \infty$, the integrals understood in the Lebesgue sense. In the subspace Λ of Ω, consisting of all solutions of $\Delta\phi = P\phi$, choose a complete system $\{\phi_\nu(Z)\}$ orthonormalized with respect to the D-metric. Its kernel function $K(Z, W) = \sum_{\nu=1}^{\infty} \phi_\nu(Z)\phi_\nu(W)$ satisfies $K(Z, W) = R(Z, W) - G(Z, W)$; $G(Z, W)$ is Green's function of the differential equation and $R(Z, W)$ the fundamental solution satisfying on C: $\partial R/\partial n = \lambda R$. In the case $\lambda = a = b = 0$, $R(Z, W)$ equals the Neumann function (cf. Bergman-Schiffer, Duke Math. J. vol. 14 (1947) pp. 609–638). The kernel function is always non-negative in B and satisfies $K(Z, W)^2 \leqq K(Z, Z)K(W, W)$. If $R_\nu(Z, W)$ satisfies $\partial R_\nu/\partial n = \lambda_\nu R_\nu$ on C $(\nu = 1, 2)$ and if $\lambda_1 > \lambda_2$, one has $R_2(Z, W) \geqq R_1(Z, W)$ in B. (Received December 22, 1947.)

124*t*. Stefan Bergman and M. M. Schiffer: *On nonlinear differential equations of elliptic type.*

We use the notations of the preceding abstract. Let $P_1(Z) > P_2(Z) > 0$ be two continuous functions in B. Consider the two differential equations $\Delta\phi = P_\nu\phi$ $(\nu = 1, 2)$. For a fixed continuous function $\lambda(Z) > 0$ on C we have two corresponding fundamental solutions $R_\nu(Z, W)$. There holds: $R_1(Z, W) < R_2(Z, W)$ in B. The same inequality holds for the corresponding Green's functions and kernel functions. Using these facts, the following theorem is proved: Let $P(Z; t, t', t'')$ be defined for $Z \in B$ and real t, t', t''; suppose that a Lipschitz condition $|P(Z; t, t', t'') - P(Z; t_1, t_1', t_1'')| < K \cdot \max \{|t - t_1|, |t' - t_1'|, |t'' - t_1''|\}$ and the inequality $M \geqq P(Z; t, t', t'') \geqq m > 0$ hold. Then the differential equation $\Delta\phi = P(Z; \phi, \partial\phi/\partial x, \partial\phi/\partial y) \cdot \phi$ possesses always solutions for continuously prescribed boundary values on C. Generalizations of this theorem in various directions are indicated. (Received December 22, 1947.)

125. L. M. Blumenthal: *Metric methods in linear inequalities.*

Irreducible systems of linear inequalities in $n+1$ indeterminates $(n \geqq 1)$, of arbitrary power not greater than c, are studied by applying metric-theoretic methods and results to the subset C of the unit n-sphere $S_{n,1}$ formed by normalizing the coefficient set, and the closed, convex subset $\Sigma(C)$ of $S_{n,1}$ whose elements correspond to the normalized solutions of the system. Among properties established for the set function Σ are: (1) $\Sigma(C)=\Sigma(\overline{C})=\Sigma(C^*)$, where $\overline{C}$, C^* denote closure and convex extension of C, respectively; (2) $\Sigma(C)=C$ provided C is closed, convex, and $p\in B(C)$, the boundary of C, implies $\text{diam}_p B(C)=\pi/2$; (3) $\Sigma(C)=C$ implies $B(C)$ of constant (spherical) breadth, but not conversely; (4) $\Sigma^3(C)=\Sigma(\Sigma(\Sigma(C)))=\Sigma(C)$ for every C. New criteria for existence of solutions (and geometric derivations of some previously known) are given. A chain of existence theorems is obtained by proving that a system of inequalities, in the wide sense, has a nontrivial solution if and only if each $n+k+1$ of them has a nontrivial solution in common with an arbitrarily selected subsystem of $n-k+1$ inequalities whose representative points of $S_{n,1}$ do not lie on an $S_{n-k-1,1}$, where k may be any one of the numbers $0, 1, \cdots, n+1$. (Received November 19, 1947.)

126t. Robert Breusch: *On the distribution of the values of $|f(z)|$ in the unit circle.*

Let $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ be a polynomial with $\prod_{\nu=1}^{n}|z_\nu|=1$. If $A(f)$ is the area of the region of the unit circle where $|f(z)|\geqq 1$, then obviously $0\leqq A(f)\leqq\pi$. It is shown that this estimate cannot be improved. That is, if α and β are two non-negative real constants such that $\alpha\leqq A(f)\leqq\pi-\beta$ for every $f(z)$, then $\alpha=\beta=0$. That $\beta=0$ is trivial; for example consider $f(z)=1+N\cdot z+z^2$, with large N. To prove that $\alpha=0$, it is shown that to every polynomial $f(z)$ another polynomial $f_1(z)$ may be constructed such that $A(f_1)<2A(f)/3$. (Received November 21, 1947.)

127t. K. Chandrasekharan. *On Fourier series in several variables.*

If $f_p(t)$ is the spherical mean of order p of the function $f(x_1, \cdots, x_k)$ at a given point, and $S^\delta(R)$ is the Riesz mean of order δ of the multiple Fourier series of $f(x_1, \cdots, x_k)$ at that point (Proc. Indian Acad. Sci. vol. 24 (1946) pp. 229–232), a formula is established, which expresses $f_p(t)$ as a transform of $S^\delta(R)$ for almost all t, and for $p>\delta-(k-1)/2$. As consequences of the formula, theorems are obtained which affirm the existence of the mean limit of a function on condition that its Fourier series be summable. Some of the results seem to be new even in the case of functions of a single variable. (Received November 10, 1947.)

128. F. E. Clark: *Positive representation of forms.*

Let F be a form with real coefficients and degree p in the variables $x_1, \cdots, x_m$, which range independently over all non-negative values except $0, \cdots, 0$. It is known that the form F is positive if and only if there exists a form f such that f and fF have positive coefficients. For $m=p=2$, Meissner (*Über positive Darstellungen von Polynomen*, Math. Ann. vol. 70 (1911) pp. 223–235) gave a system of inequalities on the coefficients of f, found the minimum n ($=\deg f$) for which that system is consistent, and solved the system in terms of an integral. His method was geometric throughout. In the present paper the author solves this system by means of a finite sum. Pólya has proved the general result, using an f with multinomial coefficients but giving no information about the value of n. For the case of multinomial coefficients and $m=p=2$, the author gives a formula for the minimum n, which is shown to be odd and in general much larger than that for the Meissner problem. (Received November 21, 1947.)

129. L. W. Cohen and Casper Goffman: *A theory of transfinite convergence.*

A theory of convergence based on the definition of a sequence as a function x_ξ on the domain of ordinals $\xi<\xi^*$ to a uniform space is given. It yields the category theorem for complete spaces and covering theorems of Lindelöf and Borel-Lebesgue in which $\aleph_0$ is replaced by the cardinal of ξ^*. The ordinal ξ^* is required to be the initial ordinal of its cardinal Ξ^* and have the property that if $\eta^*<\xi^*$ and ξ_η is single-valued on $\eta<\eta^*$ to $\xi<\xi^*$ then sup $[\xi_\eta|\eta<\eta^*]$. Examples are ω, the classical case, and Ω. The uniform space is a set $S=[x]$ with neighborhoods $U_\xi(x)$, $\xi<\xi^*$, having the following properties: (1) $\bigcap_{\xi<\xi^*}U_\xi(x)=x$, (2) $U_\xi(x)$ decreases on $\xi<\xi^*$, (3) if $\eta<\xi^*$ there is a $\xi(\eta)$ such that $U_{\xi(\eta)}(y)\cap U_{\xi(\eta)}(x)\neq 0$ implies $U_{\xi(\eta)}(y)\subset U_\eta(x)$, (4) if $U_{\xi_\eta}(x_\eta)$ decreases on $\eta<\eta^*<\xi^*$ then $\bigcap_{\eta<\eta^*}U_{\xi_\eta}(x_\eta)$ is a nonempty open set. Examples are given and it is shown that if (4) is replaced by (4') in which $x_\eta=x$ for $\eta<\eta^*$ then there are spaces with $\xi^*>\omega$ which are ξ^*-complete and of the first ξ^*-category. (Received November 24, 1947.)

130. Aryeḥ Dvoretzky: *On sections of power series.*

Through the use of the theory of normal families many extensions of the theorem of Jentzsch are obtained. The following results are typical. (1) Let $f(z)=\sum_0^\infty a_n z^n$ have 1 as radius of convergence, let $\{\rho_n\}$ be a sequence of positive numbers satisfying (*) $\log\rho_n=o(n)$, let z_0 be any number with $|z_0|=1$ and $\epsilon>0$ arbitrarily small. Then there exist infinitely many n for which $w=s_n(z)=\sum_1^n a_\nu z^\nu$ maps $|z-z_0|<\epsilon$ on a domain completely covering $|w|<\rho_n$. (2) Let $m=m(n)$ be $o(n)$; then under the same conditions as in (1) there exist infinitely many n for which $w=\sum_m^n a_\nu z^\nu$ maps $|z-z_0|<\epsilon$ on a domain completely covering $1/\rho_m<|w|<\rho_n$. (3) If in addition to the assumptions made in (1), z_0 is a regular point of $f(z)$, then there exist infinitely many n for which $w=f(z)-s_n(z)$ maps $|z-z_0|<\epsilon$ on a domain completely covering $1/\rho_n<|w|<\rho_n$. It is easy to show that (*) cannot be essentially relaxed; the exclusion of the neighborhood of $w=0$ in (2) and (3) is also essential. Finally, a result of another nature may be mentioned. (4) Let $0<\limsup_{n\to\infty}|a_n|<\infty$, let z_0 with $|z_0|=1$ be a regular point of $\sum_0^\infty a_n z^n$ and d_n the distance from z_0 to the nearest root of $\sum_0^n a_\nu z^\nu=0$, then lim inf $nd_n/\log n<\infty$. (Received November 24, 1947.)

131. Tomlinson Fort: *The value of a product.*

In this paper the product $L_w=(1-L_{1w})(1-L_{2w})\cdots(1-L_{ww})$ and the infinite product, where w becomes infinite, are considered. Both products are transformed into series; the finite product into the sum obtained by performing the indicated multiplication and the infinite product into the corresponding infinite series. Known relations among the elementary symmetric functions are used in studying the resulting series. Several theorems are proved. Formulae to facilitate the calculation of the first few terms of the series are developed. (Received November 3, 1947.)

132*t*. Evelyn Frank: *Orthogonality properties of C-fractions.*

Let the polynomials $B_p^*(z)=z^nB_p(1/z)=\beta_0^{(n)}z^n+\beta_1^{(n)}z^{n-1}+\cdots+\beta_n^{(n)}$, $n=h_p=\alpha_1+\alpha_3+\cdots+\alpha_{2q-1}$, if $p=2q-1$, or $n=g_p=\alpha_2+\alpha_4+\cdots+\alpha_{2q}$, if $p=2q$, be determined by the formulas $B_0^*(z)=1$, $B_1^*(z)=z^{\alpha_1}B_0^*(z)=z^{\alpha_1}B_1(1/z)$, $B_{2p-1}^*(z)=z^{h_p-h_{p-1}}B_{2p-2}^*(z)+a_{2p-1}B_{2p-3}^*(z)=z^{h_p}B_{2p-1}(1/z)$, $B_{2p}^*(z)=z^{g_p-h_p}B_{2p-1}^*(z)+a_{2p}B_{2p-2}^*(z)=z^{g_p}B_{2p}(1/z)$. The $B_p(z)$ are the denominators of the approximants of an α-regular C-fraction $C(z)$ (Frank, Amer. J. Math. vol. 68 (1946)). If S is defined as the operator which replaces every

s^p by c_{p+1} in any polynomial upon which it operates, then the orthogonality relations $S(B_{2p}^*B_{2k}^*)=0$ if $k\neq p$, $\omega=0$, $S(B_{2p}^*B_{2k}^*)\neq 0$ if $\omega\neq 0$, hold with respect to the c_p, and similar conditions are shown to hold for $S(B_{2p}^{**})$, $S(B_{2p-1}^*B_{2k-1}^*)$, $S(B_{2p-1}^{**})$, $S(B_{2p}^*B_{2p-1}^*)$, $S(B_{2p-2}^*B_{2p-1}^*)$. Orthogonality relations are also shown for regular C-fractions (cf. Amer. J. Math. vol. 68 (1946) p. 94). If the even approximants of $C(1/s)$ are regarded as generalizations of the approximants of a J-fraction (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 144–157), the above orthogonality relations reduce to the known relations $S(B_pB_q)=0$, $p\neq q$, $S(B_pB_q)\neq 0$, $p=q$. (Received November 24, 1947.)

133*t*. R. E. Fullerton. *Integral distances in Banach spaces.*

Let x_n be an infinite set of points in a Banach space B such that $\|x_i-x_j\|$ is an integer for each i and j. If B is not strictly convex there exists such a set which does not lie on a line. If B is strictly convex, no infinite subset lies on a line unless all the points of the set lie on the line. (Received November 24, 1947.)

134. Casper Goffman: *New proof of a theorem of Saks and Sierpinski.*

A simple proof is given of the following theorem of Saks and Sierpinski: If $f(x)$ is an arbitrary real function on the closed interval $[0, 1]$, there is a function $\phi(x)$ of Baire class 2 at most, such that for every $\epsilon>0$, $|\phi(x)-f(x)|<\epsilon$ on a set of exterior measure 1. The proof given rests on the following lemma which seems to be new. If $f(x)$ is an arbitrary real function on the closed interval $[0, 1]$, $\epsilon>0$, and $\phi(x)$ a continuous function such that $|\phi(x)-f(x)|<\epsilon$ on a set of exterior measure greater than $1-\epsilon$, then for every $\eta>0$, there is a continuous function $\psi(x)$ such that $|\psi(x)-\phi(x)|<\epsilon$ on a set of measure greater than $1-\epsilon$ and $|\psi(x)-f(x)|<\eta$ on a set of exterior measure greater than $1-\eta$. (Received November 24, 1947.)

135. M. O. Gonzalez: *On the solution of ordinary differential equations of the first order invariant under contact transformations.*

In the classical Lie theory it is shown how to construct a differential equation invariant under a given group, and how to solve an equation when a group leaving the equation invariant is known. However, little is said about the problem of determining the group for a given differential equation, which is by far the most interesting problem. In the present paper, necessary and sufficient conditions for the existence of an infinitesimal contact transformation leaving a given equation invariant are determined along with the general form of the characteristic function of the group. It will also be shown how to reduce, by a proper change of variables, the infinitesimal contact transformation to a point transformation. This enables one to solve the transformed differential equation by Lie's methods. Passing back to the original variables, a new differential equation is obtained which combined with the original equation gives its solution in parametric form. (Received November 13, 1947.)

136. H. L. Krall: *On orthogonal polynomial solutions of certain differential equations of orders four and six.*

Let $\rho(x)$ represent the weight functions of the classical orthogonal polynomials of Legendre, Jacobi and Laguerre: 1, $(1-x)^\alpha(1+x)^\beta$, e^{-x} respectively (in the Jacobi case assume $\alpha\beta=0$). The author studies the orthogonal polynomials having the Stieltjes weight function $\rho(x)d\psi(x)$, where $\psi(x)$ has a jump at the end or ends of the interval, and $\psi(x)=x$ in the interior. These polynomials satisfy a differential equation of the form $\sum_{i=0}^{r}\sum_{j=0}^{i}l_{ij}x^{j}y_n^{(i)}(x)=\lambda_n y_n(x)$ with $r=4$, and also with $r=6$. By elimination, differential equations of the second order are obtained for these polynomials. (Received November 14, 1947.)

137*t*. Norman Levinson: *A simple second order differential equation with singular motions.*

Cartwright and Littlewood have announced the equation $\ddot{x}+k(x^2-1)\dot{x}+x=bk\lambda\cos\lambda t$ has singular motions. Here a simple proof for singular motions is given for $\ddot{y}+p(y)\dot{y}+y=c\sin t$ where $p(y)$ is a polynomial and c constant. In (*) $\epsilon\ddot{x}+\phi(x)\dot{x}+\epsilon x=b\sin t$ let $\epsilon>0$ be small and let $\phi(x)=1$, $|x|>1$, $\phi(x)=-1$, $|x|<1$. Clearly (*) can be solved explicitly. Certain families of solutions starting at $t=\pi+\tau$, $|\tau-\epsilon/b|\leq k\epsilon$ where k is small, with $x=-1$ all eventually cross $x=1$ from above to below over a range in t that includes $t=2n\pi+\tau$ and $t=2(n+1)\pi+\tau$ where $|\tau-\epsilon/b|\leq 2k\epsilon$. Each of the τ intervals on $x=1$ then map into two others on $x=-1$, and so on. Therefore, there exists (Cartwright and Littlewood) a motion corresponding to any infinite array of $(2n\pm1)\pi$. Next let $\epsilon\dot{x}_1=x_2-\Phi(x_1)$, $\dot{x}_2=-\epsilon x_1+b\sin t$ where $\Phi'(x)=\phi(x)$. Here x_1 is x (*). Let $P(x)$ be a polynomial such that $P(x)-\Phi(x)$ is small for $|x|<6$. Then the solutions (y_1, y_2) of $\epsilon\dot{y}_1=y_2-P(y_1)$, $\dot{y}_2=-\epsilon y_1+b\sin t$ are near (x_1, x_2). If $y=y_1$, this implies $(y, \dot{y})$ is near $(x, \dot{x})$ and therefore the same mapping of τ intervals holds for y as for x thereby giving singular motions for y just as for x. (Received November 13, 1947.)

138. R. J. Levit: *A lower bound for the maximum absolute value of a real function on a finite set.*

Let S_n be a finite set of real numbers, $x_0<x_1<\cdots<x_n$, and $f(x)$ a function which is real and single-valued over S_n and has the nth divided difference $f(x_0, \cdots, x_n)=f_n$. In this paper it is shown that the maximum absolute value of $f(x)$ over S_n cannot be less than a certain number $M_n=M(f_n; x_0, \cdots, x_n)$. Results of this kind for the case of $f(x)$ a polynomial have been obtained by G. Pólya (Jber. Deutschen Math. Verein. vol. 28 (1919) pp. 31–40) and T. Tatuzawa (Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 253–254). A lower bound is found for the maximum absolute increment, $|f(x_j)-f(x_i)|$, also. Both bounds are in a certain sense "best possible." If $f(x)$ has a finite derivative of order n, the foregoing results yield a generalization of the mean value theorem for derivatives. A geometric interpretation of the bound M_n is given, and various applications of the theory are made. (Received November 24, 1947.)

139*t*. E. P. Miles: *An integral formula for harmonic functions*. Preliminary report.

Let $u\in L$ on the surface σ of a sphere of radius a about 0. Let U be the function, harmonic in the interior of S of σ, given by the Poisson integral of u. Let $|\nabla U|$ be the magnitude of the gradient of U. The author proves that (1) $\iiint_S|\nabla U|^2dV=(1/4\pi)\cdot\iint_\sigma\iint_\sigma[u(P)-u(Q)]^2d\sigma_Pd\sigma_Q/\overline{PQ}^4=\sum_{n=1}^{\infty}(2an\pi/2n+1)\{2A_n^2+\sum_{m=1}^{n}[(n+m)!/(n-m)!](A_{n,m}^2+B_{n,m}^2)\}$, where A_n, $A_{n,m}$, and $B_{n,m}$ are coefficients in the Laplace series for u. The equalities (1) are analogues for three dimension of similar equalities for two dimensions obtained by J. Douglas in his paper *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. vol. 33 (1931) pp. 263–321. (Received October 24, 1947.)

140t. Harold Shniad: *On the convexity of mean value functions.*

Let a set of n distinct positive numbers $a_1, a_2, \cdots, a_n$ $(n \geqq 2)$ be given and labeled so that $a_1 < a_2 < \cdots < a_n$. Let $M_t(a, \xi)$ denote the mean value function $(\sum_{\nu=1}^{n} \xi_\nu a_\nu^t)^{1/t}$, where (ξ_ν) is a set of n fixed positive numbers with $\sum_{\nu=1}^{n} \xi_\nu = 1$. The convexity and concavity of the functions $M_t(a, \xi)$ and $\Lambda(t)$ $(= \log M_t(a, \xi))$ are studied. If there exists a point $\bar{t}$ such that a function $g(t)$ defined on the range of reals is convex for $t < \bar{t}$ and concave for $t > \bar{t}$, we say that $g(t)$ is convex-concave. It is shown that there exists a point t_1 such that $\Lambda(t)$ [and therefore $M_t(a, \xi)$] is convex for $t < t_1$, that there exists a point t_2 such that $\Lambda(t)$ is concave for $t > t_2$, but that neither the function $\Lambda(t)$ nor the function $M_t(a, \xi)$ is necessarily convex-concave. If $\xi_1 \geqq 1/2$, then $\Lambda(t)$ is convex for all negative t; and if $\xi_n \geqq 1/2$, then $\Lambda(t)$ is concave for all positive t. Finally it is shown that even if the condition $\sum_{\nu=1}^{n} \xi_\nu = 1$ is removed, the zeros of the second derivative of $\Lambda(t)$ are bounded. (Received November 21, 1947.)

141. Otto Szász: *On Tauberian theorems for generalised Lambert series and integrals.*

Consider the integral transform $I_\beta(t) = \int_0^\infty p_\beta(tu) dA(u)$, where $p_\beta(x) = x^{\beta+1} \sum_1^\infty n^\beta e^{-nx}$, and $A(u)$ is such that $I_\beta(t)$ exists for $t > 0$. If $A(u)$ is a step function with jumps a_n at points λ_n, then the transform reduces to $L_\beta(t) = \sum a_n p_\beta(t\lambda_n)$. In particular for $\beta = 0$, $p_0(x) = xe^{-x}/(1 - e^{-x})$, $a_0(t) = t\sum a_n \lambda_n e^{-\lambda_n t}/(1 - e^{-\lambda_n t})$. A Tauberian theorem for this transform, when the coefficients are positive, was given by V. Ganapathy Iyer on using the general Tauberian theorem of N. Wiener. The author gives a Tauberian theorem for the transform $I_\beta(t)$, when $A(u)$ is monotone, by using the following theorem on closure: the sequence $p_\beta(nt)/p_\beta(t)$, $n = 1, 2, \cdots$, $-1 < \beta \leqq 2$, is closed in $C(0, \infty)$. He gives an elementary proof for this theorem with the aid of Möbius' inversion formula utilizing a remark due to J. Korevaar. In the generalization of a theorem of Hardy and Littlewood, however, transcendental methods are required. (Received November 15, 1947.)

142t. Olga Taussky: *A boundary value problem for a hyperbolic differential equation arising in the theory of the non-uniform supersonic motion of an aerofoil.*

A solution of the linearized equation for the velocity potential of a non-uniform supersonic two-dimensional motion is obtained in terms of the normal derivative along the aerofoil. The problem is treated as a Cauchy problem since the values of the potential along the aerofoil can be eliminated by a reflexion. The expression coincides with a formula of C. Possio. The treatment is a generalization of a procedure used by G. Temple and H. A. Jahn in the case of harmonic motion. (Received November 15, 1947.)

143. H. P. Thielman: *On commutative functions and functional equations.*

Two functions $f(x)$ and $g(x)$ are said to be commutative if $f\{g(x)\} = g\{f(x)\}$. The functional product $f\{g(x)\}$ is designated by $fg(x)$, or simply by fg. The product of ordinary multiplication of $f(x)$ by $g(x)$ is indicated by $f(x) \cdot g(x)$. Use is made of the notation and certain methods and results of A. G. Walker's papers *Commutative functions* (Quarterly Journal of Mathematics vol. 17 (1946) pp. 65–92). A function $f(x)$ is said to be expressed in canonical form if $f(x) = \psi^{-1}\alpha\psi(x)$, where α is a constant and ψ^{-1} is the inverse of $\psi(x)$. A number of theorems are proved on the equivalence of canonical forms. The functional equation $f[\phi(x, y)] = f(x) \cdot f(y)$ is considered with $\phi(x, y) = \psi^{-1}[\psi(x) + \psi(y)]$. It is proved that all continuous solutions of the given functional equation are of the form $A^{\psi(x)}$, where A is a constant. A number of generalizations of Cauchy's functional equations found in the mathematical literature are special cases of the one considered here. (Received November 24, 1947.)

144t. Alexander Weinstein: *On generalized axially symmetric potential.*

Continuing previous investigation (Bull. Amer. Math. Soc. vol. 53 (1947) p. 59) it is shown that the generalized axially symmetric potential $\phi(x, y) = y^{-q}\int_0^\infty \exp(-|x|t)J_q(yt)J_{q+m}(bt)dt$; $q > -1/2$, $b > 0$, $m = 0, 1, 2, \cdots$, has a logarithmic singularity at the point $x = 0$, $y = b$. The corresponding stream function is many-valued and has a branch which admits, for $x > 0$, the integral representation $\psi(x, y) = -y^{q+1}\int_0^\infty \exp(-xt)J_{q+1}(yt)J_{q+m}(bt)dt$. (Received November 20, 1947.)

145t. Bertram Yood: *Banach algebras of bounded functions.*

Let B be the Banach algebra of all bounded complex-valued functions defined on a set S. The maximal ideals of B are shown to be in a natural 1-1 correspondence with the two-valued bounded additive set functions defined on the class of all subsets of S and with the maximal additive ideals of the Boolean algebra of subsets of S. Maximal ideals of subalgebras of B are investigated. A generalized Weierstrass theorem for such algebras is given. The theory is used to give a proof of Stone's theorem on the topological representation of distributive lattices. (Received November 24, 1947.)

APPLIED MATHEMATICS

146t. Garrett Birkhoff: *Remarks on streamlines of discontinuity.*

Kreisel, Gilbarg and Rock have obtained the flow pattern for two-dimensional cavity motion past a flat plate, with a reentrant jet, at any cavitation number K. The present note relates the cavity drag D with the fluid density ρ, the jet cross-section A^*, the jet velocity v, and the free stream velocity V, by $D = \rho v(v + V)A^*$. This is valid for two-dimensional or axially symmetric three-dimensional flow with reentrant jet, past any symmetric obstacle. If A is the obstacle cross-section, then $C_D = 2(1 + K + (1 + k)^{1/2})A^*/A$; if $K = 0$, this reduces to $A^*/A = C_D/4$. The general velocity relation across a streamline of discontinuity ("stationary slipstream") is shown to be $v'^2 = Av^2 + B$, whenever the equations of state have the forms $\rho = kp^\alpha$, $\rho' = k'p'^\alpha$ on the two sides of the slipstream. Using a formula obtained jointly with T. E. Caywood, the flow pattern for penetration of an incompressible fluid of density ρ by a two-dimensional jet of density ρ' is obtained as a special application, with $B = 0$. Examples with $B \neq 0$, having circular and spherical slipstreams, are also sketched. (November 13, 1947.)

147. H. E. Goheen: *The critical radius of a heat-generating cylinder.*

Using methods similar to the heuristic methods described in his paper, *A non-rigorous suggestion for the solution of certain problems in heat generation*, the author discusses a problem in the design of cylindrical catalytic beds. The partial differential equation is $\partial u/\partial t = \partial^2 u/\partial r^2 + (1/r)(\partial u/\partial r) + e^{-1/u}$ in dimensionless variables. The initial conditions are at $t = 0$, $u = \theta_0$ if $r < a$ and $u = \theta_1$ if $r > a$. The problem is to determine a critical radius A such that if $a>A$ overheating occurs, while if $a<A$ it does not. (Received October 29, 1947.)

148*t*. M. Z. Krzywoblocki. *On the boundary layer motion along a periodically oscillating plane in compressible viscous fluid.*

The following assumptions are made: the fluid outside the plane at rest is in a periodically oscillating motion (or vice versa, the fluid is at rest and the plane in motion). The flow outside the boundary layer is a potential flow. The following equations were taken into account as concerning the two-dimensional motion in the boundary layer: two equations of motion, equations of continuity, state and energy. The coefficients of viscosity and heat conductivity are assumed to be functions of the temperature only. The solutions are presented as trigonometric series, in which each term may be found by the successive approximation method. By suitably chosen simplifications the nonlinear equations are transformed into linear partial differential equations of the second order with constant coefficients. The first two terms in each series were found, satisfying all the required boundary conditions. (Received November 17, 1947.)

149. P. F. Neményi and R. C. Prim: *On helicoidal flow patterns of a perfect gas.*

Flow patterns for which the reduced velocity W satisfies the condition $W\times\text{curl } W=0$ are discussed with reference to those cases in which the streamlines are coaxial helices. The special case when the consecutive curls satisfy the same condition ($\text{curl } W\times\text{curl curl } W=0$, and so on) is discussed and leads to a solution in terms of Bessel function. The case when all helices have a common slope and the case when all have a common pitch are also analyzed and the connection of the latter case with the irrotational helicoidal flow of an incompressible fluid discovered by von Laue and Hamel is established. (Received November 20, 1947.)

150*t*. S. S. Penner and Seymour Sherman: *Heat flow through composite cylinders.*

A formal solution of the differential equation for a time dependent flow of heat through a composite cylinder is given. The composite cylinder is assumed to be thermally insulated at the outer boundary and to consist of a cylindrical core of one material, initially at the uniform temperature T_0, surrounded by a cylindrical shell of another material, initially at temperature zero. This work is based on OSRD Report no. 4963 (PB Report no. 50864) *On the diffusion of nitroglycerin in wrapped powder grains*. The paper appears in the August number of the Journal of Chemical Physics. (Received October 7, 1947.)

151. Harry Polachek: *An extension of the method of least squares to systems of ordinary differential equations.*

The method of least squares is extended to apply to problems in which the governing relations are given in the form of a system of ordinary differential equations. In such problems, the experimental data may consist of three types of quantities: (a) measured values for the initial conditions to the differential equations, (b) parameters entering in the equations, or (c) values obtained in the solution. The reduction of trajectory data is a good illustration of a problem of this type. A nonlinear system of two second order differential equations is involved, which has no known solution in closed form. The experimental information to which this system must be fitted usually consists of the components of the initial velocities (initial conditions) and the final range and time of flight (solution to the differential equations). The problem is to determine certain ballistic coefficients which appear as parameters in the equations. (Received November 20, 1947.)

152. R. C. Prim: *On the existence of steady gas flow in plane isothermal streamline patterns.*

An investigation is made of the possibility of a steady plane flow of an ideal gas in the absence of body forces having a streamline pattern which can be mapped conformally onto a family of parallel lines. Irrotationality of the gas flow is not assumed. It is proved that the only possible flows of this type are those having streamlines which are concentric circles, parallel straight lines, or radial straight lines. (Received November 20, 1947.)

153. L. V. Robinson: *The solution of partial differential equations by differential operator methods.* Preliminary report.

It is shown how all linear partial differential equations, and some nonlinear ones, can be solved by operator methods involving exp (gD), where the g's are particular functions of $x, y, z, \cdots$, and where $D=\partial/\partial x, \partial/\partial y, \partial/\partial z, \cdots$. In general these methods are simpler than the usual ones, and Laplace transformations are not necessary to solve second-order linear equations. (Received November 21, 1947.)

154. J. L. Synge: *Canonical coordinates for an electro-magnetic field.* Preliminary report.

The author deals with an electromagnetic field in vacuo, and the basic idea is to present a relativistic theory without introducing a metric tensor or other structural tensor in space-time. The field is represented by two skew-symmetric covariant tensors F_{mn} and F^*_{mn}; these correspond respectively to the usual vector pairs (B, E) and (D, H). We can at once write down covariant Maxwellian equations $F_{mn,r}+F_{nr,m}+F_{rm,n}=0$ (the comma denoting partial differentiation), and the same equations with a star, but six equations are still lacking to make the number of equations equal to the number of unknowns. The question discussed is that of the existence of coordinate systems (called canonical) for which $F^*_{23}=iF_{14}$, $F^*_{31}=iF_{24}$, $F^*_{12}=iF_{34}$, $F^*_{14}=iF_{23}$, $F^*_{24}=iF_{31}$, $F^*_{34}=iF_{12}$. (The presence of the factor i is due to the imaginary character of the fourth coordinate.) The problem breaks into two parts; the first is algebraic, and concerns the existence of coordinates canonical at a point; the second part involves questions of integrability. (Received November 24, 1947.)

155. C. A. Truesdell: *The transport of vorticity.*

The mechanism by which vorticity is transported in the motion of continuous media is discussed in the light of a general kinematic theorem for the rate of change of the total vorticity of a finite volume. The general theorem is first applied to four kinematic situations, one being the infinite region treated originally by Poincaré for the special case of plane flow of an incompressible viscous fluid. It is shown that Poincaré's theorem, if slightly rephrased, holds in the three-dimensional motion of an arbitrary continuous medium. The general theorem is then applied to the discussion of diffusion of vorticity in a compressible viscous fluid. (Received November 20, 1947.)

156. A. H. Van Tuyl: *The distribution of electricity on two neighboring charged spheres in the presence of an outside point charge.*

In order to study the general problem it is sufficient to consider three special cases. For this investigation, the cases chosen are the following: Two spheres at unit potential when no outside influence acts, two such spheres when the potentials are one and minus one, and two spheres at zero potential in the presence of an outside unit point charge. Without loss of generality, unity and b are taken as the radii. The charge densities are expressed as definite integrals whose integrands contain elliptic functions, and with the aid of these integral representations the charge distributions are studied in the limit as the spheres approach contact. Denoting the distance between the spheres by ϵ, Kirchhoff found that for two spheres of radius unity and potential unity, the charge density at the point of final contact is asymptotically equal to $\pi\epsilon^{-3/2}\exp\left[-\pi^2(2\epsilon^{1/2})^{-1}\right]$. A similar result is found here for two spheres at zero potential in the presence of an outside point charge. Various expansions are found for the limiting charge densities, one of which is a generalization of some expansions given by Poisson. An application of the preceding results is made to the study of circles of equilibrium in the limit. (Received November 20, 1947.)

GEOMETRY

157. J. M. Clarkson: *An irreversible line transformation.*

An arbitrary line l of space meets each of two generators h_1, h_2 belonging to the same regulus K of a quadric surface S. These lines h_1, h_2 each meet a ruled cubic surface C in three points at which there are tangent planes to C. These triads of tangent planes meet in two points which determine a line l'. The line l' is called the transform of the line l under the line transformation T. The irreversibility of T, the order of T and the singular lines of T are discussed. (Received December 16, 1947.)

158. N. A. Court: *Skewly cevian tetrahedrons.*

If a tetrahedron $(T') = A'B'C'D'$ is inscribed in a tetrahedron $(T) = ABCD$ and the lines AA', BB', CC', DD' form a hyperbolic group, the tetrahedron (T') may be said to be *skewly cevian* to, or for, (T). The lines AA'', BB'', CC'', DD'' passing through the vertices of (T) and meeting the lines AA', BB', CC', DD' form a second hyperbolic group whose traces in the faces of (T) are the vertices of a second tetrahedron $(T'') = A''B''C''D''$ which is also skewly cevian to (T). The two tetrahedrons (T'), (T'') may be said to constitute a *twin pair of skewly cevian tetrahedrons for* (T). The feet of the twelve cevians of the four vertices of (T') for the respective faces of (T) coincide with the twelve analogous points obtained for the vertices of (T''). These twelve points lie on a quadric surface. The lines joining corresponding vertices of a twin pair of skewly cevian tetrahedrons (T'), (T'') form a hyperbolic group. The anharmonic ratio of these four hyperbolic lines is equal to that of the pencil of lines $T''(T'ABC)$ in the plane ABC. (Received November 17, 1947.)

159. G. B. Huff: *Diophantine problems in geometry and elliptic ternary forms.*

Several diophantine problems in elementary geometry are equivalent to the determination of the rational solutions of the elliptic cubic C: $ax_1(x_2^2-x_3^2)=bx_2(x_1^2-x_3^2)$. It is shown that any elliptic cubic with coefficients in a field k of algebraic numbers and possessing a certain configuration of eight points may be represented by C. The theory of C is developed by the methods of Hurwitz and Mordell. Certain nonexistence theorems are proved when k is the field of rational numbers and applications are made to the problems in geometry. (Received November 7, 1947.)

160t. M. O. Reade: *Extension to minimal surfaces of a theorem of Lindelöf.*

Let $f_j(z)$ be analytic for $|z| = |u+iv| < 1$ and continuous for $|z| \leq 1$, $j = 1, 2, 3$, and let $\sum_{j=1}^{3} [f_j'(z)]^2 = 0$. Then the surfaces $S: x_j = Rf_j(z)$ and $S^*: y_j = If_j(z)$, $j = 1, 2, 3$, are adjoint minimal surfaces given in isothermic representation. Then an extension of a result due to Lindelof (4th Scandinavian Mathematical Congress, Stockholm, 1916, pp. 59–90) is the following theorem. If the boundaries of S and S^* are Jordan curves, and if these curves have tangents at the respective images of $z = e^{i\theta_0}$, then the mappings from $|z| \leq 1$ to S and S^* remain isogonal—in the wide sense—for $z = e^{i\theta_0}$. Applications of this extension are considered. (Received November 24, 1947.)

161. E. B. Shanks: *Homothetic correspondences between Riemannian spaces of two dimensions.*

Necessary and sufficient conditions (in terms of invariants) that there exist a homothetic correspondence between two Riemannian spaces of two dimensions are obtained. The sufficient conditions are deduced from a lemma which states the conditions for a to be a constant when two such spaces have linear elements in the relation $\bar{g}_{ij}(u, v) = e^{2a}g_{ij}(u, v)$. The lemma makes it possible to state the necessary and sufficient conditions without the use of an arbitrary constant. (Received November 20, 1947.)

STATISTICS AND PROBABILITY

162. A. C. Cohen: *Generalized I_n functions.*

The relations $I_0(\xi) = \int_\xi^s \psi(t)dt$ and $I_n(\zeta) = \int_\zeta^s I_{n-1}(\xi)dt$, where $\psi(t)$ is an unspecified frequency function characterized by parameters $\alpha, \beta, \gamma, \cdots$, define a set of functions which are related to the moments of a general class of truncated frequency distributions. $\psi(t)$ is defined over the range $r \leq t \leq s$, and ζ is any value of t within this range. If $\psi(t) = (1/2\pi^{1/2})e^{-t^2/2}$, these functions become those used by R. A. Fisher (*Introduction to mathematical tables*, vol. 1, British Association for the Advancement of Science, 1931, pp. 26–35) in connection with his solution of truncated normal distributions. Properties of the general I_n functions are investigated, and it is shown how they might be employed in solutions of multi-parameter truncated frequency distributions. Differential and difference equations satisfied by these functions are developed for two cases in which $\psi(t)$ assumes the forms $\psi(t) = t^\alpha e^{-t}$ and $\psi(t) = (1-t)^\alpha(1+t)^\beta$ respectively. R. A. Fisher previously stated similar results when $\psi(t)$ is the normal frequency function. (Received November 14, 1947.)

163t. Aryeh Dvoretzky: *On smoothing through repeated averaging.* Preliminary report.

Let $\{A_n\}$ ($n = 0, \pm 1, \pm 2, \cdots$) be a statistical distribution sequence, that is, $A_n \geq 0$ for all n and $\sum A_n = 1$. Let k be a positive integer and put $A_n^{(0)} = A_n$, $A_n^{(m+1)} = (\sum_{\nu=-k}^{k} A_{n+\nu}^{(m)})/(2k+1)$. It was proved by E. Slutsky (Econometrica vol. 5 (1937)) and by M. Faesi (Mitteilungen der Vereinigung Schweitzerischer Versicherungs Mathematiker vol. 40 (1940)) that under certain restrictions the points

$(n/m^{1/2}, m^{1/2}A_n^{(m)})$ approach, as $m\to\infty$, the curve $(*) y=(k/\pi^{1/2})e^{-k^2x^2}$ where $k=3/(2h^2+2h)$. Using simple results of the theory of probability it is easily established that the above result subsists without any restrictions (other than $A_n\ge 0$, $\sum A_n=1$). Indeed, the same result holds even when the averaging process is replaced by $A_n^{(m+1)}=\sum_{r=-h}^{h} g_r A_{n+r}^{(m)}$, with $g_r\ge 0$, $\sum_{-h}^{h} g_r=1$, $\sum_{-h}^{h} r g_r=0$, provided we replace k in $(*)$ by $1/(2^{1/2}\sigma)$ where $\sigma^2=\sum_{-h}^{h} r^2 g_r$. Moreover, $\{g_r\}$ may even be an infinite sequence provided it satisfies some weak conditions. The sequence $\{g_r\}$ may be replaced by a sequence of sequences $\{g_r^{(m)}\}$. Analogous results can be obtained for the averaging of continuous functions. (Received November 24, 1947.)

164*t*. Kenneth May: *Variation of the probability of unfair election results*. Preliminary report.

The paper continues a previous study (Bull. Amer. Math. Soc. Abstract 53-9-346) by dropping the assumption of uniform distribution of outcomes in the districts. Let x_i be the vote of one party in the ith district and let x_i for all i follow the law given by p_x $(x=0, 1, \cdots, m$ where $m=2\mu-1)$. Let $y_i=0$ when $x_i<m/2$ and $y_i=1$ when $x_i>m/2$. Assume $e(x_i)=m/2$ and $e(y_i)=1/2$. Then as the number of districts becomes large, the probability of an unfair result approaches $P(m,\infty)=1/2-(1/\pi)\arctan\{R/(1-R^2)^{1/2}\}$, where R is the correlation coefficient of x_i and y_i. Consider the variation of R and hence of $P(m,\infty)$ with variations in the p_x. R has no relative extrema in the region of interest. It attains an absolute maximum of 1 when for some k, $p_k=p_{m-k}=1/2$. Its absolute minimum, $2m^{1/2}/(m+1)$, is attained at the single point given by $p_0=p_m=1/2(m+1)$, $p_{\mu-1}=p_\mu=m/2(m+1)$, and all other p_x equal to zero. With varying frequency distributions the probability of an unfair result ranges from zero to a maximum which approaches $1/2$ as $m\to\infty$. (Received November 17, 1947.)

165*t*. G. R. Seth: *The variance of sequential estimates.*

Let $X_1, X_2, \cdots$ be a sequence of chance variables whose distribution depends upon an unknown parameter θ and possibly also on a finite number of other parameters. Assume that the X's are either absolutely continuous or take only discrete values and let $p_n(x_1, x_2, \cdots, x_n, \theta)$ denote either the probability density or the probability. Let a sequential process be given, and the chance variable n be the number of observations. Let $\theta^*(x_1, x_2, \cdots, x_n)$ be an unbiased estimate of θ and write $\phi_{in}=(1/p_n)\partial^i p_n/\partial\theta^i$. Finally let λ_{ij} be the covariance between ϕ_{in} and ϕ_{jn} and $[\lambda^{ij}]=[\lambda_{ij}]^{-1}$ $(i, j=1, 2, 3, \cdots, m)$. Under certain regularity conditions and the linear independence of ϕ_{in} the author proves $\sigma^2(\theta^*(x_1, x_2, \cdots, x_n))\ge\lambda^{11}$. This lower bound is not less than the lower bound K obtained by Wolfowitz (Ann. Math. Statist. (1947)). If and only if for some integer i $(i\ge 2)$ $\lambda_{1i}\ne 0$, then $\lambda^{11}>K$. The latter is true, for example, in some very important Wald sequential tests. When $\theta^*(x_1, x_2, \cdots, x_n)$ is a biased estimate, that is, $E[\theta^*(x_1, x_2, \cdots, x_n)]=\theta+b(\theta)$, then $\sigma^2(\theta^*)\ge\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda^{ij}(d^i(b(\theta)+\theta)/d\theta^i)(d^j(b(\theta)+\theta)/d\theta^j)$. (Received October 23, 1947.)

166*t*. W. R. Thompson: *Direct probability sequential analysis*. Preliminary report.

The general subject of sequential analysis was broached elsewhere with illustrations of systematically associated apportionment (Biometrika vol. 25 (1933) p. 285; Amer. J. Math. vol. 57 (1935) p. 450; Annual APHA Meeting, October 8, 1947). Also, it was shown that if ϕ is the unknown probability that x, a random element of

U, be less than a constant c, and if $\{x_i\}$ is a random sample of n observed values, only one of which *may* equal c, and a of them are above and b below; then, for $0<p<1$, $I_p(b+1,\ n-b) \leq \overline{P}(\phi<p) \leq I_p(n-a,\ a+1)$. Thus if $0<p' \leq p''<1$ and α and β are fixed tolerable risks, then U may be classified as in A, B, or C according as data shows: $\overline{P}(\phi<p') \leq \alpha$; $\overline{P}(\phi>p'') \leq \beta$; or otherwise, whereupon another observation is taken. This process *will* end with U in either A or B for some n not exceeding a predeterminable n_0 provided $p' \neq p''$. Risks approximate α and β, respectively, for decisions made at any stage, the acceptance numbers depending respectively on $(p',\ \alpha)$ or $(p'',\ \beta)$ only. These properties contrast favorably with those of corresponding probability-ratio sequential tests. (Received November 17, 1947).

TOPOLOGY

167*t*. Mariano García and G. A. Hedlund: *The structure of minimal sets.*

A minimal set M is a topological space X acted on by a transformation group T such that the orbit closure of every point of X coincides with X. Let M be a minimal set for which X is a compact metric space and T is an abelian topological group. The subgroup G of T is relatively dense in T if there exists a compact subset C of T such that $T=G+C$. The orbit closure under G of a point of X may or may not coincide with X. It is shown that the orbit closures under G constitute a continuous decomposition of X. Several of the possibilities with regard to the totality of decompositions obtained by consideration of all relatively dense subgroups of T are analyzed. There is displayed an example of a regularly almost periodic point such that not all points in the orbit closure are regularly almost periodic. (Received October 23, 1947.)

168*t*. W. H. Gottschalk. *Transitivity and equicontinuity.*

A group of homeomorphisms on a topological space is called *algebraically* (*topologically*) *transitive* in case the orbit (orbit-closure) of some point of the space is the space. The following theorem is proved: If X is a compact metric space, if G is a topologically transitive abelian group of homeomorphisms on X and if H is the group of all homeomorphisms on X which commute with every element of G, then the following statements are pairwise equivalent: (1) H is algebraically transitive; (2) H is equicontinuous; (3) G is equicontinuous. (Received November 24, 1947.)

169. W. H. Gottschalk and G. A. Hedlund: *Dynamics of transformation groups.*

Let X be a topological space, T an abelian topological group, and f a continuous transformation of the Cartesian product $X \times T$ onto X. If $x \in S$ and $t \in T$, denote the image under f of $x \times t$ by xt. Assume that T is a transformation group acting on X in the sense that (a) $xe=x$ if $x \in X$ and e is the identity in T, and (b) $(xs)t=x(st)$ if $x \in X$ and $s,\ t \in T$. The semigroup S in T is *replete* if, C being any compact set in T, an element $t \in T$ exists such that $Ct \subset S$. The subset B of T is *extensive* if B intersects every replete semigroup in T. The group T is *recurrent* at x if, corresponding to any neighborhood U of x, there exists an extensive set B in T such that $xB \subset U$. The group T is *regionally recurrent* (nonwandering) *at* x if, corresponding to any neighborhood U of x, an extensive set B in T exists such that $b \in B$ implies $U \cap Ub \neq \emptyset$. The group T is *permissible* if it is locally compact and contains a compact generating system. In this paper the properties of recurrence are developed in detail and in particular it is shown that if X is metrizable and T is separable, permissible and pointwise regionally recurrent, then the set of recurrent points is a G_δ set residual in X. This is the topological analogue of the Poincaré recurrence theorem. (Received November 24, 1947.)

170*t*. S. T. Hu: *Mappings of a normal space into an absolute neighborhood retract.*

The author generalizes the theorems of Eilenberg (Ann. of Math. vol. 41, p. 231, vol. 42, p. 459) to the mappings of a normal space into an ANR and to the homotopy type of compact ANR. The chief methods which he used are his fundamental bridge theorems. A bridge of a mapping $f:X\rightarrow Y$ is a finite open covering α such that there exists a mapping ψ of the nerve A into Y and a canonical-mapping $\phi:X\rightarrow A$ such that $\psi\phi$ is homotopic with f. Every mapping of a normal space into an ANR has a bridge, and every refinement of a bridge is also a bridge. Further, for any two bridges of the mapping f, the mappings in the definition of bridge satisfy some homotopy relations. By means of the bridge operation and Čech cohomology theory, he carries out the said generalizations. (Received November 24, 1947.)

171. F. B. Jones: *Concerning non-aposyndetic continua.*

Let M be a compact, metric continuum. A point p of M is said to be a weak cut point of M if $M-p$ is not strongly (continuum-wise) connected. It is shown that if M is not semi-locally-connected at any point of an open subset U of M, then U contains a weak cut point of M. Now suppose that M is cyclic (that is, M contains no separating point). G. T. Whyburn has shown that if M is semi-locally-connected, then M contains no weak cut point. The converse is not true. It is shown, however, that if M contains no weak cut point, then the set of points at which M is both aposyndetic and semi-locally-connected is a dense inner-limiting subset (G_δ-set) of M. (Received November 24, 1947.)

172*t*. F. I. Mautner: *Decomposition of unitary representations of groups.* Preliminary report.

Using von Neumann's theory of generalized direct sums (=direct integrals) of Hilbert spaces and rings of operators it is proved that any self-adjoint family of bounded operators in a separable Hilbert space is generalized direct sum of irreducible families, that is, families which do not leave any proper linear closed subspaces invariant. It follows that any unitary representation of a group is generalized direct sum of irreducible unitary representations. Corollary: If U is any faithful unitary representation of a locally compact group G in a separable Hilbert space then the irreducible components of U form a separating system of irreducible unitary representations of G. (Cf. Gelfand and Raikov, C. R. (Doklady) Acad. Sci. URSS. vol. 42 (1944) pp. 199–201.) Also I. E. Segal's existence of irreducible representations of operator algebras follows (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 73–88). The above results are applied to a generalization of the Peter-Weyl theory to arbitrary locally compact separable groups. (Received November 25, 1947.)

173. A. N. Milgram: *Isomorphisms of the semigroup of continuous functions on a bicompact Hausdorff space.*

Let S be a bicompact Hausdorff space and C the set of continuous functions defined on S. C is regarded as a semigroup under multiplication (that is, $f\cdot g(x)=f(x)\cdot g(x)$ for each $x\in S$). It is shown that two spaces with isomorphic semigroups are homeomorphic. This is used to prove that if S is connected then the automorphisms σ of C (that is, 1-1 mappings σ of C on C such that $\sigma(f\cdot g)=\sigma f\cdot\sigma g$) must have a particularly simple form. For each such automorphism σ there exists a homeomorphism H of S on S and a continuous nowhere zero function $p(x)$ defined for $x\in S$ as $\sigma f(x)=\pm\left|f(H(x))\right|^{p(x)}$ where the sign $\pm$ is the same as that of $f(H(x))$. (Received December 30, 1947.)

174t. G. D. Mostow: *On the extensibility of local Lie groups of transformations and groups of the plane.* Preliminary report.

Let $G:R$ denote a local transitive Lie group of transformations (cf. Pontrjagin, *Topological groups*, p. 286) defined in an open set R of an n-dimensional manifold. If G is a global group and the transformations of $G:R$ are defined and transitive on all of R, then $G:R$ is called a "global transitive group of transformations." Theorem. Let G be a local Lie group and U_n a Euclidean n-dimensional neighborhood. If $n<5$, $G:U_n$ can be extended to a global transitive group of transformations $\overline{G}:M_n$ over a manifold M_n, that is, there is a local isomorphism between $G:U_n$ and $\overline{G}:M_n$. If $n\geq 5$, the theorem is not true. Equivalent to this theorem is the following: Any $(r-k)$-dimensional Lie subgroup of a simply connected r-dimensional Lie group is closed, if $k<5$. All global Lie transitive groups of transformations on (2-dimensional) surfaces are determined and also all surfaces on which global Lie transitive groups of transformations act. The surfaces are the plane, cylinder, torus, sphere, projective plane, möbius strip, and Klein bottle. (Received November 15, 1947.)

175. J. H. Roberts: *Open transformations of Peano continua.*

Suppose that $f(A)=B$, where f is an open transformation (continuity is not assumed), and f is at most k-to-1, for some integer k. Then if A is a Peano continuum which contains no continuum of condensation, and B is any compact continuum, it follows that B is actually a Peano continuum. This permits a strengthening of a theorem by Rhoda Manning (Duke Math. J. vol. 13 (1946) pp. 179–184). Other theorems are developed, with the aim of characterizing all possible compact connected images of a linear graph under at most k-to-1 open transformations. (Received November 24, 1947.)

176. H. D. Ursell and L. C. Young: *Remarks on the theory of prime ends.* I.

The beautiful theory of prime ends of a simply-connected plane domain D, conceived by Carathéodory some thirty-five years ago, has hardly been touched by subsequent advances in plane topology. For instance, their relation to the other complementary domains of the boundary of D has not previously been studied. The results established by the present writers include: "I. Let X be a prime end of a domain complementary to the boundary B of D. Then there exist two prime ends Y, Z of D which together include each point of X." "II. Let $\{D_n\}$ be the domains complementary to a bounded continuum W and suppose given in W a system of subcontinua called 'links,' each of which meets at least 3 prime ends of the same or different domains D_n. Then there exists an enumerable selection $\{X_k\}$ of prime ends of the various D_n and an enumerable selection $\{\Lambda_k\}$ of the given links, such that every link of the given system meets at least one X_k or Λ_k." As a corollary, when links are single points: "The boundary points of multiplicity not less than 3 of D lie on a certain enumerable selection of the prime ends of D." (Received November 21, 1947.)

177*t*. H. D. Ursell and L. C. Young: *Remarks on the theory of prime ends*. II.

In a prime end, Carathéodory distinguished between principal points and subsidiary points. The authors distinguish further between two *halves* of a prime end (each half containing the principal points but not necessarily all subsidiary points) and obtain: "III. Let x, y be two points of a same half of a prime end of D. Then the boundary B of D contains a sequence $\{K_n\}$ of disjoint continua such that (a) the distances $\rho(x, K_n)$, $\rho(y, K_n)\rightarrow 0$ as $n\rightarrow\infty$; (b) given $\delta>0$, there exist n_0 so that each subcontinuum C of B with both distances $\rho(x, C)$, $\rho(y, C)\geqq\delta$ meets at most one K_n where $n>n_0$." As an application of III it follows, for instance, that: "IV. Let $\phi(w)$ be a continuous vector function for w in B whose values are points of Euclidean space, and suppose that $\phi(w)$ is not constant on a certain prime end of D. Then B contains a sequence W_n of disjoint continua, no two of which meet a same subcontinuum of constancy of $\phi(w)$ in B, such that $\sum_n \text{osc}\,(\phi, W_n)=+\infty$." This requires that whenever B is transformed continuously into a system whose length (suitably defined) is finite, the prime ends are transformed into single points. A number of further results are also obtained by defining a certain "order of priority" of the points on each half of a prime end. (Received November 21, 1947.)

178*t*. A. D. Wallace: *Continua invariant under solvable groups*.

Let X be a Hausdorff continuum, Z a topological group and f a map of $Z\times X$ into X such that $f(x, e)=x$ (e the neutral element of Z) and $f(s, f(s', x))=f(ss', x)$ ($s, s'\in Z$). Define $s(x)$ as $f(s, x)$ so that s is a homeomorphism of X onto X. It is shown that, if Z is solvable, there exists a prime chain ($=$cyclic element when X is metric and locally connected) which is invariant under each element of Z. This had previously been shown when Z is Abelian or finite. (Received November 3, 1947.)

179. A. D. Wallace: *The fundamental group as a Galois group*.

Let X be a Hausdorff space satisfying certain local regularity conditions and which is compact. Let Y be the universal covering space for X of f the natural map of X onto Y so that f is a uniform local homeomorphism. Denote by $C(X)$, $C(Y)$ the ring of continuous real functions on X, Y. Then f determines an isomorphism of $C(Y)$ into $C(X)$. Let C_0 be the image of $C(Y)$. Finally denote by $G(X)$ the Galois group of $C(X)$ over C_0. Under rather stringent conditions it is shown that $G(X)$ is an isomorph of the fundamental group of X. This result points the way toward possibly interesting generalizations of the fundamental group. There does not seem to be an obvious and simple extension of this to the Hurewicz groups. (Received November 3, 1947.)

J. R. KLINE,
Secretary

APPENDIX

EXCERPTS FROM REPORT OF TREASURER

December 15, 1947

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1947, with such comments as have become customary.

Investment Portfolio

On November 30, 1947, the market value of securities held for Invested Funds exceeded book value by $6,508, and the market value of securities held for Current Funds exceeded book value by $94. In view of these figures and of the large amount of U. S. Government bonds in the portfolio, the reserves held in accounts "Reserve for Investment Losses" and "Profit on Sales of Securities" may still be considered adequate protection against contingent depreciation in market value.

The following is a summary of the changes in security holdings made during the year.

Acquired by Purchase:

40 shares	American Can Co. com.
100 shares	Crane Co. com.
35 shares	Radio Corp. of America cum. pfd.
50 shares	Union Oil of California $3.75 cum. pfd.
25 shares	Northern States Power Co. of Minnesota $3.60 cum. pfd.
10 shares	Bethlehem Steel 7% pfd.
30 shares	Aetna Insurance Co. cap.
20 shares	Texas Co. cap.

Sold or Recalled:

100 shares	Sears Roebuck and Co. com.
30 shares	Firestone Tire and Rubber Co. 4½% cum. pfd.
50 shares	Tidewater Associated Oil Co. $3.75 cum. pfd.

The investment portfolio, valued at market November 30, 1947, now includes Government bonds 36.7 per cent, other bonds 7.7 per cent, preferred stock 12.9 per cent, common stock 37.4 per cent, cash in savings bank 5.3 per cent.

Income from Investments

Income received during the year from investment of Current Funds amounted to $2,231, exclusive of $95 earmarked for International Congress. This represents a return of 2.8 per cent computed on average book value of investments. Income on Invested Funds amounted to $7,837, representing a return of 4.0 per cent. Total investment income from all sources was $10,163, representing a return over 3.6 per cent. These rates of return are higher than in 1946.

Income from the Henderson Estate was $4,850; in 1946 it was $4,800.

Changes in Net Assets

Net assets increased by $2,655 during the year. In interpreting this figure, it should be noted that among the receipts is a net amount of $3,225 for account of the Birkhoff Memorial Project. Except for this item net assets would have shown a decrease even though a bequest of $1,000 from the Estate of Mrs. J. I. Hutchinson is included among the receipts, as well as $1,903 net profit received from sales of securities during the year. Furthermore over $23,000 of assets represent reserves designated toward the publication of six volumes (other than the Birkhoff Papers) which are in process, and whose cost is expected to be not less than $30,000. Difficulties in the printing industry are the sole reason that substantial payments have not been made from these reserves.

Decrease in Surplus

Surplus account for the first time in many years shows a decrease of $6,878, and it has been necessary to draw upon savings accounts for funds to meet expenses. For this, heavily increased printing costs for Bulletin and Transactions were largely responsible, though additional salary expense to meet increasing demands for clerical assistance played a part. Also, it became necessary to rent additional office space in New York, and finally, during the year the Rockefeller Foundation felt obliged to discontinue their generous grants toward the expenses of the Policy Committee. On the income side, receipts from dues were somewhat more than in the previous year, as were initiation fees. A high volume of publication sales was maintained, even though stocks of books have been undesirably short. But even with increased investment income the gains were not sufficient to balance increased general disbursements.

Mathematical Reviews

For the second consecutive year, operation of Mathematical Reviews resulted in a deficit, this time of $5,782, although receipts for this account exceed expectations by some $2,700. Again, the heavily increased printing costs were a principal factor. A contract with the Office of Naval Research, Department of the Navy, negotiations for which are just being completed, is expected to provide funds accrued in the fiscal year 1947 which will wipe out this deficit. This contract is to cover the editorial expenses for Mathematical Reviews until March, 1949.

I trust that these remarks may prove helpful to the Board in planning financial policies for the next fiscal year.

Respectfully submitted,
BENNINGTON P. GILL,
Treasurer

BALANCE SHEET

Assets

	November 30, 1947	November 30, 1946
CURRENT FUNDS:		
Cash	$ 16,315.50	$ 26,130.60
Investments	70,559.50	60,837.08
	$ 86,875.00	$ 86,967.68
INVESTED FUNDS:		
Cash	$ 1,287.90	$ 652.40
Investments	194,987.15	192,874.73
	$196,275.05	$193,527.13
TOTAL ASSETS	$283,150.05	$280,494.81

Liabilities

	November 30, 1947	November 30, 1946
CURRENT FUNDS:		
Mathematical Reviews	$ 1,897.50	$ 7,389.34
Colloquium	11,708.12	9,495.90
Mathematical Surveys	3,734.70	1,796.87
Symposia on Applied Mathematics	2,700.00	
Birkhoff Memorial Project	3,061.68	
International Congress	6,413.49	6,321.92
Committee on Aid to Devastated Libraries	1,320.50	
Policy Committee		913.34
Prize Funds and Other Special Funds Accumulated Income	6,848.12	5,591.42
Reprinting Funds	6,165.74	5,780.67
Sinking Fund	1,044.54	960.65
Profit on Sales of Securities	982.60	982.60
Miscellaneous	300.45	159.24
Surplus	40,697.56	47,474.73
	$ 86,875.00	$ 86,967.68
INVESTED FUNDS:		
Endowment Fund Principal	$ 71,000.00	$ 71,000.00
Prize Funds and Other Special Funds	32,033.22	31,033.22
Life Membership and Subscription Reserve	3,407.61	3,563.40
Colloquium	5,000.00	5,000.00
Mathematical Reviews	65,000.00	65,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	15,448.33	13,544.62
	$196,275.05	$193,527.13
TOTAL LIABILITIES	$283,150.05	$280,494.81

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1946–1947

	1947		1946	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS:				
Dues—Ordinary Memberships	$21,628.19		$20,761.05	
Dues—Contributing Memberships	931.86		951.54	
Dues—Institutional Memberships	8,011.63		8,362.60	
Initition Fees	1,195.10		1,139.22	
Investment Income	11,055.52		10,130.97	
Miscellaneous	288.19		290.79	
Grant from Rockefeller Foundation for Policy Committee	1,500.00		3,000.00	
GENERAL DISBURSEMENTS:				
Secretaries		$10,596.01		$ 7,372.96
Treasurer		1,469.54		1,200.37
Officers' Traveling		705.05		629.56
Furniture and Fixtures		608.53		601.14
Library		1,477.86		1,473.46
Policy Committee		2,452.14		2,229.91
Printing Contracts Committee				.59
Committee on Aid to Devastated Libraries	2,068.22	747.72		41.20
Rent—Editorial Office		305.32		
Emergency		65.33		120.65
Total	$46,678.71	$18,427.50	$44,636.17	$13,669.84
Excess of General Receipts	$28,251.21		$30,966.33	
PUBLICATION:				
Bulletin	$ 3,179.61	$21,940.26	$ 2,498.87	$16,425.94
Bulletin Reprinting	846.51	486.11	928.75	
Transactions	7,397.86	13,065.22	7,114.07	10,215.58
Transactions Reprinting	1,517.05	1,492.38	1,541.68	
Colloquium	8,135.42	5,923.20	5,829.18	5,644.08
Mathematical Reviews	19,817.73	25,309.57	17,365.28	23,613.28
Mathematical Surveys	2,489.12	551.29	621.51	75.59
Birkhoff Memorial Project	3,225.32	163.64		
Semicentennial Publications	93.15	9.26	147.71	7.97
Symposium on Applied Mathematics	2,700.00			
American Journal		2,250.00		2,500.00
Library Catalogue				1,868.68
Total	$49,401.77	$71,190.93	$36,047.05	$60,351.12
Excess Cost of Publication		$21,789.16		$24,304.07
OTHER:				
Special Funds (including Congress)	$ 2,351.87	$ 3.60	$ 1,267.91	$ 204.39
Profit/Loss on Sales of Securities	1,903.71		1,023.22	
Miscellaneous	141.21		67.64	
Surplus:				
Mathematical Reviews		1,000.00		1,000.00
Mathematical Surveys		2,000.00		
Symposium on Applied Mathematics		2,700.00		
Committee on Aid to Devastated Libraries		1,500.00		
Annals of Mathematics		1,000.00		
Total	$ 4,396.79	$ 8,203.60	$ 2,358.77	$ 1,204.39
Difference		$ 3,806.81	$ 1,154.38	
Net Change in Assets	$ 2,655.24		$ 7,816.64	
ASSETS BEGINNING OF YEAR		$280,494.81		$272,678.17
ASSETS END OF YEAR		283,150.05		$280,494.81

BOOK REVIEWS

Die Bewegungsgruppen der Kristallographie. By J. J. Burckhardt. (Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften.) Basel, Birkhauser, 1947. 186 pp. 24.50 S. fr.; bound 29 S. fr.

The systematic study of the discrete space groups and their application to crystallography dates back to Schoenflies (1891) and Federow (1892). Bieberbach in 1910 and Frobenius, with a simpler proof, in 1911 showed that there exist only a finite number of these groups having a finite fundamental region. Treatises listing the 230 space groups and the corresponding point lattices have been written by P. Niggli, *Geometrische Kristallographie des Diskontinuums* (Leipzig, 1919) and R. W. G. Wyckoff, *The analytical expression of the results of the theory of space groups* (Carnegie Institution of Washington Publication 318, 2d ed., 1930). Other contributions to the theory of space groups have been made by D. Hilbert, C. Jordan, L. Schläfli, H. S. M. Coxeter, W. Nowacki, A. Speiser, G. Wintgen, and others to whom references are given. The purpose of the author is to develop the theory of space groups systematically with primary emphasis not on the geometric crystal classes but on the arithmetic theory of space lattices.

Chapter I opens with a careful presentation of the fundamental notions of vectors and matrices, and the particular properties of certain orthogonal and integral unimodular matrices which represent the symmetry operations of a point lattice. The point groups are orthogonal in rectangular coordinates, but integral unimodular when referred to lattice coordinates. If y, x, a are ν-dimensional column vectors, A is a ν by ν matrix and E the unit matrix, then the general rigid motion $y=Ax+a$ is denoted by (A, a). In any of the 230 space groups G, the transformations (E, a) form the invariant abelian subgroup T of translations, and the quotient group G/T is isomorphic to a finite group G_0 of integral unimodular matrices, called a crystal class.

Chapter II contains a description and classification of the crystal classes—the symmetry groups of a given point lattice which leave fixed one lattice point. Two crystal classes are considered *geometrically* equivalent if one can be transformed into the other by a *nonsingular* transformation. They are *arithmetically* equivalent only if a transforming matrix can be chosen which is *integral* and *unimodular*.

Great stress is laid on this distinction. The 14 space lattices give rise to 32 ternary geometric classes, but 73 ternary arithmetic classes. The latter fall into seven systems as follows: 2 triclinic, 6 monoclinic, 13 orthorhombic, 16 tetragonal, 15 cubic, 5 rhombohedral, and 16 hexagonal. The author derives them first from the lattices and their geometric classes in a manner which generalizes to spaces of more than three dimensions, and then gives a simpler proof of the completeness of the three-dimensional classification, based on a study of the symmetry of the fundamental cell. A lucid diagrammatical representation of the crystal classes and their subgroups is shown on p. 72. Typographical errors appear on p. 39, matrix E in (2), and on pp. 98–99 where in three equations (3) the expression $xy/2$ should read $2xy$.

Chapter III is concerned with the derivation and classification of the 230 non-equivalent space groups having finite fundamental regions. Should no distinction be made between right and left-handed screw motions, this number would reduce to 219. Two space groups are to be considered equivalent only if they belong to the same arithmetic class G_0. If, referred to lattice coordinates, the elements of G_0 are the integral unimodular matrices A_i, and if representatives of the n cosets of the space groups G with respect to T (subgroup of translations) are (A_i, a_i), then according to Frobenius a choice of origin for which $\sum a_i \equiv 0 \pmod 1$ will imply that na_i are integral vectors whose components can be chosen to lie between 0 and $n-1$. For each arithmetic class there is always the null solution $a_i = 0$, and there may be at most a finite number of other solutions of the Frobenius congruence $A_i a_k + a_i \equiv a_l$ (where $A_i A_k = A_l$). For a given point group G_0 with matrices A_i the two space groups given by (A_i, a_i) and (A_i, b_i) are called equivalent solutions if and only if the congruences $a_i - b_i \equiv (E - A_i)s$ have a common solution s for each i. It is clear then that if G_0 is reducible and contains the identical representation as a component, and if the corresponding component of $a_i - b_i$ is $\neq 0$, then the two space groups (A_i, a_i) and (A_i, b_i) are not equivalent. For example, the two plane crystal classes C_2 and C_s are generated respectively by diag $(-1, -1)$ and diag $(1, -1)$. The former gives rise to but one group of motions (the null solution) but the latter admits also a second group containing a sliding reflection whose translation vector is either $(1/2, 0)$ or $(1/2, 1/2)$. Following the complete derivation and classification of the 17 groups of motions of the plane (with finite fundamental region) arising from 13 arithmetic classes of which all except C_s, C_{2v} and C_{4v} contribute only the null solution, a few simple space groups are derived immediately by adjoining the identity representation to G_0. The remaining space groups are then derived in detail and classified within the rhombohedral, hexagonal, monoclinic, rhombic, tetragonal, and cubic systems.

The two final sections are devoted to the study of special families of space groups in n dimensions, such as those arising from the cyclic, symmetric, and alternating groups on n symbols.

The book is clearly written and self-contained, except in the section beginning on p. 91 where the ternary arithmetic classes are listed. Here the reader without previous knowledge of the notations of crystallography may have some difficulty reading the rather condensed summary of the 73 ternary arithmetic classes. The groups of motions in the plane are illustrated by excellent figures, but no attempt is made to illustrate the 230 space groups by drawings such as are given by Wyckoff. The emphasis in the book is clearly on the mathematical derivation rather than the pictorial representation of the 230 space groups.

J. S. FRAME

Methods of mathematical physics. By Harold Jeffreys and Bertha S. Jeffreys. New York, Macmillan; Cambridge University Press, 1946. 9+679 pp. $15.00.

The book starts with a substantial chapter on real variable—Dedekind sections, sequences, series, continuity, integration, mean value theorems. Chapters 2, 3, and 4 cover vectors, cartesian tensors, and matrices, and these are followed by chapters on multiple integrals and potential theory. Operational methods and their applications occupy two chapters, and a long chapter is devoted to numerical methods. A short chapter on calculus of variations brings us to what may be regarded as the mid-point of the book, attained almost entirely without the use of complex numbers.

The essential elements of the theory of functions of a complex variable are covered in two chapters. This opens up a wide field, and chapters follow on conformal representation, Fourier's theorem, factorial (gamma) functions, linear differential equations of the second order, asymptotic expansions, equations of wave motion and heat conduction (three chapters), Bessel functions and applications, confluent hypergeometric functions, Legendre functions, elliptic functions. The book ends with explanatory notes, an appendix on notation, and an index.

Each chapter has a set of examples, a stimulating collection culled from examinations of the Universities of Cambridge, London, and Manchester.

It is obviously impossible to cover in a single volume, even a large one, all those methods of mathematical physics which a wide variety of readers might wish to see included. It is natural that authors should allow their particular interests to influence their choice of material, and one is not surprised to find in the book much of the contents of two short books by Harold Jeffreys—*Operational methods in mathematical physics* (now out of print) and *Cartesian tensors*. The authors admit that the choice of subject-matter has been difficult. Their stated intention is to "provide an account of those parts of pure mathematics that are most frequently needed in physics," and they set themselves the general rule to include a method if it has applications in at least two branches of physics. They have not tried to give a detailed account of any branch of physics, regarding that as a matter for special text books.

The authors' style is crisp and at times rather condensed, but it expands as the occasion requires in a humorous, historical, or caustic vein. This gives the book a human atmosphere rare in mathematical publications, but naturally enough leads at times to statements which may arouse the antagonism of the reader; for example, the statement on p. 76 (footnote) that the reduction of a general force system to a force and a couple is the sort of thing that occurs only in examination questions! In most books with a scope comparable to that of the book under review, intrinsic evidence is present to show that the authors are either mathematicians or physicists; there is a rawness or incompleteness in the domain less familiar to the authors. In the present instance the authors have preserved a good balance. Physics is the goal, but the mathematics is handled with feeling and care. To quote from the Preface: "We maintain therefore that careful analysis is more important in science than in pure mathematics, not less. We have also found repeatedly that the easiest way to make a statement reasonably plausible is to give a rigorous proof."

Nevertheless the cloven hoof of the physicist peeps out from time to time, and is very much in evidence in the first few pages. Once the mathematical reader has found nests of intervals on p. 6 and a reassuring epsilon on p. 7, he knows that all is well. But to reach this haven of rest he has first to weather a stormy sea. On p. 1 it is written that the formulae of algebra "may still be correct when we replace the letters in them by something other than numbers, and it is to this fact that the possibility of mathematical physics is due." The authors do not have in mind here replacement by matrices or operators, but rather replacement by the quantities of physics in which a unit of measurement is involved, for example, replacement of the number 3 by 3 cm. "When a particular application to a measured system is made we naturally give the symbols their actual values in terms of the measures, which will include a statement of the units; but in the general theory the unit is irrelevant. The symbols will then be said to stand, not for numbers, but for *physical magnitudes.*" Does this mean that when we write $e^x = 1+x+x^2/2+\cdots$, we are to have in the backs of our minds the idea that x stands for 3 cm. or 5 gm. rather than a pure number? Surely this is not an honest picture of the psychology of the modern mathematical physicist. If it is, we have traced to its source a subtle divergence of points of view which keeps mathematicians and physicists from enjoying a full measure of mutual esteem. It is a divergence which takes place at the level of high school algebra, and must be felt as soon as a student wonders whether he should write for his answer "$x=3$" or "$x=3$ miles." Venturing an opinion apparently in direct contradiction to that of the authors, it seems to me that the power of modern mathematical physics lies in the fact that it can and does formulate its problems in terms of pure numbers. Nevertheless, mathematicians should not dismiss the question of units as a storm in a teacup; a little experience with electromagnetic units will reduce the arrogant to a spirit of respectful humility. The best bridge between measured quantities and pure mathematics seems to lie in the formulation of a vector algebra of measured quantities, as was once pointed out to me by Professor L. Infeld. On the whole, the authors should not be criticized for introducing the matter, which is obviously of importance, but it is to be regretted that they did not deal with it more completely and satisfactorily. Confusion on an elementary level is the worst confusion of all.

A few small points may be noted. The juxtaposition of vectors, tensors, and matrices brings together a happy family group. As refresher material, it is excellent. But will the novice understand what a scalar is from the following: "Any physical measurement is the assignment of a single magnitude. Such magnitudes are called *scalars*" (p. 49), and "A scalar is a single quantity, the same for all axes" (p. 53)? What about the x-coordinate of a point? It is a scalar by first definition, but not by the second. The definition of a scalar function of position on p. 82 is not as clear as it might be; we have merely to say that the function has at each point a value independent of the coordinate system, but the authors somewhat obscure this simple idea

The statement following equation (21) on p. 109 is not correct; the adjugate of an antisymmetric matrix a is antisymmetric if the order of a is even, symmetric if the order of a is odd.

The index appears to be adequate; one misprint in it was accidentally noticed: Group velocity, for 477 read 479.

The production conforms to the excellent standards of the Cambridge University Press. Those in whom the conservatism of publishers engenders a deep rage will note with satisfaction that the historic Cambridge Press starts sentences with mathematical symbols (lower case!) and adheres to the practice (anathema to some publishers) of cutting in figures on the right hand sides of both odd and even pages, so as to leave an unbroken margin. Very few misprints have been noticed, and they are not likely to confuse the reader. In places a dearth of commas forces a rereading, but this is no harm.

The authors show good sense with regard to terminology and notation, and are not afraid to make suggestions. In the case of a theorem bearing a hyphenated name (for example, Newton-Bessel) they make it clear that the second name is merely a label, and does not imply a division of priority. It is to be hoped that this practice may prevail; the creators of new theorems should welcome it: reference to the Smith-Robinson theorem would imply (a) that Smith has created at least two theorems, and (b) that Robinson vouches for the correctness of one of them. The word "limit" is overworked, and the authors suggest "termini" for "limits of integration." For Hankel functions they use $\mathrm{Hs}_n(x)$ and $\mathrm{Hi}_n(x)$, the "s" and the "i" meaning respectively "superior" and "inferior" in reference to paths of integration in the Schläfli integrals; they use also $\mathrm{Kh}_n(x)$ for Heaviside's function, equal to $2/\pi$ times the usual $K_n(x)$.

The book assumes a knowledge of calculus, and might serve as a text book for students who proceed from that basis in the direction of applied mathematics or mathematical physics. One hesitates to affirm or deny its merits as a text; so much depends in such cases on the establishment of a triple bond of sympathy between student, teacher, and author. Under favorable conditions it should work well; as for unfavorable conditions, let me requote the quotation from Oliver Heaviside which heads the chapter on Operational methods: "Even Cambridge mathematicians deserve justice."

J. L. SYNGE

Applied Bessel functions. By F. E. Relton. London and Glasgow, Blackie, 1946. 7+191 pp. 17s. 6d.

This text furnishes an introduction to those properties of the Bessel and allied functions which are of use in solving boundary value problems. The role of these properties in applied mathematics and engineering is fully illustrated by numerous examples both in the text and in the exercises. Prerequisites for the reading of the text are some elementary calculus and a course in elementary (or possibly intermediate) differential equations. However, the author states (and frequently proves) those properties of differential equations which are used in the text.

Essentially, the text treats four topics: (1) zeros and growth properties of Bessel functions of the first and second kinds; (2) growth properties of related functions; (3) integral expansions of Bessel functions; (4) applications to ordinary and partial differential equations.

After defining the error, gamma, and beta functions, the author discusses those properties of linear differential equations which are to be used in the treatment of the Bessel functions. Considerable emphasis is placed on normal forms and zeros of second order equations. The cylinder functions C_n (functions which satisfy a particular form of the Bessel differential equation) are defined by means of two recurrence relations. By means of these relations, the author obtains the structure of the zeros of the cylinder functions and then the nature of the graph of these functions. Further, various transformations of the dependent and independent variables which lead to cylinder functions are discussed. In Chapter IV, the power series expansion for the Bessel function of the first kind J_n ($J_n = x^{-n}C_n$) is obtained. Some numerical values for the first zero of different integral J_n are determined and Bessel identities involving Lommel integrals are discussed. Before attempting to obtain the second integral Y_n of the Bessel equation for integral n, the author merely states various properties of the indicial equation. Proofs of these statements will be found in texts on intermediate or advanced differential equations (see *Intermediate differential equations* by E. D. Rainville, Wiley, 1943). Actually, the second integral of the zero order is obtained from the fact that $(dC_n/dn)_{n=0}$ is a solution of the zero order Bessel equation. The function Y_n for integral n is obtained by a similar device.

The properties of modified Bessel functions (imaginary argument) and the definitions of the "ber" and "bei" functions are studied in the latter part of the text. The first class of functions are related to the Bessel functions in the same manner that the hyperbolic functions are related to the circular functions. Hence, they can possess zeros only at the origin and only the growth of these functions is discussed. This is done by use of the recurrence formulas.

Various integral expansions for Bessel functions are furnished in Chapters IX and X. They include: (1) the well known relations between integrals of the circular functions and the Bessel functions of the first kind; (2) the Lipschitz and Sonine integrals for Bessel functions; (3) Weber's discontinuous integral; (4) some asymptotic formulas for the various Bessel functions.

Following each discussion of a type of Bessel function, the author furnishes an application to problems in applied mathematics. In particular, Chapter V is devoted to five boundary value problems whose solutions involve Bessel functions of the first kind. Of these, the following should be of interest to the engineer: (1) transverse vibrations of a non-uniform string; (2) stability of a vertical wire and deep cantilever beam. Again, in Chapter VIII, some applications to hydrodynamics (tidal motion), elasticity (buckling of a circular plate, vibrations of a disk), and heat conduction are given. In each case, the differential equation is solved by the separation of variables and then the appropriate Bessel functions are introduced in studying the solutions of the resulting ordinary differential equation.

The reviewer feels that this book should furnish an excellent introduction to Bessel functions and their applications in applied mathematics.

N. COBURN

NOTES

At the present time the American Mathematical Society has reciprocity agreements with the following societies: Deutsche Mathematiker Vereinigung, London Mathematical Society, Matematisk Forening, Société Mathématique de France, Swiss Mathematical Society, and the Unione Matematica Italiana. In accordance with a reciprocity agreement, members of the Society may join any one of the organizations mentioned by the payment of one-half the annual dues of the organization. On the other hand, members of the organizations mentioned above (with certain restrictions as to residence) may become members of the Society and are required to pay only half the regular annual dues, namely five dollars. The Board of Trustees voted recently that, during the present difficult financial situation abroad, the usual initiation fee should be waived in case of persons joining the Society under reciprocity agreements. Further details may be secured by writing the Secretary of the Society.

The Mathematical Association of America has announced the election of Professor Saunders MacLane of the University of Chicago as first vice president, Dean W. L. Ayres of Purdue University and Professor C. R. Adams of Brown University as members of the Board of Governors, and Professor H. M. Gehman of the University of Buffalo as secretary-treasurer.

A conference on *Topological groups and their representations* was held at the University of Chicago on January 22–24, 1948. During the five sessions of the Conference fifteen papers were presented.

The National Council of Teachers of Mathematics will hold its twenty-sixth annual meeting in Indianapolis, Indiana, on April 2–3, 1948.

The French Academy of Sciences has announced the following awards for 1947: Bordin Prize to Professor René Lagrange of the University of Dijon; Carrière Prize to Mr. René Gosse of the University of Grenoble; Dickson Prize to Mr. Charles Pisot of the University of Bordeaux; Montyon Prize to Professor Daniel Dugué of the University of Algiers; Petit D'Ormoy Prize to Professor Henri Cartan of the University of Paris; Saintour Prize to Professor Hubert Delange of the University of Clermont-Ferrand; Thebault Prize to Professor Gaston Benneton of the Carnot Lycée.

The Royal Society of London has awarded the Copley Medal to the late Professor Emeritus G. H. Hardy of the University of Cambridge for his part in the development of mathematical analysis in Britain during the last thirty years.

The Chauvenet Prize has been awarded by the Mathematical Association of America to Dr. P. R. Halmos of the University of Chicago for his paper *The foundations of probability* which was published in the American Mathematical Monthly for November 1944.

The Duodecimal Society of America has conferred the 1948 Annual Award upon Mr. H. C. Robert of Atlanta, Georgia.

Associate Director Joseph Slepian of Westinghouse Research Laboratories has received the Edison medal for 1947 from the American Institute of Electrical Engineers.

Professor William Findlay of McMaster University has retired with the title emeritus.

Professor Hassler Whitney of Harvard University received in 1947 an honorary doctorate of science from Yale University.

Professor R. L. Wilder of the University of Michigan has been elected vice president of Section A of the American Association for the Advancement of Science.

Professor Arne Beurling of the University of Uppsala will be a visiting lecturer at Harvard University for the coming academic year.

Professor Henri Cartan of the University of Paris is a visiting lecturer at Harvard University for this term.

Professor Alessandro Terracini of the University of Tucumán has returned to the University of Turin.

Assistant Professor H. G. Apostle of Amherst College has been appointed to an assistant professorship in the philosophy department at the University of Chicago.

Mr. L. C. Bagby of the Jam Handy Corporation has been appointed to a professorship at the Lawrence Institute of Technology, Detroit, Michigan.

Assistant Professor F. L. Celauro of Lehigh University has been appointed to an assistant professorship at Newark College of Engineering.

Professor W. W. S. Claytor of Hampton Institute has been appointed to an associate professorship at Howard University.

Professor V. G. Grove of Michigan State College is a visiting professor at the University of Puerto Rico during the present term.

Assistant Professor Rufus Isaacs of the University of Notre Dame has accepted a position as research engineer with the North American Aviation Corporation in Los Angeles, California.

Mr. P. B. Johnson of the University of Illinois has been appointed to an assistant professorship at Occidental College.

Professor C. E. Love of the University of Michigan has retired.

Mr. C. R. Morris of Indiana University has accepted a position as mathematician with the National Union Radio Corporation, Orange, New Jersey.

Professor Joseph Seidlin of Alfred University has been appointed Dean of the Graduate School.

Professor Gerhard Tintner of Iowa State College of Agriculture and Mechanic Arts will be on leave of absence during the coming academic year and will attend the Institute of Economics at Cambridge University.

Professor J. L. Walsh of Harvard University is on leave of absence during this term.

Professor D. V. Widder of Harvard University will be on leave of absence during the coming academic year.

The following promotions are announced:

E. W. Anderson, Iowa State College of Agriculture and Mechanic Arts, to a research professorship in mechanical engineering.

L. A. Aroian, Hunter College, to an assistant professorship.

H. G. Ayre, Western Illinois State College, to a professorship.

J. W. Beach, Iowa State College of Agriculture and Mechanic Arts, to an assistant professorship.

F. E. Bortle, Iowa State College of Agriculture and Mechanic Arts, to an assistant professorship.

A. T. Brauer, University of North Carolina, to a professorship.

A. B. Brown, Queens College, to an associate professorship.

C. L. Dolph, University of Michigan, to an assistant professorship.

G. N. Garrison, Lehigh University, to an associate professorship.

C. H. Lindahl, Iowa State College of Agriculture and Mechanic Arts, to an assistant professorship.

Sallie Pence, University of Kentucky, to an associate professorship.

J. Shibli, Pennsylvania State College, to a professorship.

D. E. South, University of Kentucky, to a professorship.

R. R. Stoll, Lehigh University, to an associate professorship.

H. W. Syer, Boston University, to an assistant professorship.

C. A. Truesdell, Research Department, Naval Ordnance Laboratory, White Oak, Maryland, to the position of chief of the theoretical mechanics section.

F. E. Ulrich, Rice Institute, to an associate professorship.

The following appointments to instructorships are announced: University of Kentucky: Miss Elsie T. Church, Mrs. L. C. Cooper, Mr. D. L. Daly, Mr. D. C. Rose, Miss Genevieve Snider; Oregon State College: Miss Ella M. Sowder; University of Pennsylvania: Miss Jean B. Walton; Syracuse University: Mr. E. O. Allen.

Professor Emeritus G. H. Hardy of Trinity College, Cambridge, England died on December 1, 1947. He had been a member of the Society since 1915.

Mr. G. E. Bulloch of the New Jersey Bell Telephone Company died on December 25, 1947.

Professor Emeritus E. E. DeCou of the University of Oregon died on October 15, 1947. He had been a member of the Society since 1903.

Mr. G. W. Evans, Berkeley, California, died on February 27, 1947. He had been a member of the Society since 1910.

Dr. A. M. Harding, formerly president and professor of mathematics and astronomy of the University of Arkansas, died December 24, 1947.

Professor Emeritus Vladimir Karapetoff of Cornell University died January 11, 1948, at the age of seventy-two years. He had been a member of the Society since 1925.

Dr. Jacob Westlund, formerly of Purdue University, died on March 27, 1947, at the age of seventy-nine years. He had been a member of the Society since 1898.

Professor A. N. Whitehead of Harvard University died on December 30, 1947.

NEW PUBLICATIONS

Colombo, S. See Humbert, P.

Douglass, R. D., and Zeldin, S. D. *Calculus and its applications*. New York, Prentice-Hall, 1947. 8+568 pp. $5.15.

Gross, G. L. See Rosser, J. B.

Hodge, W. V. D., and Pedoe, D. *Methods of algebraic geometry*. Vol. I. Cambridge University Press, 1947. 8+440 pp. $6.50.

Humbert, P., and Colombo, S. *Le calcul symbolique et ses applications à la physique mathématique*. (Mémorial des sciences mathématiques, no. 105.) Paris, Gauthier-Villars, 1947. 52 p. 150 fr.

Kelly, T. L. *Fundamentals of statistics*. Cambridge, Harvard University Press, 1947. 16+755 pp. $10.00.

Kiss, S. A. Transformations on lattices and structures of logic. New York, 1947. 10+322 pp. $7.50.

Leja, Franciszek. Rachunek różniczkowy i całkowy ze wstępem do równań różniczkowych. Cracow, 1947. 8+300 pp.

Lichnerowicz, A. *Algèbre et analyse linéaires*. Paris, Masson, 1947. 316 pp. 800 fr.

Newton, R. R. See Rosser, J. B.

Pedoe, D. See Hodge, W. V. D.

Ritt, J. F. Theory of functions. New York, Kings Crown Press, 1947. 10+181 pp. $3.00.

Rosser, J. B., Newton, R. R., and Gross, G. L. Mathematical theory of rocket flight. New York, McGraw-Hill, 1947. 8+276 pp. $4.50.

Schillinger, J. *The mathematical basis of the arts*. New York, Philosophical Library, 1947. 703 pp. $12.00.

Selected techniques of statistical analysis. Statistical research group, Columbia University. New York, McGraw-Hill, 1947. 14+473 pp. $6.00.

Tables of the Bessel functions of the first kind of orders ten, eleven, and twelve. (Annals of the Computation Laboratory of Harvard University, vol. 7.) Cambridge, Harvard University Press, 1947. $10.00.

Tables of the Bessel functions of the first kind of orders thirteen, fourteen, and fifteen. (Annals of the Computation Laboratory of Harvard University, vol. 8.) Cambridge, Harvard University Press, 1947. $10.00.

Vaidyanathaswamy, R. Treatise on set topology. Madras, 1947. 6+306 pp. Rs 16-4-0.

Zeldin, S. D. See Douglass, R. D.

ON RINGS OF ANALYTIC FUNCTIONS

LIPMAN BERS

Let D be a domain in the complex plane (Riemann sphere) and $R(D)$ the totality of one-valued regular analytic functions defined in D. With the usual definitions of addition and multiplication $R(D)$ becomes a commutative ring (in fact, a domain of integrity). A one-to-one conformal transformation $\zeta=\phi(z)$ of D onto a domain Δ induces an isomorphism $f \to f^*$ between $R(D)$ and $R(\Delta)$: $f(z)=f^*[\phi(z)]$. An anti-conformal transformation

$$\zeta = \overline{\phi(z)}$$

also induces an isomorphism:

$$\overline{f(z)} = f^*[\overline{\phi(z)}].$$

The purpose of this note is to prove the converses of these statements.

THEOREM I. *If $R(D)$ is isomorphic to $R(\Delta)$, then there exists either a conformal or an anti-conformal transformation which maps D onto Δ.*[1]

THEOREM II. *If D and Δ possess boundary points, then every isomorphism between $R(D)$ and $R(\Delta)$ is induced by a conformal or an anti-conformal transformation of D onto Δ.*

Theorem I may be regarded as a complex variable analogue of theorems characterizing a topological space in terms of the family of its continuous functions. If $R(D)$ is made into a topological ring by defining $f_n \to f$ to mean that $f_n(z) \to f(z)$ uniformly in every bounded closed subset of D, then Theorem II implies that, except for a trivial special case, every isomorphism between $R(D)$ and $R(\Delta)$ is of necessity a homeomorphism.

To prove the theorems we consider a fixed isomorphism between $R(D)$ and $R(\Delta)$. It takes a function $f(z)$, $z \in D$, into a function $f^*(\zeta)$, $\zeta \in \Delta$, a set $S \subset R(D)$ into a set $S^* \subset R(\Delta)$. Let c be a complex constant.

Presented to the Society, November 2, 1946; received by the editors May 27, 1947.

[1] After this paper was completed the author learned about a closely related unpublished result which was obtained by C. Chevalley and S. Kakutani several years ago. Chevalley and Kakutani proved that if to each boundary point W of B there exists a bounded analytic function defined in B and possessing at W a singularity then B is determined (modulo a conformal transformation) by the ring of all bounded analytic functions. The author is indebted to Professor Chevalley for the opportunity of reading a draft of the paper containing the proof.

For the sake of brevity we denote the element of $R(D)$ corresponding to the functions $f(z) \equiv c$ (or the element of $R(\Delta)$ corresponding to the function $g(\zeta) \equiv c$) by the letter c. We call a complex number rational if its real and imaginary parts are rational.

LEMMA 1. *Either* $i^* = i$ *and for every rational complex constant* $r: r^* = r$, *or* $i^* = -i$ *and* $r^* = \bar{r}$.

The proof is clear.

LEMMA 2. *If* c *is a constant, so is* c^*.

PROOF. If c is rational the assertion is contained in the preceding lemma. Irrational constants c are characterized by the existence of the inverse of the element $c-r$ for every rational constant r.

LEMMA 3. *All elements of* $R(D)$ *are constants if and only if* D *is the whole complex plane including the point at infinity.*

The proof is clear.

Lemmas 2 and 3 contain the proof of Theorem I for the case when D is the domain $0 \leq |z| \leq \infty$. In what follows we consider only domains possessing boundary points. Without loss of generality we assume that neither D nor Δ contains the point at infinity. We also assume that $i^* = i$; the case $i^* = -i$ can be treated in the same way.

We denote the set of all functions belonging to $R(D)$ and vanishing at a point $a \in D$ by I_a. The set $I_\alpha \subset R(\Delta)$, $\alpha \in \Delta$, is defined similarly.

LEMMA 4. *There exists a one-to-one mapping* $z \rightarrow z' = \phi(z)$ *of* D *onto* Δ *such that* $I_a^* = I_{\phi(a)}$.

PROOF. Every element of $R(D)$ generates a principal ideal (f), that is, the set of all elements of the form fh, $h \in R(D)$. (f) is said to be a maximal principal ideal if $(f) \neq R(D)$ and if $(f) \subset (g) \neq R(D)$ implies that $(f) = (g)$. It is clear that $(f)^* = (f^*)$ and that (f^*) is a m.p. ideal if and only if (f) is. Hence Lemma 4 is an immediate consequence of the following lemma.

LEMMA 5. (f) *is a m.p. ideal if and only if* $(f) = I_a$.

PROOF. I_a is the principal ideal generated by the function $z-a$. If $I_a \subset (g) \neq R(D)$, then $g(z)$ must possess zeros in D. Since $z-a = g(z)h(z)$, $h \in R(D)$, $g(a) = 0$ and $g \in I_a$. On the other hand, if $f(z)$ has no zeros in D, then $(f) = R(D)$, and if $f(a) = f'(a) = 0$, or if $f(a) = f(b) = 0$, $a \neq b$, then (f) is contained in and different from the principal ideal generated by the function $z-a$. It follows that (f) is a m.p. ideal if and only if $f(z)=(z-a)e^{h(z)}$, $a\in D$, $h\in R(D)$.

LEMMA 6. *For every point* $z_0\in D, f(z_0)^*=f^*[\phi(z_0)]$.

PROOF. If c is a constant such that $f(z_0)=c$, then $c-f$ belongs to I_{z_0}, so that c^*-f^* belongs to $I_{\phi(z_0)}$ and $f^*[\phi(z_0)]=c^*$.

The following two lemmas are immediate consequences of Lemma 6.

LEMMA 7. *If* $z_0\in D$ *and* $f(z_0)$ *is a rational number, then* $f(z_0)=f^*[\phi(z_0)]$.

LEMMA 8. *If* $f(z)$ *is univalent in* D, *then* $f^*(\zeta)$ *is univalent in* Δ.

LEMMA 9. *Let* $f(z)$ *be a univalent function defined in* D, *let* $f(D)$ *be the image of* D *under the transformation* $w=f(z)$, *and let* W *be the (finite) limit of a convergent sequence of distinct rational points* $\{w_n\}$ *belonging to* $f(D)$. W *is a boundary point of* $f(D)$ *if and only if there exists a function* $g(z)\in R(D)$ *such that* $g[h(w_n)]=n$, $h(w)$ *being the function inverse to* $w=f(z)$.

PROOF. If W is a boundary point of $f(D)$, choose an entire function $F(Z)$ such that $F[(W-w_n)^{-1}]=n$. The function $g(z)=F\{[W-f(z)]^{-1}\}$ satisfies the conditions of the lemma. On the other hand, if W is an interior point of $f(D)$, $W=f(a)$, $a\in D$, and for every $g(z)\in R(D)$, $\lim g[h(w_n)]$ exists and is finite.

LEMMA 10. *Let* $f(z)$ *be a univalent function defined in* D, *so that* $f^*(\zeta)$ *is a univalent function defined in* Δ. *The domains* $f(D)$ *and* $f^*(\Delta)$ *are identical.*

PROOF. It follows from Lemma 7 that the rational points belonging to $f(D)$ are identical with the rational points belonging to $f^*(\Delta)$. $f(D)$ is the set of all limit points of its rational points, except those limit points which lie on the boundary of $f(D)$. A similar remark applies to $f^*(\Delta)$. But Lemmas 7 and 9 imply that if a sequence of rational points from $f(D)$ converges to a boundary point W, W is a boundary point of $f^*(\Delta)$.

Lemma 10 contains the proof of Theorem I for domains possessing boundary points.

The proof of Theorem II depends on the following lemma.

LEMMA 11. *If* D *possesses boundary points, then* $c^*=c$ *for every constant* c.

We prove this lemma in several steps. Let B be any domain. By

$m[B]$ we denote the set of all complex numbers d such that the translation $Z=z+d$ maps B onto itself.

LEMMA 12. *For every univalent function $f\in R(D)$ and for every constant c the difference $c-c^*$ belongs to $m[f(D)]$.*

PROOF. The function $f_1=f+c$ is univalent in D, and the functions f^* and $f_1^*=f^*+c^*$ are univalent in Δ (Lemma 8). By virtue of Lemma 10, $f(D)$ is identical with $f^*(\Delta)$, and $f_1(D)$ is identical with $f_1^*(\Delta)$. But a translation by c takes $f(D)$ into $f_1(D)$ and a translation by $-c^*$ takes $f_1^*(\Delta)$ into $f^*(\Delta)$. Thus the translation by $c-c^*$ leaves $f(D)$ invariant.

LEMMA 13. *If D possesses finite boundary points, then there exists a univalent function $f\in R(D)$ such that $m[f(D)]$ is a discrete set.*

PROOF. Let f be a fixed univalent function defined in D. The set $m[f(D)]$ is closed and a modul, that is, it contains d_1-d_2 whenever it contains d_1 and d_2. Assume that $m[f(D)]$ is not discrete. Then it either contains all points, or all points of a straight line. If W is a finite boundary point of $f(D)$, every point $W+d$, $d\in m[f(D)]$, is a boundary point. It follows that the boundary of $f(D)$ contains a finite straight segment S. Let $Z(w)$ be the function which maps the domain exterior to S conformally onto $|Z|<1$. The function $g(z)=Z[f(z)]$ is univalent and bounded in D. It follows that $m[g(D)]$ contains only the point 0.

Now we can prove Lemma 11 under the hypothesis of Lemma 13. Let $f\in R(D)$ be univalent and such that $m[f(D)]$ is discrete. For every positive number t set $f_t=tf$. By Lemma 12 the difference $c-c^*$ belongs to $m[f_t(D)]$, that is, the number $(c-c^*)/t$ belongs to $m[f(D)]$. It follows that $c-c^*=0$.[3]

It remains to establish Lemma 11 for the case of the whole finite plane.

LEMMA 14. *If D is the domain $|z|<\infty$ and Δ the domain $|\zeta|<\infty$, then $c_n\to\infty$ implies $c_n^*\to\infty$.*

PROOF. Set $f(z)=z$. Then $f^*(\zeta)$ is univalent in Δ, that is, $f^*(\zeta)=A\zeta+B$, A, $B=$const., $A\neq 0$. For this f and for $z_0=c$, Lemma 6 yields $c^*=A\phi(c)+B$. Hence $c_n^*\to\infty$ whenever $\phi(c_n)\to\infty$. But $c_n\to\infty$

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[3] An alternative argument was suggested to the author by C. Loewner. Assume that $c-c^*=re^{i\theta}$, $r\neq 0$. It is easy to see that for any univalent f and for any domain D satisfying the hypothesis of Lemma 13 there exists a (finite or semi-infinite) straight segment S whose interior points belong to $f(D)$ and whose end points are boundary points of $f(D)$. By a linear transformation we can achieve that S be the semi-infinite segment $z=te^{i\theta}$, $t>0$. It follows that $c-c^*$ does not belong to $m[f(D)]$.

implies the existence of a function $g \in R(D)$ such that $g(c_n)$ is rational and $g(c_n) \rightarrow \infty$. By Lemma 7, $g^*[\phi(c_n)] \rightarrow \infty$, from which it follows that $\phi(c_n) \rightarrow \infty$.

In order to prove Lemma 11 under the hypothesis of Lemma 14 we note that the transformation $c \rightarrow c^*$ is an automorphism of the complex field. This automorphism is continuous by virtue of Lemma 14. Hence $c^* = c$, for we assumed that $i^* = i$.

Lemma 11 being established, Lemma 6 yields the following lemma.

LEMMA 15. *If D possesses boundary points, then $f(z_0) = f^*[\phi(z_0)]$ for every $z_0 \in D$.*

This lemma would contain Theorem II if we would know that $\phi(z)$ is analytic in D. To show this we select for f the function $f(z) = z$. Lemma 15 shows that ϕ is the function inverse to the univalent (analytic) function f^*.

In the statement of Theorem I the ring of *all* analytic functions cannot be replaced by the subring $B(D)$ of all bounded analytic functions defined in D, even if $B(D)$ is treated as a normed ring (with $\|f\| = \text{l.u.b.}\,|f(z)|$), and the isomorphism between $B(D)$ and $B(\Delta)$ is required to be norm preserving.[3] In fact, let D be the domain $|z| < 1$, and let Δ be the domain $0 < |z| < 1$. The normed rings $B(D)$ and $B(\Delta)$ are identical.[4]

Neither is it possible to replace $R(D)$ by the linear space $L(D)$ of all analytic functions defined in D, even if $L(D)$ is considered to be a topological space (with the topology defined above) and the isomorphism between $L(D)$ and $L(\Delta)$ is required to be a homeomorphism. In fact, let D be the domain $0 < r < |z| < 1$, and let D_1 and D_2 denote the domains $|z| < 1$ and $|z| > r$, respectively. Every function $f \in L(D)$ admits a unique Laurent decomposition: $f(z) = g(z) + h(z)/z$, $g \in L(D_1)$, $h \in L(D_2)$. It is easy to see that $f_n \rightarrow f$ if and only if $g_n \rightarrow g$ and $h_n \rightarrow h$. On the other hand, the linear subspace of $L(D_2)$ consisting of all functions of the form $h(z)/z$, $h \in L(D_2)$, is topologically isomorphic to $L(D_2)$, and $L(D_2)$ is topologically isomorphic to $L(D_1)$. It follows that $L(D)$ is topologically isomorphic to the direct sum of two spaces $L(D_1)$. Thus $L(D)$ considered as an abstract linear topological space is independent of r.

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[3] As a matter of fact, every isomorphism is.

[4] Cf., however, footnote 1.

THE CENTER OF A JORDAN RING[1]

N. JACOBSON

If $\mathfrak{A}$ is an arbitrary associative ring we can symmetrize and antisymmetrize the multiplication defined in $\mathfrak{A}$ to obtain two non-associative rings. We set

$$\{ab\} = ab + ba, \qquad [ab] = ab - ba \tag{1}$$

and call the former the *Jordan product* and the latter the *commutator* or *Lie product* of a and b. If we use $\{ab\}$ as product in place of the originally defined ab we obtain the *Jordan ring* $\mathfrak{A}_j$ determined by $\mathfrak{A}$. Similarly the *Lie ring* $\mathfrak{A}_l$ is obtained by using $[ab]$ in place of ab. Naturally if $\mathfrak{A}$ has characteristic 2 then $\mathfrak{A}_j = \mathfrak{A}_l$. It is customary to exclude this case from consideration but in most of our discussion we shall not find it necessary to do so. Clearly $\{ab\} = \{ba\}$, $[ab] = -[ba]$. Also we recall the following well known identity of Jacobi's:

$$[[ab]c] + [[bc]a] + [[ca]b] = 0. \tag{2}$$

If $\mathfrak{R}$ is any non-associative ring one defines the center of $\mathfrak{R}$ to be the totality of elements c that commute,

$$c \cdot a = a \cdot c, \tag{3}$$

and associate,

$$\begin{gathered} (a \cdot b) \cdot c = a \cdot (b \cdot c), \qquad (a \cdot c) \cdot b = a \cdot (c \cdot b), \\ (c \cdot a) \cdot b = c \cdot (a \cdot b), \end{gathered} \tag{4}$$

with all a, b in $\mathfrak{R}$.[2] It is known that the center is a subring of $\mathfrak{R}$. Clearly this subring is associative. It is also known that the center of a simple ring is either 0 or a field. It is easy to see that the middle condition in (4) is a consequence of (3) and the other conditions in (4). Also it is clear that if $\mathfrak{R}$ is commutative then the first condition of (4) characterizes the center.

We consider now the centers $\mathfrak{C}_j$ and $\mathfrak{C}_l$ respectively of $\mathfrak{A}_j$ and $\mathfrak{A}_l$.

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[1] I am indebted to A. H. Clifford for a number of valuable conversations on the subject of this note.

[2] See Jacobson, *Structure theory of simple rings without finiteness assumptions*, Trans. Amer. Math. Soc. vol. 57 (1945) p. 239, or T. Nakayama, *Über einfache distributive Systeme unendlicher Ränge*, Proc. Imp. Acad. Tokyo vol. 20 (1944) p. 62 for this definition and for the results quoted in this paragraph.

First let $c \in \mathfrak{C}_l$. Then by (1) and (3), $2[ca] = 0$. By (4) and Jacobi's identity, $[[ca]b] = 0$. Hence

$$2[ca] = 0, \qquad [[ca]b] = 0 \tag{5}$$

holds. It is also easy to see that these conditions are sufficient that $c \in \mathfrak{C}_l$.

Next let $c \in \mathfrak{C}_j$. We introduce the *Jordan associator*

$$A(a, b, c) = \{\{ab\}c\} - \{a\{bc\}\}$$

and we can verify that

$$A(a, b, c) = [[ca]b]. \tag{6}$$

Hence

$$[[ca]b] = 0 \tag{7}$$

is a necessary and sufficient condition that $c \in \mathfrak{C}_j$. Thus we see that $\mathfrak{C}_l \subseteq \mathfrak{C}_j$. If we denote the center of $\mathfrak{A}$ by $\mathfrak{C}$, $c \in \mathfrak{C}$ if and only if $[ca] = 0$. Hence $\mathfrak{C} \subseteq \mathfrak{C}_l \subseteq \mathfrak{C}_j$.

Let $c \in \mathfrak{C}_j$ and a, b, d be arbitrary in $\mathfrak{A}$. Then $[[c, ab]d] = 0$. Since

$$[c, ab] = [ca]b + a[cb]$$

we obtain

$$[ca][bd] + [ad][cb] = 0. \tag{8}$$

This simple relation has a number of interesting consequences. In the first place if we set $d = c'$, a second element in $\mathfrak{C}_j$, we obtain $[ca][bc'] + [ac'][cb] = 0$. If we use the fact that $[cb] \in \mathfrak{C}$ this reads

$$[ca][c'b] + [cb][c'a] = 0. \tag{9}$$

It is easy to see that this implies

$$[[cc', a]b] = 0. \tag{10}$$

Thus $cc' \in \mathfrak{C}_j$. It is clear also that if $2[ca] = 0$ and $2[c'a] = 0$ then $2[cc', a] = 0$. Hence we have the following theorem.

THEOREM 1. *The Jordan center $\mathfrak{C}_j$ and the Lie center $\mathfrak{C}_l$ are (ordinary) subrings of $\mathfrak{A}$.*

If we specialize $d = a$ and $b = c$ successively in (8) we obtain

$$[ca][ba] = 0, \qquad [ca][cd] = 0. \tag{11}$$

In particular

$$[ca]^2 = 0. \tag{12}$$

This implies the following theorem.

THEOREM 2. *If $\mathfrak{A}$ is an associative ring whose center $\mathfrak{C}$ contains no nilpotent elements not equal to* 0 *then the Jordan center $\mathfrak{C}_J$ coincides with $\mathfrak{C}$.*

For if $c \in \mathfrak{C}_J$, $[ca]^2 = 0$ for any $a \in \mathfrak{A}$. Since $[ca] \in \mathfrak{C}$ this implies that $[ca] = 0$. Hence $c \in \mathfrak{C}$.

It is easy to see that if a ring contains a nilpotent element in its center then it contains a nilpotent two-sided ideal. Hence we have the following corollary.

COROLLARY. *If $\mathfrak{A}$ is a ring that has no nilpotent two-sided ideals then its Jordan center $\mathfrak{C}_J$ coincides with the ordinary center.*

It is clear from Jacobi's identity that if $c \in \mathfrak{C}_J$ then $[[ab]c] = 0$ for all a, b. Thus c commutes with every commutator. Clearly the elements that have this property form a subring $\mathfrak{B}$ of $\mathfrak{A}$. It will be shown in a forthcoming paper by Kaplansky that if $\mathfrak{A}$ is a semi-simple ring then $\mathfrak{B} = \mathfrak{C}$.[3] Thus Kaplansky's result has both a stronger hypothesis and a stronger conclusion than our Theorem 2. The following examples will serve to illustrate these results.

EXAMPLE 1. Let $\mathfrak{A}$ be the ring of triangular matrices

$$\begin{bmatrix} \alpha_{11} & & * \\ & \alpha_{22} & \\ & & \ddots & \\ 0 & & & \alpha_{nn} \end{bmatrix} \tag{13}$$

with elements in a field Φ. It is easy to see that $\mathfrak{C}$ is the set of scalar matrices. Hence $\mathfrak{C}_J = \mathfrak{C}$. On the other hand it can be seen that $\mathfrak{B}$ is the set of matrices

$$\begin{bmatrix} \alpha & 0 \cdots 0 & \beta \\ & \ddots & 0 \\ & & \ddots \; 0 \\ & & \alpha \end{bmatrix}. \tag{14}$$

EXAMPLE 2. Let $\mathfrak{A}$ be the set of triangular matrices for which $\alpha_{11} = \alpha_{22} = \cdots = \alpha_{nn} = \alpha$. Here $\mathfrak{C}$ is the set of matrices of the form

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[3] *Added in proof.* This paper has now appeared: *Semi-automorphisms of rings*, Duke Math. J. vol. 14 (1947) pp. 521–527.

(14) and $\mathfrak{B}$ is the set of matrices of the form

$$\begin{pmatrix} \alpha & 0 & \cdots & 0 & \gamma & \beta \\ & \ddots & & & \epsilon & \delta \\ & & \ddots & & & 0 \\ & 0 & & \ddots & & \vdots \\ & & & & \ddots & 0 \\ & & & & & \alpha \end{pmatrix}. \tag{15}$$

Finally $\mathfrak{C}_j$ is the subset of $\mathfrak{B}$ of the matrices of the form (15) in which $\epsilon=0$.

We derive next another property of the element c in $\mathfrak{C}_j$. If a, b, d are arbitrary in $\mathfrak{A}$,

$$\begin{aligned}[][ca, b]d] &= [(a[cb] + c[ab]), d] \\ &= [ad][cb] + [cd][ab] + c[[ab]d].\end{aligned}$$

If we interchange a and d in (8) we see that

$$[ad][cb] + [cd][ab] = 0.$$

Hence

$$[[ca, b]d] = c[[ab]d].$$

This implies also that $[[a, cb]d]=c[[ab]d]$. By Jacobi's identity we obtain finally $[[ab]cd]=c[[ab]d]$. Thus

$$c[[ab]d] = [[ca, b]d] = [[a, cb]d] = [[ab]cd]. \tag{16}$$

This can also be written in the following form in terms of the associator $A(b, d, a)=[[ab]d]$:

$$cA(b, d, a) = A(b, d, ca) = A(cb, d, a) = A(b, cd, a). \tag{16$'$}$$

In the remainder of this note we consider rings $\mathfrak{U}$ that are subrings of a Jordan ring of the form $\mathfrak{A}_j$. By the enveloping (associative) ring of $\mathfrak{U}$ in $\mathfrak{A}$ we mean the subring of $\mathfrak{A}$ generated by $\mathfrak{U}$. If $\mathfrak{E}$ is the enveloping ring of $\mathfrak{U}$ clearly $\mathfrak{U}$ is also a subring of the Jordan ring $\mathfrak{E}_j$. Hence we can suppose that $\mathfrak{E}=\mathfrak{A}$.

If c is in the Jordan center $\mathfrak{C}_j(\mathfrak{U})$ of $\mathfrak{U}$ then $[[ca]b]=0$ holds for all $a, b\in\mathfrak{U}$. Since $\mathfrak{U}$ is a Jordan ring

$$[[c, \{ab\}]d] = 0$$

for all a, b, d in $\mathfrak{U}$. Also we have $[c[ab]]=0$ so that $[[c, [ab]]d]=0$. Hence by addition

$$2[[c, ab]d] = 0. \tag{17}$$

We shall assume now that the ring $\mathfrak{A}$ contains no element that has order two in the additive group. Then by (17), $[[c, ab]d] = 0$ for all a, b, d in $\mathfrak{U}$. As before this leads to the relation (8) and to its consequence $[ca]^2 = 0$. On the other hand since $[[ca]b] = 0$, $[ca]$ commutes with every b in $\mathfrak{U}$. Hence it commutes with every b in the enveloping ring $\mathfrak{A}$. Thus $[ca]$ is in the center $\mathfrak{C}$ of $\mathfrak{A}$. This shows that if some $[ca] \neq 0$ then the center of $\mathfrak{A}$ possesses nilpotent elements. Hence we have the following theorem.

THEOREM 3. *Let $\mathfrak{A}$ be an associative ring that has no elements of order 2 and that has no nilpotent elements ($\neq 0$) in its center. Let $\mathfrak{U}$ be a subring of the Jordan ring $\mathfrak{A}_j$ such that the enveloping ring of $\mathfrak{U}$ is $\mathfrak{A}$. Then the Jordan center $\mathfrak{C}_j(\mathfrak{U})$ coincides with the totality of elements c of $\mathfrak{U}$ that commute with every a in $\mathfrak{U}$ (or in $\mathfrak{A}$).*

A. A. Albert has recently studied the structure of the subalgebras $\mathfrak{U}$ of a Jordan algebra $\mathfrak{A}_j$, $\mathfrak{A}$ a matrix algebra Φ_n over a field Φ.[4] In the course of this study he has defined the center of $\mathfrak{U}$ to be the totality of elements c such that $[ca] = 0$ for all a in $\mathfrak{U}$. We shall show in an example that Albert's definition is unsatisfactory for arbitrary Jordan algebras since it is not invariant under isomorphism. On the other hand it will also be shown that in the case of simple algebras over a field of characteristic 0—and this case is the only one for which Albert uses his definition—Albert's center coincides with the Jordan center as defined in the present paper. We remark that Theorems 2 and 3 give other cases for which this identity holds. In all of these cases Albert's definition can not lead to any difficulties. We consider now the following example.

EXAMPLE 3. Let $\mathfrak{A}$ be the algebra over Φ that has the basis z_1, z_2 such that $z_i z_j = 0$. The Jordan ring $\mathfrak{A}_j$ has the basis z_1, z_2 with the multiplication table

$$\{z_1 z_1\} = 0, \qquad \{z_1 z_2\} = 0, \qquad \{z_2 z_2\} = 0.$$

Let $\mathfrak{B}$ be the algebra over Φ that has the basis Z_1, Z_2, Z_3 with the multiplication table

$$Z_1^2 = Z_2^2 = Z_3^2 = 0, \qquad Z_1 Z_2 = -Z_2 Z_1 = Z_3, \qquad Z_2 Z_3 = Z_3 Z_2 = 0,$$
$$Z_3 Z_1 = Z_1 Z_3 = 0.$$

Since any product of three Z's is 0, $\mathfrak{B}$ is an associative algebra. We have the Jordan multiplication table

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[4] *On Jordan algebras of linear transformations*, Trans. Amer. Math. Soc. vol. 59 (1946) p. 540.

$$\{Z_1Z_1\} = 0, \qquad \{Z_1Z_2\} = 0, \qquad \{Z_2Z_2\} = 0.$$

Hence $\mathfrak{U} = (Z_1, Z_2)$ is a subalgebra of $\mathfrak{B}_j$ isomorphic to $\mathfrak{A}_j$. The totality of elements of $\mathfrak{A}_j$ that commute with all the elements of $\mathfrak{A}_j$ is $\mathfrak{A}_j$ itself. However, the elements Z_1 and Z_2 do not commute with all the elements of $\mathfrak{U}$. Evidently the Jordan center of $\mathfrak{A}_j$ is $\mathfrak{A}_j$.

We shall now consider the Jordan center of any simple Jordan algebra with a finite basis over a field of characteristic 0. At first we consider any associative algebra $\mathfrak{A}$ over a field Φ. As before let $\mathfrak{A}_j$ denote the Jordan algebra obtained from $\mathfrak{A}$ by using the operations $a+b$, αa for α in Φ and $\{ab\} = ab+ba$. If $\mathfrak{U}$ is a subalgebra of $\mathfrak{A}_j$ it is clear that the center $\mathfrak{C}_j(\mathfrak{U})$ is a subalgebra. Let $c \in \mathfrak{C}_j(\mathfrak{U})$ and let a be any algebraic element of $\mathfrak{U}$. Then there exists a polynomial $\phi(\lambda) \neq 0$ in $\Phi[\lambda]$ such that $\phi(a) = 0$. Since $[ca]$ commutes with a, $[c, a^k] = ka^{k-1}[ca]$. Hence

$$0 = [c, \phi(a)] = \phi'(a)[ca] \tag{18}$$

where $\phi'(\lambda)$ is the derivative of the polynomial $\phi(\lambda)$. We assume next that c is algebraic and that a is arbitrary. Then a similar argument shows that if $\psi(\lambda)$ is a polynomial not equal to 0 such that $\psi(c) = 0$ then

$$\psi'(c)[ca] = 0. \tag{19}$$

We suppose now that $\mathfrak{A} = \Phi_n$, Φ of characteristic 0. Let $\mathfrak{U}$ be a simple subalgebra of $\mathfrak{A}_j$. Then it has been shown by Albert that $\mathfrak{U}$ has an identity e.[5] Evidently e is in the Jordan center $\mathfrak{C}_j(\mathfrak{U})$. Hence $\mathfrak{C}_j(\mathfrak{U}) \neq 0$ and it follows that $\mathfrak{C}_j(\mathfrak{U})$ is a field. Since $ea + ae = a$, $e^2 = e/2$. If we set $e' = 2e$ then $(e')^2 = e'$ and $(e'a + ae')/2 = a$. Hence

$$e'a/4 + e'ae'/2 + ae'/4 = (e'a + ae')/2$$

and

$$a = (e'a + ae')/2 = e'ae'.$$

It follows that $e'a = a = ae'$. Thus e' acts as an identity for all the elements of $\mathfrak{U}$ and therefore for all the elements of the enveloping algebra $\mathfrak{E}$ of $\mathfrak{U}$.

Suppose now that $c \in \mathfrak{C}_j(\mathfrak{U})$ and that $\mu(\lambda)$ is the minimum polynomial of c regarded as an element of $\mathfrak{C}_j(\mathfrak{U})$. Then if $\mu(\lambda) = \lambda^m$

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[5] Albert (loc. cit., footnote 3) uses the multiplication $a \cdot b = \{ab\}/2$. Clearly an algebra is simple relative to the dot multiplication if and only if it is simple relative to $\{\ \}$. Moreover, if e' is an identity relative to $\cdot$, then $e = e'/2$ is one relative to $\{\ \}$.

$+\gamma_1\lambda^{m-1}+\cdots+\gamma_m$,

$$\{c\}^m+\gamma_1\{c\}^{m-1}+\cdots+\gamma_{m-1}c+\gamma_m e=0 \tag{20}$$

where $\{c\}^r$ is defined inductively by $\{c\}^r=\{\{c\}^{r-1}, c\}$. Since $\{c\}^r=2^{r-1}c^r$, (20) yields

$$c^m+\frac{\gamma_1}{2}c^{m-1}+\cdots+\frac{\gamma_{m-1}}{2^{m-1}}c+\frac{\gamma_m}{2^m}e'=0. \tag{21}$$

Thus if $\psi(\lambda)=2^{-m}\mu(2\lambda)$ then $\psi(c)=0$. Since $\mathfrak{C}_J(\mathfrak{U})$ is a separable field, $\mu'(\lambda)$ is prime to $\mu(\lambda)$. Hence $\psi'(\lambda)$ is prime to $\psi(\lambda)$. It follows that

$$\psi'(c)=mc^{m-1}+(m-1)\frac{\gamma_1}{2}c^{m-2}+\cdots+\frac{\gamma_{m-1}}{2^{m-1}}e'$$

has an ordinary inverse relative to e'. Hence by (19), $[ca]=0$ for all a. This proves the following theorem.

THEOREM 4. *Let $\mathfrak{A}=\Phi_n$ the ring of $n\times n$ matrices over a field of characteristic* 0 *and let* $\mathfrak{U}$ *be a simple Jordan subalgebra of $\mathfrak{A}_J$. Then the center of* $\mathfrak{U}$ *coincides with the totality of elements c of* $\mathfrak{U}$ *that commute in the ordinary multiplication with every a in* $\mathfrak{U}$.

THE JOHNS HOPKINS UNIVERSITY

ON PRIME POWER ABELIAN GROUPS

YENCHIEN YEH

G. A. Miller in two papers [1, 2][1] has discussed the number of subgroups of prime power abelian groups. He has obtained a formula concerning the number of cyclic subgroups of a given prime power abelian group G,[2] and proved some theorems relating to subgroups of special types. He predicted there that the general formula for the determination of the number of subgroups of any type could be obtained without difficulty except complexity of calculations. In this paper, however, we get the general formula with a method thoroughly independent of his approach, and it seems that our method is somewhat simpler than his would be.

LEMMA 1. *If G is a prime power abelian group of order $p^{k_1+k_2+\cdots+k_n}$, type $(k_1, k_2, \cdots, k_n)$, where the k's are arranged in ascending order of magnitude, then the number of cyclic subgroups of order p^k in G is*

$$\frac{p^{n-r_k}-1}{p-1}\,p^{(n-r_k-1)(k-1)+a},\tag{1}$$
$$\text{where}\quad a=\sum_{\mu=0}^{r_k}k_\mu,\ k_{r_k}<k\leqq k_{r_k+1},\ k_0=0.$$

DEFINITION. Two cyclic subgroups of order p^k are called isomers in G if a generator of one group is the product of an element in G of order less than p^k and a generator of the other group.

The number of elements of order not greater than p^k in G is easily seen to be

$$p^{(n-r_k)k+a},$$

a defined as in (1). From Lemma 1, the number of elements of order p^k is

$$\frac{p^{n-r_k}-1}{p-1}\,p^{(n-r_k-1)(k-1)+a}\cdot p^{k-1}(p-1)=(p^{n-r_k}-1)p^{(n-r_k)(k-1)+a}.$$

Hence the number of elements of order less than p^k is

$$p^{(n-r_k)k+a}-(p^{n-r_k}-1)p^{(n-r_k)(k-1)+a}=p^{(n-r_k)(k-1)+a}.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] [1, p. 259]. We state this here as Lemma 1.

A cyclic group S of order p^k contains p^{k-1} elements of order less than p^k, hence corresponding to a given generator g of S there exist just p^{k-1} elements of order p^k, each of them is the product of g and an element of order less than p^k in S. We have therefore the following lemma.

LEMMA 2. *Every cyclic subgroup of order p^k has*

$$\frac{p^{(n-\nu_k)(k-1)+e}}{p^{k-1}} = p^{(n-\nu_k-1)(k-1)+e}$$

isomers in G (including itself).

It is clear that the product of two cyclic subgroups of order p^k is a group of order p^{2k} if and only if these two cyclic subgroups are not isomers.

THEOREM. *Let G be the prime power abelian group as in Lemma* 1.

$$\begin{aligned} h_1 = h_2 = \cdots = h_{m_1} > h_{m_1+1} = \cdots = h_{m_1+m_2} > \cdots \\ > h_{m_1+m_2+\cdots+m_{r-1}+1} = \cdots = h_{m_1+m_2+\cdots+m_r} \end{aligned} \tag{2}$$

where $m_1+m_2+\cdots+m_r=m\leqq n$ are m positive integers not greater than k_n, with $k_{\nu_i}<h_i\leqq k_{\nu_i+1}$ $(i=1, 2, \cdots, m; k_0=0)$. Then the number of subgroups of type (2) *is given by*

$$p^H \prod_{i=1}^{m} (p^{n-\nu_i-i+1} - 1) \Big/ \prod_{\mu=1}^{r} \prod_{\nu=1}^{m_\mu} (p^\nu - 1) \tag{3}$$

where

$$\begin{aligned} H = & \sum_{i=1}^{m} (n - \nu_i + 1 - 2i)(h_i - 1) \\ & + \frac{1}{2} (m_1^2 + m_2^2 + \cdots + m_r^2 - m^2) + \sum_{i=1}^{m} \sum_{\mu=0}^{\nu_i} k_\mu . \end{aligned}$$

PROOF. We prove this by induction. For $m=1$,

$$H = (n - \nu_1 - 1)(h_1 - 1) + \sum_{\mu=0}^{\nu_1} k_\mu,$$

$$\prod_{i=1}^{m} (p^{n-\nu_i-i+1} - 1) = p^{n-\nu_1} - 1, \qquad \prod_{\mu=1}^{r} \prod_{\nu=1}^{m_\mu} (p^\nu - 1) = p - 1,$$

(3) reduces to (1), the theorem is therefore true by Lemma 1.

Assume it to be true for the integer m, then the number of subgroups of type (2) is given by (3). Let h_{m+1} be a positive integer less than h_m, the number of cyclic subgroups of order $p^{h_{m+1}}$ is

$$\frac{p^{n-\nu_{m+1}}-1}{p-1}p^{(n-\nu_{m+1}-1)(h_{m+1}-1)+b}, \qquad b=\sum_{\rho=0}^{\nu_{m+1}} k_\rho,$$

where $k_{\nu_{m+1}}<h_{m+1}\leqq k_{\nu_{m+1}+1}$. From Lemma 2, every cyclic subgroup of order $p^{h_{m+1}}$ has

$$p^{(n-\nu_{m+1}-1)(h_{m+1}-1)+b},$$

isomers in G, hence there are

$$\frac{p^{n-\nu_{m+1}}-1}{p-1}$$

sets of isomers of order $p^{h_{m+1}}$ in G. With the same reason, we see that the number of sets of isomers of order $p^{h_{m+1}}$ in a group of type (2) is

$$\frac{p^m-1}{p-1}p^{(m-1)(h_{m+1}-1)}/p^{(m-1)(h_{m+1}-1)}=\frac{p^m-1}{p-1}=M$$

$$(\nu_{m+1}=0, \text{ since } h_{m+1}<h_m).$$

Now, let G^* be a subgroup of type (2) in G, let $S_1, S_2, \cdots, S_M$ be M cyclic subgroups of order $p^{h_{m+1}}$ in G^* such that no two of them are isomers in G^* and, a fortiori, no two of them are isomers in G. The product of G^* and any cyclic subgroup S in G, which is an isomer of a certain S_ν, is a group of order less than

$$p^c, \qquad c=\sum_{\nu=1}^{m+1} h_\nu.$$

The number of such subgroups S is easily seen to be

$$\frac{p^m-1}{p-1}p^{(n-\nu_{m+1}-1)(h_{m+1}-1)+b}$$

(when $m=n-\nu_{m+1}$, this represents the number of all cyclic subgroups of order $p^{h_{m+1}}$ in G; when $m>n-\nu_{m+1}$, subgroup of type (2) does not exist at all). With the exception of these cyclic subgroups, the product of any cyclic subgroup of order $p^{h_{m+1}}$ (in G) and G^* is a group of order p^c, type $(h_1, h_2, \cdots, h_m, h_{m+1})$.

On the other hand, every group K of type $(h_1, h_2, \cdots, h_m, h_{m+1})$ contains $p^{mh_{m+1}}$ subgroups of type (2). One can find the proof of this in [**2**; p. 366]. Or, using directly formula (3) (since the theorem is assumed to be true for the integer m), we have here

$$
\begin{aligned}
H = {} & \sum_{i=1}^{m}(m+2-\nu_i-2i)(h_i-1) \\
& + \frac{1}{2}(m_1^2+m_2^2+\cdots+m_r^2-m^2) \\
& + \sum_{\mu=1}^{r} m_\mu(m_{\mu+1}h_{m_1+\cdots+m_{\mu+1}} + m_{\mu+2}h_{m_1+\cdots+m_{\mu+2}} + \cdots \\
& + m_r h_m + h_{m+1}) \\
= {} & - m_1m_2(h_{m_1+m_2}-1) - m_3(m_1+m_2)(h_{m_1+m_2+m_3}-1) - \cdots, \\
& - m_r(m-m_r)(h_m-1) + \frac{1}{2}(m_1^2+\cdots+m_r^2-m^2) + mh_{m+1} \\
& + \sum_{\mu=1}^{r-1} m_\mu(m_{\mu+1}h_{m_1+\cdots+m_{\mu+1}} + \cdots + m_r h_m) \\
= {} & mh_{m+1} + \frac{1}{2}(m_1+m_2+\cdots+m_r)^2 - \frac{1}{2}m^2 \\
= {} & mh_{m+1},
\end{aligned}
$$

and

$$\prod_{i=1}^{m}(p^{m+2-\nu_i-i}-1)\Big/\prod_{\mu=1}^{r}\prod_{\nu=1}^{m_\mu}(p^\nu-1) = 1,$$

hence the result.

Now let $H_1, H_2, \cdots, H_N$ $(N = p^{mh_{m+1}})$ be the N subgroups of type (2) in K, let

$$K = H_\nu \times L_\nu \qquad (\nu = 1, 2, \cdots, N)$$

where L_ν is a cyclic subgroup of order $p^{h_{m+1}}$ with generator l_ν. In every H_ν there are $p^{mh_{m+1}}$ elements of order not greater than $p^{h_{m+1}}$, say $g_{\nu 1}, g_{\nu 2}, \cdots, g_{\nu N}$. The N elements $l_\nu g_{\nu\mu}$ $(\mu = 1, 2, \cdots, N)$ generate N different cyclic subgroups $L_{\nu 1}, L_{\nu 2}, \cdots, L_{\nu N}$ of order $p^{h_{m+1}}$ such that

$$H_\nu \times L_{\nu\mu} = H_\nu \times L_\nu = K \qquad (\mu = 1, 2, \cdots, N).$$

Hence every group of type $(h_1, h_2, \cdots, h_m, h_{m+1})$ can be obtained by $p^{mh_{m+1}}$ cases of direct production of two groups of type $(h_1, h_2, \cdots, h_m)$ and (h_{m+1}) respectively. The number of subgroups of type $(h_1, h_2, \cdots, h_m, h_{m+1})$ in G is therefore

$$
\begin{aligned}
&R_m\left(\frac{p^{n-r_{m+1}}-1}{p-1}-\frac{p^m-1}{p-1}\right)p^{(n-r_{m+1}-1)(k_{m+1}-1)+b}/p^{2mk_{m+1}} \\
&= R_m\frac{p^m(p^{n-m-r_{m+1}}-1)}{p-1}p^{(n-r_{m+1}-1)(k_{m+1}-1)-2mk_{m+1}+b} \\
(4)\qquad &= R_m\frac{p^{n-m-r_{m+1}}-1}{p-1}p^{(n-r_{m+1}+1-2(m+1))(k_{m+1}-1)+(1+m^2-(m+1)^2)/2+b} \\
&= R_{m+1}, \qquad\qquad b=\sum_{\mu=0}^{r_{m+1}}k_\mu,
\end{aligned}
$$

where R_m denotes Formula (3) for subgroups of type (2), R_{m+1} denotes the same formula for subgroups of type $(k_1, k_2, \cdots, k_m, k_{m+1})$.

When $k_{m+1}=k_m$, every group of type $(k_1, k_2, \cdots, k_m, k_{m+1})$ contains

$$
\frac{p^{m_r+1}-1}{p-1}p^{mk_{m+1}-m_r}
$$

subgroups of type $(k_1, k_2, \cdots, k_m)$, so it can be obtained by

$$
\frac{p^{m_r+1}-1}{p-1}p^{mk_{m+1}-m_r}\cdot p^{mk_{m+1}}=\frac{p^{m_r+1}-1}{p-1}p^{2mk_{m+1}-m_r}
$$

cases of direct production. Hence instead of (4) we have

$$
\begin{aligned}
&R_m\frac{p^m(p^{n-r_{m+1}-m}-1)}{p-1}p^{(n-r_{m+1}-1)(k_{m+1}-1)+b}\bigg/\frac{p^{m_r+1}-1}{p-1}p^{2mk_{m+1}-m_r} \\
&= R_m\frac{p^{n-r_{m+1}-m}-1}{p^{m_r+1}-1}p^{(n-r_{m+1}+1-2(m+1))(k_{m+1}-1)+((m_r+1)^2-m_r^2)/2-((m+1)^2-m^2)/2+b} \\
&= R'_{m+1},
\end{aligned}
$$

which represents Formula (3) for subgroups of type $(k_1, k_2, \cdots, k_m, k_{m+1})$, when $k_{m+1}=k_m$.

The theorem is therefore proved by induction.

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2. ———, *Independent generators of the subgroups of an abelian group*, Ibid. pp. 364–367.

NATIONAL UNIVERSITY OF CHEKIANG

SYMMETRY OF ALGEBRAS OVER A NUMBER FIELD

SAUNDERS MACLANE

1. **Introduction.** If the field N is a finite normal extension of the field k, and if K is a normal subfield with $N\supset K\supset k$, a fundamental theorem of Galois theory asserts that every automorphism λ of K over k can be extended to an automorphism of N. As Teichmüller in [7][1] and Jacobson [6, p. 36] have shown, the development of a Galois theory for a simple algebra A with center K leads naturally to a related question: can a given automorphism λ of K be extended to an automorphism of the algebra A? In the event that all automorphisms λ of a finite group Q of automorphisms of K are so extendable, we say that the algebra A is Q-normal. Since any total matric algebra over K is Q-normal for any Q, it follows that any algebra A similar to a Q-normal algebra is Q-normal, and hence that "Q-normality" is a property of algebra classes. Furthermore, if k is the subfield of all elements of K invariant under each automorphism λ of Q, any simple algebra B with center k yields a scalar extension B_K with center K which is Q-normal. The algebra class of any B_K (that is, the algebra classes obtained by scalar extension from k) may thus be termed *trivially* Q-normal. The further investigation of these properties thus raises the problem: are there any algebras which are Q-normal but not trivially so?

If $K\supset k$ are p-adic fields, Köthe [5] has shown that every algebra class over K may be obtained by scalar extension from k, so that in this case all Q-normal algebra classes are trivial. If K is an algebraic number field, he shows that there are algebra classes over K which cannot be obtained by scalar extension. If Q is cyclic, and if K is an algebraic number field, Deuring [2] showed that every Q-normal algebra class is trivially Q-normal. By using three-dimensional cocycles, the same results may be proved for Q cyclic and any field K (Teichmüller, op. cit. p. 149 or Eilenberg-MacLane [3, Corollary 7.3]). In case Q is not cyclic, the answer to our question apparently depends on the arithmetic properties of the field K. In case K is an algebraic number field, the algebra classes can be described completely by the usual arithmetic invariants (cf. for example, Deuring [1, chap. VII]). Using these invariants and the above facts about the cyclic case we obtain in Theorem 3 a complete description of the

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

group of nontrivial Q-normal algebra classes over a number field. In particular, the existence of nontrivial Q-normal algebras follows (§5), for Q a four group and K a suitable field. In view of the possibility of describing Q-normal algebras by cocycles [3, 7] this also shows that there exist three-dimensional cocycles of Q in K which are not coboundaries.

2. Invariants of scalar extensions.

A simple algebra B with center an algebraic number field k has for each finite or infinite prime divisor p of k (that is, for each valuation of k) a rational number (B/p) (mod 1) as local invariant. It is known (cf. Deuring [1, p. 119]) that the mapping

$$B \to \{ \cdots, (B/p), \cdots \}$$

yields an isomorphism of the group of algebra classes over k into a subgroup of the direct sum of groups of rational numbers $\rho(p)$ mod 1, with one summand for each p. The isomorphic image consists of those elements $\{\rho(p)\}$ in the direct sum such that:

(1) only a finite number of $\rho(p)$ are $\not\equiv 0$ (mod 1);

(2a) if p is a real infinite prime divisor of k, $2\rho(p) \equiv 0$ (mod 1);

(2b) if p is a complex infinite prime divisor of k, $\rho(p) \equiv 0$ (mod 1);

(3) $\sum_p \rho(p) \equiv 0$ (mod 1).

(The third condition expresses the reciprocity theorem in terms of the invariants of an algebra.)

Consider Q, a finite group of automorphisms of the algebraic number field K, and k, the subfield of elements of K invariant under Q. Each prime divisor p in k has $S = S(p)$ factors $P_1, \cdots, P_S$ in K, and these prime divisors P_i of K are all conjugate under automorphisms of Q. The complete field K_{P_i} of K in the valuation associated with P_i is a finite extension of the complete field k_p. The degrees $[K_{P_i}:k_p]$ of these extensions are equal, for $i = 1, \cdots, S$. This degree $M(p)$, called the local degree of p, satisfies

$$M(p)S(p) = n = [K:k]. \tag{4}$$

For any algebra B over k, the scalar extension B_K to K has as invariants

$$(B_K/P_i) = M(p)(B/p), \qquad P_i \mid p \tag{5}$$

(cf. Deuring [1, p. 113, Theorem 4] or Köthe [5, Theorem 3]).

We introduce the integers

$$s = \text{g.c.d. } S(p), \qquad m = \text{l.c.m. } M(p), \qquad \text{over all } p. \tag{6}$$

Then (4) gives $n = sm$.

LEMMA. *The algebra class of A over K can be obtained by scalar extension from an algebra class of an algebra B over k if and only if the invariants (A/P) of A satisfy the conditions*

$$\text{(i)} \qquad (A/P) = (A/P') \quad \textit{for } P, P' \textit{ conjugate over } k,$$

$$\text{(ii)} \qquad \sum_{p} (m/M(p))(A/P) \equiv 0 \pmod 1, \qquad P \mid p,$$

where the sum is taken over all p of k, using some one factor P in K for each p.

PROOF. First suppose that the class of A is that of B_K. Then $(A/P) \equiv (B_K/P) \pmod 1$, and condition (i) follows from (5) above. After choosing some rational number $(A/P)/M(p)$, the invariants of B must be

$$(7) \qquad (B/p) \equiv (A/P)/M(p) + i(p)/M(p) \pmod 1, \qquad P \mid p,$$

for suitable integers $i(p)$. Since $\sum(B/p) \equiv 0 \pmod 1$, summation of (7) over all p and multiplication by m gives (ii).

Conversely, given an A which satisfies (i) and (ii), use (7) to define the invariants of a prospective algebra B. Such an algebra will exist, provided only that $\sum(B/p) \equiv 0 \pmod 1$. By (ii), $\sum(A/P)/M(p)$ is a rational number with denominator m. By suitable choice of integers $i(p)$, the (finite) sum $\sum i(p)/M(p)$ can be made equal to the negative of this quotient. The algebra B which thus exists has $(B_K/P) = (A/P)$ by (5) and (7), hence the class of A is obtained by scalar extension.

COROLLARY. *If Q is cyclic, condition* (ii) *of the lemma may be omitted.*

PROOF. Let λ be a generator of the cyclic group Q. The Tchebotareff density theorem (Hasse [4, p. 133]) shows that there is a prime ideal p in k which has its decomposition group in K generated by λ. This prime p is then unramified and undecomposed in K; hence $M(p) = n$ and $m = n$. Thus $m/M(p) = S(p)$, and in this case condition (ii) reduces simply to the condition (3) which must be satisfied in any event by the invariants of A.

3. Conditions for Q-normality.

THEOREM 1. *The central simple algebra A over K is Q-normal if and only if, for every automorphism $\lambda \in Q$, the algebra class of A can be obtained by scalar extension from an algebra class over K_λ, the subfield of elements of K fixed under λ.*

The proof does not require that K be a number field. In case A is obtained by scalar extension from an algebra B over K_λ, the automorphism λ can clearly be extended to B_K and hence also to the similar algebra A. This gives the required Q-normality. Conversely, the Q-normality of A implies that A is normal for the cyclic subgroup generated by λ. But, by the result quoted in the introduction, every cyclically normal algebra class is trivially such, so that the algebra class of A can be obtained by scalar extension from K_λ, q.e.d.

The condition for normality over a number field now takes the following very simple form.

THEOREM 2. *The algebra A is Q-normal if and only if its invariants satisfy the condition* $(A/P)\equiv(A/P')$ (mod 1) *for every pair of prime divisors P, P' of K conjugate under Q.*

PROOF. Assume that A is Q-normal. Given conjugate divisors P, P', select a $\lambda\in Q$ mapping P into P'. By Theorem 1, the algebra class of A is obtained by scalar extension from an algebra B_λ over K_λ. Since λ maps P into P', P and P' are both factors of the same prime divisor p_λ in K_λ. The condition then follows from the corollary in §2. The converse proof is similar.

4. The group of normal algebra classes.

THEOREM 3. *The group of Q-normal algebra classes over K, modulo the subgroup of those algebra classes which are trivially Q-normal, is a cyclic group of order s, where $s=$* g.c.d. $S(p)$ *is the greatest common divisor for all p of the numbers $S(p)$ of distinct prime factors in K of the prime divisors p of k. To calculate s from the $S(p)$, it suffices to consider only the finite primes (that is, the prime ideals) of k.*

PROOF. Consider first the last remark. An infinite prime divisor p_∞ has $S(p_\infty)=n$ factors in K unless $n\equiv 0$ (mod 2), p_∞ is real, and its factors P_∞ in K are all complex; in this case $S(p_\infty)=n/2$. By the Tchebotareff density theorem, there then exists a prime ideal P in K with cyclic decomposition group of order 2, so that $S(p)=n/2$ for $P\mid p$, and $S(p_\infty)$ may be omitted in forming the g.c.d.

For each Q-normal algebra A, define

$$J(A)\equiv m\sum_{p}(A/P)/M(p)\ (\text{mod }1),\qquad P\mid p,$$

where P is any one selected factor of p in K. Since $ms=n$,

$$sJ(A)\equiv\sum_{p}(A/P)[n/M(p)]\equiv\sum_{p}S(p)(A/P)\equiv\sum_{P}(A/P)\equiv 0,$$

hence $J(A)$ is a rational number with denominator a factor of s.

Conversely, let t/s be a given rational number, where t is an integer. Since s is the g.c.d. of the $S(p)$, we can find a finite number of finite prime divisors $p_1, \cdots, p_r$ with integers μ_i such that

$$ts = \sum \mu_i S(p_i).$$

In particular, $\sum(\mu_i/s)S(p_i) \equiv 0 \pmod 1$; hence there exists an algebra A with invariants

$$(A/P_{ij}) = \mu_i/s \qquad \text{for each } P_{ij} \text{ with } P_{ij} \mid p_i.$$

By construction, A has equal invariants at conjugate divisors, hence is Q-normal by Theorem 2. Furthermore

$$J(A) \equiv m\sum_i \mu_i/(sM(p_i)) \equiv (m/sn)\sum_i \mu_i S(p_i)$$
$$\equiv tm/n \equiv tm/ms \equiv t/s \pmod 1.$$

Thus $J(A)$ may be any rational number with denominator s.

Therefore J is a homomorphic mapping of the group of Q-normal algebra classes onto the rationals with denominator s, mod 1. By the lemma, the kernel of this homomorphism is precisely the subgroup of trivially Q-normal algebra classes; hence the theorem.

5. **Construction of examples.** By Theorem 3, the construction of nontrivial Q-normal algebras requires only the construction of a normal algebraic number field K over some k with $s>1$. For example, it would suffice to have K with the four group as Galois group such that every prime divisor in k has at least two (and therefore either 2 or 4) prime divisors in K. In other words, one must find a K in which the factorizations $p=P^4$, $p=P^2$, and $p=P$ are impossible. In the third case, the Galois group of K over k would be the group of the local extension K_P/k_p, which is cyclic, and not the four group. There remain only the ramified cases $p=P^4$, $p=P^2$.

Consider $K=R(13^{1/2}, 17^{1/2})$, where R is the field of rationals. By considering the possible inertial groups of a ramified prime, it follows that any ramified prime must be ramified in at least two cyclic subfields, and hence that 13, 17 are the only (finite) ramified primes in K. But 13 has two distinct factors in $R(17^{1/2})$ and hence in K, while 17 has two distinct factors in $R(13^{1/2})$, and hence in K. Thus in both cases $p=(P_1P_2)^2$, and $s=2$ for this field. There thus exists one nontrivial Q-normal algebra over this field. A similar result holds for $R(2^{1/2}, 17^{1/2})$, and so on.

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DETERMINATION OF ALL CLASSES OF POSITIVE QUATERNARY QUADRATIC FORMS WHICH REPRESENT ALL (POSITIVE) INTEGERS[1]

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1. Introduction. It was apparently known to Diophantus and first proved by Lagrange [8][2] that the form $x^2+y^2+z^2+u^2$ represents all positive integers. Examples of other integral forms

$$\Phi = ax^2 + by^2 + cz^2 + du^2 \tag{1.1}$$

which represent all positive integers were first obtained by Jacobi [4], Liouville [9], and Pepin [13]. Ramanujan [14] proved that there are only 54 sets of positive integers a, b, c, d, such that (1.1) represents all positive integers. Dickson called all such forms universal [3].

Among the 54 universal forms of type (1.1) the form $x^2+2y^2+4z^2+14u^2$ has the largest determinant, 112 [1].

It is of interest to ask what would be the largest determinant of the universal form if we should allow cross product terms. If we should consider forms whose cross product terms may have odd coefficients then the value of the determinant may not exceed 861 [17], and is of that order of magnitude. If, however, we should consider only the so-called "classic" forms, that is, forms whose cross product terms have necessarily even integral coefficients, then we find the value of the largest determinant of such universal forms is, remarkably enough, again 112 [15].

The number of universal forms without cross products, whose coefficients are arranged in the order of increasing magnitude, is equal to 54 [2]. Since such forms are reduced forms we may rephrase this result to say that the number of classes of forms which are equivalent to a form without cross products is equal to 54.

Pursuing this idea further we may ask what is the number of classes of classic universal forms, or extending still farther the set of admissible forms, we may attempt to determine the number of universal classes of non-classic integral forms. In the last case this number is very large as one can see from the work of Morrow [10]. However since in the case of classic forms the upper bound for the determinant is small (112), one should expect that the number of uni-

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

versal classes of classic forms would not be large. It is of interest, therefore, to determine this number exactly.

Employing an extension of the method of Ramanujan we find:

THEOREM 1. *There are exactly* 178 *classes of universal classic positive quaternary quadratic forms* [18].[3]

2. **General theory.** It was shown in [15] that every classic universal quaternary quadratic form is equivalent to a quadratic form

$$f = x'Ax = x_1^2 + \sum_{i,j=2}^{4} a_{ij}x_ix_j = x_1^2 + \psi(x_2, x_3, x_4) \tag{2.1}$$

where the ternary form $\psi(x_2, x_3, x_4)$ is a reduced positive ternary form of determinant less than or equal to 112.

Jones' table [6] of these forms shows that there are 1046 such forms and gives the first rough upper bound for the number of classes of classic universal quaternary quadratic forms.

To obtain the exact count of the number of universal classes one must examine each one of these 1046 forms. Many of these forms are eliminated immediately by showing that they do not represent some small integer. Forms which pass these tests are scrutinized more carefully and submitted to more powerful number theoretic tests for proving universality. To prove the remaining forms universal it was necessary to use three somewhat different methods.

3. **Tests for universality: Method 1.** The first method proceeds essentially as follows. We complete the squares in the form (2.1) by multiplying f by a suitable integer n in such a way that if $F=nf$ represents all positive integral multiples of n, then f represents all positive integers.

Let S be the regular or almost regular ternary section of F which represents all positive multiples of n not of a certain form Q, Q being the arithmetic progression associated [2, 6] with S. We write $F=S+cx_4^2$. Now if it can be shown that for every integer of the form Q, S represents $Q-cx_4^2$ for a suitable value of x_4 and the given value of c, then F represents Q and therefore every positive integral multiple of n. Hence f is universal.

4. **Tests for universality: Method 2.** In case the form ψ in (2.1) is a regular ternary form [16] one may proceed very much as in the first method. Since ψ is regular it represents all positive integers not

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[3] A table of these forms may be found in a more complete version of this dissertation deposited in the American Mathematical Society Library.

of a certain form Q, Q being the arithmetic progression associated [2, 6] with ψ. Now if it can be shown that for every integer of the form Q, ψ represents $Q - x_1^2$, for a suitable value of x_1, then f represents Q and all positive integers.

5. Tests for universality: Method 3. In some forms due to the irregularity of ψ or the ternary sections of the corresponding F form, or because the representation of positive integral multiples $a \cdot n$ of n by $F = nf$ does not insure an integral representation of a by f, universality is tested by a different method.

In this method two ternary forms S_1 and S_2 are obtained by the substitution $x = Hy$ where $H = [h_{ij}]$, $i = 1, 2, 3, 4$; $j = 1, 2, 3$.

By referring to a table giving the generic characters of such ternary forms [11] or by computing the generic characters in the cases of the improperly primitive forms it is possible to determine whether or not the ternary sections S_1 and S_2 are in a genus of one class. Since such forms are regular [7], it may be concluded that S_1 and S_2 are regular.

The progressions associated with S_1 and S_2 are compared [16]. If the integers excluded by the progressions for S_1 are represented by S_2 and those excluded by the progressions associated with S_2 are represented by S_1 it follows that f represents every positive integer.

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QUADRATIC FORMS OVER FIELDS WITH A VALUATION

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1. **Introduction.** The problem of the representation of a number by a quadratic form and of the equivalence of two such forms has been solved by Hasse [2, 3, 4, 5][1] for the case where the coefficient field is the field of $\mathfrak{p}$-adic numbers. In this paper we consider the problem more generally for forms over any field with a non-archimedean valuation subject to the restriction that the field is complete with respect to the valuation and that its residue-class field has characteristic not two. A complete solution is not obtained except under certain further restrictions described below, but the general problem is shown to be reducible to the case where the forms in question have unit coefficients, and to be equivalent to the corresponding problem previously studied by the author [1] for forms over valuation rings. It is also shown that a form with unit coefficients represents zero if and only if the image form over the residue-class field represents zero, and similarly for the equivalence of two such forms.

In the latter part of the paper we obtain a complete solution for forms over certain special fields. The Hilbert norm residue symbol is introduced and conditions are given under which the Hasse function $c(f)$ is invariant. With its aid the necessary and sufficient conditions of Hasse, expressed, however, in an improved form due to Pall [7], for the representation of zero and the equivalence of two forms over a $\mathfrak{p}$-adic field are shown to apply more generally to forms over any complete field with a valuation for which the residue-class field is finite and has characteristic not two; for example, the field of formal power series over a finite field of characteristic not two. We give a new proof of the invariance of $c(f)$ which is shorter than that given by Hasse.

2. **Definitions and notations.** Let $f=\sum_1^n a_{ij}x_ix_j$ be a quadratic form whose coefficients a_{ij} are in an integral domain D. Let D' be any integral domain containing D and m any element of D. f is said to *represent* m over D' if there exist α_i in D' $(i=1, 2, \cdots, n)$, not all zero, such that $\sum a_{ij}\alpha_i\alpha_j=m$. If $m=0$, f is called a *zero form* over D'. Let $g=\sum_1^n b_{ij}y_iy_j$ be another form over D. f is said to be *equivalent* to g over D', written $f\cong g$, if there is a linear transformation $x_i=\sum_{j=1}^n p_{ij}y_j$ $(i=1, 2, \cdots, n)$, p_{ij} in D', which carries f into g

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

and such that the inverse transformation exists and has all its coefficients in D'. When explicit mention of D' is omitted it will be understood to be the same as D. The order n of f will be denoted by $n(f)$.

It is easily shown [9, Theorem 15] that if D is a field the problem of the representation of a nonzero element by a form of order n is equivalent to the problem of the representation of zero by a form of order $n+1$. In view of this and since we shall be working for the most part with fields we shall consider only the latter problem.

We shall assume that all forms used are nonsingular. When we write $f+g$ for the sum of two forms it will always be understood that f and g have no variables in common.

In this paper we shall take D to be either a field K with a non-archimedean valuation V which is complete with respect to this valuation and such that the characteristic of the residue-class field $\bar{K}$ is not two, or its valuation ring R. Thus we can assume all forms symmetric, that is, $a_{ij}=a_{ji}$ for all i and j, and each equivalent to some diagonal form [1, Theorem 1]. (For an exposition of the general theory of valuations see [6, chap. 2] or [8, chap. 10].)

Use will often be made of Witt's cancellation theorem [9, Theorem 4] which says that if f, g and h are forms over a field of characteristic not two, then $f+g\cong f+h$ implies $g\cong h$. This was extended by the author [1, Theorem 5] to forms over the valuation ring R.

The form $(x_1^2-x_2^2)+(x_3^2-x_4^2)+\cdots+(x_{k-1}^2-x_k^2)$ will be denoted by H_k. Then $H_k\cong aH_k$ over the field K for any nonzero a in K [9, p. 34], and Witt showed that every zero form f could be expressed as $f\cong f^*+H_k$ where f^* is either vacuous or a nonzero form unique to within an equivalence [9, Theorem 5]. If $f\cong H$, we shall call f a *totally zero form*. We shall frequently omit the subscript on H when there is no ambiguity about its length.

3. Forms over fields with a valuation. We shall now suppose that the domain of coefficients of our forms and transformations is the field K. $\bar{a}$ will denote the homomorphic image in $\bar{K}$ of the element a of R.

DEFINITION. If k is an even integer, by $f\cong g+H_k$ we mean $f\cong g+H_k$ if $k>0$, $f\cong g$ if $k=0$, and $g\cong f+H_{-k}$ if $k<0$.

LEMMA 1. *$f-g\cong H_{n(f)+n(g)}$ if and only if $f\cong g+H_{n(f)-n(g)}$.*

PROOF. Suppose $n(f)\geqq n(g)$. If $f-g\cong H_{n(f)+n(g)}$, then

$$f-g+g\cong f+H_{2n(g)}\cong g+H_{n(f)+n(g)}.$$

Cancelling an $H_{2n(g)}$ from each side gives us the desired result. Conversely, if $f \cong g + H_{n(f)-n(g)}$, then

$$f - g \cong g + H_{n(f)-n(g)} - g \cong H_{n(f)-n(g)} + H_{2n(g)} = H_{n(f)+n(g)}.$$

A similar argument disposes of the case $n(f) < n(g)$.

LEMMA 2. *If $f = \sum_1^n a_i x_i^2$, a_i in K, is a zero form with $\sum_1^n a_i \alpha_i^2 = 0$, which has no zero subform, then $Va_i\alpha_i^2 = Va_j\alpha_j^2$ for all i and j.*

PROOF. None of the α_i can be zero since f has no zero subform. Rearrange the terms of f so that

$$Va_1\alpha_1^2 = Va_2\alpha_2^2 = \cdots = Va_r\alpha_r^2 < Va_{r+1}\alpha_{r+1}^2 \leqq \cdots \leqq Va_n\alpha_n^2.$$

Then

$$V(a_1\alpha_1^2 + a_2\alpha_2^2 + \cdots + a_r\alpha_r^2) = V(a_{r+1}\alpha_{r+1}^2 + \cdots + a_n\alpha_n^2) > Va_1\alpha_1^2.$$

Let $b = a_2\alpha_2^2 + \cdots + a_r\alpha_r^2$ (b exists since we must have $r \geqq 2$ for f to be a zero form). Hence $V(1 + ba_1^{-1}\alpha_1^{-2}) > 0$ and $V(ba_1^{-1}\alpha_1^{-2}) = 0$. This implies that the equation

$$x^2 + \overline{ba_1^{-1}\alpha_1^{-2}} = \bar{0}$$

over $\bar{K}$ has the solutions $\pm\bar{1}$ which are distinct since the characteristic of $\bar{K}$ is different from two. By the Hensel-Rychlik theorem the equation $x^2 + ba_1^{-1}\alpha_1^{-2} = 0$ has a solution β in K and therefore $a_1(\alpha_1\beta)^2 + a_2\alpha_2^2 + \cdots + a_r\alpha_r^2 = 0$. Since f has no zero subform we must have $r = n$.

DEFINITION. Let a and b be nonzero elements of K. Then a and b are ***congruent***, written $a \equiv b$, if there is a unit u of R such that $a \cong bu$.

DEFINITION. A diagonal form $\sum_1^n a_i x_i^2$ will be called a ***unit form*** if $Va_i = 0$ for all i.

DEFINITION. A form of the type $\sum_{i=1}^t b_i f_i$, where $b_i \not\equiv b_j$ for $i \neq j$ and each f_i is a unit form, is called a *standard form.*

Every nonsingular diagonal form over K is equivalent to a standard form under transformations of the type $x_i = b_i y_i$ and a rearrangement of terms.

THEOREM 1. *A standard form $f = \sum_{i=1}^t a_i f_i$ over K is a zero form if and only if at least one of the f_i is a zero form.*

PROOF. Suppose f is a zero form. Let g be a zero subform of f which does not contain a proper zero subform. g may possibly be f itself. Write g as $g = \sum a_j g_j$, where the a_j are certain of the a_i and each g_j is a subform of some f_i, such an f_i contributing only the one subform g_j, if any. Let $g_j = \sum_k a_{jk} x_{jk}^2$, $Va_{jk} = 0$ for all j and k. Then

$\sum_j a_j(\sum_k a_{jk}\alpha_{jk}^2)=0$ for some α_{jk} in K. By Lemma 2, $V(a_j a_{j1}\alpha_{j1}^2) = V(a_m a_{m1}\alpha_{m1}^2)$ for each j and m. Therefore $a_j \equiv a_m$. But, since f is a standard form, this can happen only if $m=j$. Hence g must be a subform of some $a_i f_i$ and this f_i will be a zero form. The converse is obvious.

Corollary. *A standard form $f=\sum_{i=1}^r a_i f_i$ over K is a totally zero form if and only if each of the f_i is a totally zero form.*

Proof. By the above theorem at least one of the f_i, say f_1, is a zero form and therefore $a_1 f_1 \cong a_1 f_1' + H_2$. Since $f \cong H_{n(f)}$ we have $a_1 f_1' + H_2 + \sum_2^r a_i f_i \cong H_{n(f)}$. Cancelling an H_2 from each side leaves us $a_1 f_1' + \sum_2^r a_i f_i \cong H_{n(f)-2}$. By repeating the argument we have eventually $f_i \cong H_{n(f_i)}$ for all i.

In certain important cases, such as p-adic fields, p an odd prime, K has the property that ax^2+by^2 represents 1 whenever $Va=Vb=0$. When this is true we have the following simple criterion for the representation of zero by a given form f.

Theorem 2. *If K has the property mentioned above, then the diagonal form f represents zero if and only if it contains a binary subform ax^2+by^2 with $-ab$ a square in K or a ternary subform $ax^2+by^2+cz^2$ with $a\equiv b\equiv c$.*

Proof. By transformations of the type $x_i=d_iy_i$ and by a rearrangement of terms we can express f in a standard form $f\cong\sum_1^r b_i f_i$. If f is a zero form, then by the preceding theorem some f_i, say f_1, is a zero form. If f_1 is the binary $a'x^2+b'y^2$, we must have $-a'b'$ a square and hence if ax^2+by^2 is the corresponding binary subform of f, $-ab$ is also a square. If f_1 has order greater than two, it has a ternary subform $c_1x_1^2+c_2x_2^2+c_3x_3^2$ with $Vc_i=0$ which by our assumption on K represents zero. Since the b_1c_i have equal values the corresponding coefficients of f are congruent to each other. Conversely, if $-ab$ is a square for some binary subform ax^2+by^2 of f, this binary, and hence f, will represent zero. If the ternary subform $g=ax^2+by^2+cz^2$ has $a\equiv b\equiv c$, then $b=au_1c_1^2$ and $c=au_2c_2^2$ for some units u_1 and u_2, and hence $g\cong a(x^2+u_1y^2+u_2z^2)$. Since the ternary in the brackets represents zero so does f.

Theorem 3. *Let $f=\sum_1^r a_i f_i$ and $g=\sum_1^s b_j g_j$ be equivalent standard forms over K. If for a given a_i there is a b_j such that $a_i\equiv b_j$, then this b_j is unique and $f_i\cong u_j g_j+H$, where u_j is the unit defined by $b_j=a_i c_j^2 u_j$. If there is no such b_j, then $f_i\cong H$, and similarly if, for a given b_k, $b_k\not\equiv a_l$ for all l, then $g_k\cong H$.*

Proof. If there is a b_j such that $a_i\equiv b_j$, it must be unique since the relation of congruence is transitive. Rearrange the terms of f and g so that $a_i = b_i$ $(i = 1, 2, \cdots, t)$, $a_i \neq b_j$ $(i, j > t)$. Then, using Lemma 1,

$$H \cong f - g \cong \sum_1^t a_i(f_i - c_i^2 u_i g_i) + \sum_{t+1}^r a_i f_i + \sum_{t+1}^s b_j g_j.$$

By the corollary to Theorem 1, $f_i - c_i^2 u_i g_i \cong H$ $(i = 1, 2, \cdots, t)$, $f_i \cong H$ $(i = t+1, \cdots, r)$, $g_j \cong H$ $(j = t+1, \cdots, s)$. By Lemma 1 $f_i \cong c_i^2 u_i g_i + H \cong u_i g_i + H$ $(i = 1, 2, \cdots, t)$.

As a converse of this theorem we have the following theorem.

THEOREM 4. *Let $f = \sum_1^r a_i f_i$ and $g = \sum_1^s b_j g_j$ be standard forms over K. f is equivalent to g if*

(i) $\sum_1^r n(f_i) = \sum_1^s n(g_j)$,

(ii) *$a_i = b_j$ implies $f_i \cong u_j g_j + H_{n(f_i) - n(g_j)}$, where u_j is the unit defined by $b_j = a_i c_j^2 u_j$,*

(iii) *for a given a_i, $a_i \neq b_j$ for all j implies $f_i \cong H_{n(f_i)}$, and for a given b_k, $b_k \neq a_l$ for all l implies $g_k \cong H_{n(g_k)}$.*

PROOF. Rearrange the terms of f and g so that $a_i = b_i$ $(i = 1, 2, \cdots, t)$, $a_i \neq b_j$ $(i, j > t)$. Then

$$f_i \cong u_i g_i + H_{n(f_i) - n(g_i)} \qquad (i = 1, 2, \cdots, t),$$

$$f_i \cong H_{n(f_i)} \quad \text{and} \quad g_j \cong H_{n(g_j)} \qquad (i, j > t).$$

$$\begin{aligned}
\sum_1^r a_i f_i &\cong \sum_1^t (a_i u_i g_i + a_i H_{n(f_i) - n(g_i)}) + \sum_{t+1}^r a_i H_{n(f_i)} \\
&\cong \sum_1^t b_i g_i + \sum_1^r a_i H_{n(f_i)} + \sum_1^t a_i H_{-n(g_i)} \\
&\cong \sum_1^t b_i g_i + \sum_1^s b_j H_{n(g_j)} + \sum_1^t b_j H_{-n(g_j)} \\
&\cong \sum_1^t b_i g_i + \sum_{t+1}^s b_j H_{n(g_j)} \\
&\cong \sum_1^t b_i g_i + \sum_{t+1}^s b_j g_j \cong \sum_1^s b_j g_j.
\end{aligned}$$

For fields K in which it is possible to tell whether or not any two given elements are congruent, Theorems 1, 3 and 4 reduce the problem of the representation of a given element in K by a given form and of the equivalence of two forms to the case of unit forms. In an earlier paper [1, Theorem 4] we performed a similar reduction for forms over valuation rings. We shall now show that the two problems are equivalent by proving (Theorem 5) that two unit forms are equivalent over the field K if and only if they are equivalent over the valuation ring R.

First we extend to forms over valuation rings some of the elementary properties of forms over general fields.

If f and g are forms over R, then $f \simeq g$ will mean that f is equivalent to g over R, while $f \cong g$ will stand for equivalence over K.

LEMMA 3. *If $f = \sum_1^n a_i x_i^2$, $Va_i = Va_j \geqq 0$ (all i, j), represents zero over K, and b is any element of R for which $Vb = Va_1$, then $f \simeq by_1^2 - by_2^2 + \phi(y_3, y_4, \cdots, y_n)$, where ϕ is a quadratic form over R. Hence f represents b over R.*

PROOF. Let $\sum a_i \alpha_i^2 = 0$, α_i in K. We can assume that $V\alpha_i \geqq 0$ for all i and that some α_i, say α_1, has $V\alpha_1 = 0$. Since f is a zero form, some other α, say α_2, must also have $V\alpha_2 = 0$. The transformation over R

$$x_1 = \alpha_1 z_1, \qquad x_i = \alpha_i z_1 + z_i \qquad (i = 2, 3, \cdots, n)$$

is unimodular and carries f into

$$2z_1 \sum_{i=2}^{n} a_i \alpha_i z_i + \sum_{i=2}^{n} a_i z_i^2. \tag{1}$$

Next apply the unimodular transformation

$$w_1 = z_1, \qquad w_2 = b^{-1} \sum_{j=2}^{n} a_j \alpha_j z_j, \qquad w_i = z_i \qquad (i = 3, 4, \cdots, n)$$

which takes (1) into

$$bw_2\left(2w_1 + \frac{b}{a_2\alpha_2^2} w_2 - \frac{2}{a_2\alpha_2^2} \sum_{j=3}^{n} a_j \alpha_j w_j\right) + \phi(w_3, w_4, \cdots, w_n), \tag{2}$$

where ϕ is a form over R. The unimodular transformation

$$v_1 = 2w_1 + \frac{b}{a_2\alpha_2^2} w_2 - \frac{2}{a_2\alpha_2^2} \sum_{j=3}^{n} a_j \alpha_j w_j, \qquad v_i = w_i \qquad (i = 2, 3, \cdots, n)$$

carries (2) into

$$bv_1v_2 + \phi(v_3, v_4, \cdots, v_n). \tag{3}$$

Finally the unimodular transformation

$$v_1 = y_1 + y_2, \qquad v_2 = y_1 - y_2, \qquad v_i = y_i \qquad (i = 3, 4, \cdots, n)$$

will take (3) into $by_1^2 - by_2^2 + \phi(y_3, y_4, \cdots, y_n)$.

LEMMA 4. *If $f=\sum_1^n a_i x_i^2$, $Va_i = Va_j \geqq 0$ (all i, j), represents over K any b for which $Vb = Va_1$, then f represents b over R and $f \simeq by_1^2 + \psi(y_2, y_3, \cdots, y_n)$, where ψ is a quadratic form over R.*

PROOF. Let $\sum a_i \alpha_i^2 = b$, α_i in K. Then $\sum a_i x_i^2 - bz^2$ is a zero form over K, and by Lemma 3

$$\sum_{i=1}^n a_i x_i^2 - bz^2 \simeq by_1^2 - bz^2 + \psi(y_2, y_3, \cdots, y_n).$$

By Witt's theorem for forms over a valuation ring we can cancel $-bz^2$ from each side giving us $f \simeq by_1^2 + \psi(y_2, y_3, \cdots, y_n)$.

THEOREM 5. *Two unit forms are equivalent over K if and only if they are equivalent over R.*

PROOF. The condition is obviously sufficient. Let the two forms be

$$f = \sum_{i=1}^n a_i x_i^2 \quad \text{and} \quad g = \sum_{i=1}^n b_i x_i^2, \qquad Va_i = Vb_i = 0 \text{ for all } i,$$

and assume $f \cong g$. We shall use induction on the order n of f. The theorem is true for unary forms. Suppose that it is true for forms of order $n-1$. Since f represents a_1 over R, g represents a_1 over K and by Lemma 4

$$g \simeq a_1 x_1^2 + \sum_{i=2}^n c_i x_i^2$$

for some c_i. Since the values of corresponding terms are invariant under a transformation over R [1, Lemma 2], $Vc_i = 0$ for all i. Applying Witt's cancellation theorem for forms over a field we have

$$\sum_{i=2}^n a_i x_i^2 \cong \sum_{i=2}^n c_i x_i^2$$

which with the induction hypothesis gives us

$$\sum_{i=2}^n a_i x_i^2 \simeq \sum_{i=2}^n c_i x_i^2.$$

From this it follows that $f \simeq g$.

We shall now show how the equivalence of two unit forms and the representation of zero by a unit form are connected with corresponding problems for related forms over the residue-class field.

If f is a form $\sum_1^n a_i x_i^2$ over R, then $\bar{f}$ will stand for the form $\sum_1^n \bar{a}_i x_i^2$ over the residue-class field $\bar{K}$.

THEOREM 6. *If f and g are unit forms, then $f \cong g$ over K if and only if $\bar{f} \cong \bar{g}$ over $\bar{K}$.*

A proof of this theorem is given in [1, Lemma 1 and Theorem 2].

THEOREM 7. *A unit form f is a zero form over K if and only if $\bar{f}$ is a zero form over $\bar{K}$.*

PROOF. Let $f = \sum_1^n a_i x_i^2$, $Va_i = 0$, for all i. If f is a zero form over K, then there are α_i in K ($i = 1, 2, \cdots, n$), not all zero, such that $\sum a_i \alpha_i^2 = 0$. We can assume that $V\alpha_i \geqq 0$ for all i and that some α_i, say α_1, has $V\alpha_1 = 0$. Then $\sum \bar{a}_i \bar{\alpha}_i^2 = \bar{0}$ with $\bar{\alpha}_1 \neq \bar{0}$.

Conversely, suppose that $\bar{f}$ is a zero form over $\bar{K}$. Then there are $\bar{\alpha}_i$ in $\bar{K}$ ($i = 1, 2, \cdots, n$), not all equal to $\bar{0}$, such that $\sum \bar{a}_i \bar{\alpha}_i^2 = \bar{0}$. Suppose that $V\alpha_i = 0$ ($i = 1, 2, \cdots, r$) and $V\alpha_i > 0$ for $i > r$, where α_i is an antecedent in R of $\bar{\alpha}_i$. Since not all the $\bar{\alpha}_i$ are zero, $r \geqq 2$. Therefore $V(a_1\alpha_1^2 + a_2\alpha_2^2 + \cdots + a_r\alpha_r^2) > Va_1\alpha_1^2$, and as in the proof of Lemma 2 there is a nonzero β in K such that $a_1(\alpha_1\beta)^2 + a_2\alpha_2^2 + \cdots + a_r\alpha_r^2 = 0$. Thus f is a zero form over K.

COROLLARY. *If the unit form f represents over K any m in K for which $Vm > \min V\alpha_i^2$, where $\sum a_i \alpha_i^2 = m$, then f is a zero form over K.*

PROOF. Let $V\alpha_1 = \min V\alpha_i$. Then $\sum a_i(\alpha_1^{-1}\alpha_i)^2 = \alpha_1^{-2}m$. $V(\alpha_1^{-2}m) > 0$ and hence $\sum \bar{a}_i x_i^2$ is a zero form over $\bar{K}$.

4. **The Hasse function.** The problem of determining when a form represents zero and when two forms are equivalent was solved for the $\mathfrak{p}$-adic case by Hasse. He made extensive use of the Hilbert norm residue symbol. We shall show that his results do not depend on having a $\mathfrak{p}$-adic field for a base but can be extended to any complete field with a discrete valuation whose residue-class field has characteristic not two, and having the property that the product of any two non-square units is a square and that $ax^2 + by^2 = 1$ has a solution whenever a and b are both units of R. An example of such a field is the one obtained by completing with respect to any one of its valuations the field of rational functions over a finite field of characteristic not two.

DEFINITION. If $ax^2 + by^2$ is a form over K with a and b not zero, we define the function (a, b) to have the value 1 or -1 according as the form does or does not represent 1 over K.

It is obvious that $(a, b) = (b, a)$, $(a, -a) = 1$ and that $(a, b) = 1$ if a or b is the square of an element in K.

LEMMA 5. *Let a and b be non-squares. Then*

(i) *$a \not\sim b$ implies* $(a, b) = -1$,

(ii) *$a \not\sim 1$, $b \not\sim 1$ implies* $(a, b) = 1$ *if and only if* $-ab$ *is a square.*

PROOF. Let $f = ax^2 + by^2 - z^2$. If $a \not\sim b$, the only possible standard forms of f are $(ax^2 - z^2) + b(y^2)$, $(by^2 - z^2) + a(x^2)$ or $a(x^2) + b(y^2) + (-z^2)$. By Theorem 1 if f is a zero form, at least one of the bracketed subforms must be a zero form. But this is impossible for each of the three possibilities. Hence $(a, b) = -1$.

If $a \not\sim 1$, and $b \not\sim 1$, the only possible standard forms of f are $a(x^2 + ba^{-1}y^2) + (-z^2)$ or $a(x^2) + b(y^2) + (-z^2)$. If $-ab$ is a square then f is a zero form. Conversely, if f is a zero form, then the first of the above possibilities for a standard form is the correct one and, by Theorem 1, $x^2 + ba^{-1}y^2$ must be a zero form, implying that $-ab$ is a square.

From now on we shall assume that the valuation of K is discrete with the integers as the value group. t will denote an element whose value is one.

LEMMA 6. *The product rule* $(a, b)(a, c) = (a, bc)$ *holds for all nonzero a, b and c in K if and only if the product of every two non-square units is a square and* $Va = Vb = 0$ *implies* $(a, b) = 1$.

PROOF. Suppose that, for all nonzero a, b and c, $(a, b)(a, c) = (a, bc)$. If a or b is a square, $(a, b) = 1$. Hence let a and b be two non-square units. Chose t such that $Vt = 1$. Since $Va = Vb = 0$, $t \not\sim a$, $t \not\sim b$, $ta \not\sim b$ and, by Lemma 5, $(t, a) = (t, b) = (ta, b) = -1$. The product rule gives us $(t, b)(ta, b) = (a, b) = 1$. Also $(t, a)(t, b) = (t, ab) = 1$, and since $t \not\sim ab$ we have by Lemma 5 that ab is a square.

Conversely, suppose that the product of two non-square units is a square and that $Va = Vb = 0$ implies $(a, b) = 1$. If $(x_1, y_1, 1)$ is a solution of $ax^2 + by^2 - z^2 = 0$ and $(x_2, y_2, 1)$ a solution of $ax^2 + cy^2 - z^2 = 0$, then $(x_1 - x_2, y_1y_2, 1 - ax_1x_2)$ is a solution of $ax^2 + bcy^2 - z^2 = 0$. It is well known [9, p. 39] that if a form represents zero with some zero terms in the solution, then there is a solution in which none of the terms is zero. Thus $(a, b) = (a, c) = 1$ implies $(a, bc) = 1$. Suppose that $(a, b) = 1$ and $(a, c) = -1$. If $(a, bc) = 1$, then by the case just discussed, $(a, bc)(a, b) = (a, c) = 1$, a contradiction. Hence $(a, bc) = -1$. Similarly $(a, b) = -1$ and $(a, c) = 1$ imply $(c, bc) = -1$. Suppose finally that $(a, b) = (a, c) = -1$. Then a, b and c are non-squares. In view of our assumption that $Va = Vb = 0$ implies $(a, b) = 1$ we need consider only the following cases:

Case 1. $Va = 0$, $Vb = Vc = 1$.

Let $b = tv$, $c = tw$, where $Vt = 1$. Then $Vv = Vw = 0$ and $(a, bc) = (a, vw) = 1$.

Case 2. $Va=1$, $Vb=Vc=0$.

bc is a square and $(a, bc)=1$.

Case 3. $Va=Vb=1$, $Vc=0$.

Let $a=tu$, $b=tv$. Then $Vu=Vv=0$. $(a, b)=-1$ implies by Lemma 5 that $-ab$ and hence $-uv$ is a non-square. Therefore $-uvc$ and $-abc$ are squares and by Lemma 5 again $(a, bc)=1$.

The case $Va=1$, $Vb=0$, $Vc=1$ is treated similarly.

Case 4. $Va=Vb=Vc=1$.

Let $a=tu$, $b=tv$, $c=tw$. Then $Vu=Vv=Vw=0$. As in the preceding case $(a, b)=(a, c)=-1$ implies that $-uv$ and $-uw$ are non-squares. Hence vw and bc are squares, and $(a, bc)=1$.

This establishes the product rule.

THEOREM 8. *If $\bar{K}$ is a finite field with p^m elements, p an odd prime, then the product rule $(a, b)(a, c)=(a, bc)$ holds in K for all nonzero a, b and c.*

PROOF. The nonzero elements of $\bar{K}$ form under multiplication a cyclic group of order p^m-1. If $\bar{\eta}$ is a generator of this group, the p^m elements of $\bar{K}$ can be listed as $0, \bar{\eta}, \bar{\eta}^2, \cdots, \bar{\eta}^{p^m-1}=1$. If an odd power of $\bar{\eta}$ were a square, then $\bar{\eta}^{1/2}$ would exist in $\bar{K}$ and we should have $\bar{\eta}^{1/2}=\bar{\eta}^h$ for some h, $1\leqq h\leqq p^m-1$. This implies that $\bar{\eta}=\bar{\eta}^{2h}$ and hence that $2h\equiv 1 \pmod{p^m-1}$, an impossibility, since p^m-1 is even. Thus $\bar{K}$ has exactly $1+(p^m-1)/2$ squares and the product of two non-squares of $\bar{K}$ is a square. Using Theorem 7 we see that the product of two non-square units of K will be a square. Suppose now that $Va=Vb=0$, and consider $\bar{a}x^2+\bar{b}y^2=\bar{1}$. x^2, and hence $\bar{a}x^2-\bar{1}$, takes on $1+(p^m-1)/2$ different values as x runs through all values of $\bar{K}$. Similarly so does $-\bar{b}y^2$. Two of these must be equal if we are not to have p^m+1 different elements in $\bar{K}$. Therefore $\bar{a}x^2+\bar{b}y^2=\bar{1}$ has a solution in $\bar{K}$ which obviously is not $(\bar{0}, \bar{0})$. Hence by Theorem 7 $ax^2+by^2=1$ has a solution in K. From Lemma 6 it follows that the product rule is valid in K.

We shall now assume that not only is the valuation of K discrete but also that the product rule holds in K. Under these two assumptions it is possible to obtain a complete set of criteria for the representation of zero by a given form and for the equivalence of two forms in terms of the Hasse invariants. The conditions are the same as those for forms over the p-adic numbers as given first by Hasse [2 and 3, Theorem 2] and recently in a more convenient form by Pall [7].

For any nonsingular diagonal form $f=\sum_1^n a_i x_i^2$ we define the function, with the values ± 1,

$$c(f) = \prod_{i=1}^{n} (-d_{i-1}, d_i),$$

where $d_i = a_1 a_2 \cdots a_i$ if $i \geqq 1$ and $d_0 = 1$. This function was introduced by Hasse, his definition, though different, being equivalent to ours. It is easily shown that if f and g are diagonal forms with no variables in common that

$$c(f+g) = c(f)c(g)(|f|, |g|), \tag{4}$$

where $|f|$ is the determinant of f, and for any nonzero m

$$c(mf) = (m, (-1)^{n(n+1)/2}|f|^{n+1})c(f). \tag{5}$$

THEOREM 9. *If f and g are equivalent diagonal forms, then $c(f) = c(g)$.*

PROOF. Replacing any coefficient a in f by ac^2 for any nonzero c obviously does not change $c(f)$. If we interchange two adjacent terms of f, say the jth and $(j+1)$th, the only change in the expression for $c(f)$ will be to replace $(-d_{j-1}, d_j)$ $(-d_j, d_{j+1})$ by $(-d_{j-1}, D_j)$ $\cdot(-D_j, d_{j+1})$, where $D_j = a_1 a_2 \cdots a_{j-1} a_{j+1}$. If we use the relation $D_j = d_j a_{j+1} a_j^{-1}$, it is easy to show that these two products are equal and hence that $c(f)$ is unchanged. By transformations of these two simple types we can express f and g in their standard forms: $f \cong f_1 + tf_2$, $g \cong g_1 + tg_2$, where $V(t) = 1$ and the forms f_i and g_i have unit coefficients, and have $c(f) = c(f_1 + tf_2)$, $c(g) = c(g_1 + tg_2)$. If f_1, f_2, g_1, g_2 all exist, then $c(f_i) = c(g_i) = 1$ $(i = 1, 2)$ and by (4) and (5)

$$\begin{aligned} c(f) = c(f_1 + tf_2) &= c(tf_2)(|f_1|, |tf_2|) \\ &= (t, (-1)^{r(r+1)/2}|f_2|^{r+1})(|f_1|, t^r|f_2|) \\ &= (t, (-1)^{r(r+1)/2}|f_2|^{r+1})(|f_1|, t^r), \\ c(g) = c(g_1 + tg_2) &= (t, (-1)^{s(s+1)/2}|g_2|^{s+1})(|g_1|, t^s), \end{aligned}$$

where r and s are the respective orders of f_2 and g_2. By Theorem 3, assuming without loss in generality that $r \leqq s$, we have $f_1 \cong g_1 + H_{s-r}$, $g_2 \cong f_2 + H_{s-r}$ (since the valuation is discrete we can take $u_j = 1$ in Theorem 3). Let $h = |H_{s-r}| = (-1)^{(s-r)/2}$. Then $|g_1| \cong |f_1|h$, $|g_2| \cong |f_2|h$ and

$$c(g) = (t, (-1)^{s(s+1)/2}|f_2|^{s+1}h^{s+1})(|f_1|h, t^r),$$

since $s - r$ is even. If $r = s$, this will still be correct, though H does not exist, provided we take $h = 1$. $c(f)c(g) = (t, (-1)^{\nu})(h, t^r)$ where

$$\nu = [r(r+1) + s(s+1) + (s-r)(s+1)]/2 \equiv r(r-s)/2 \pmod 2.$$

If r, and hence s, is even, then so is ν and $c(f)c(g)=1$. If r is odd,

$$c(f)c(g) = (t, (-1)^{\nu+(s-r)/2})$$

which is unity since $\nu+(s-r)/2$ is even. If any of f_1, f_2, g_1 or g_2 are non-existent, a similar argument can be used to show that in these cases too $c(f)=c(g)$.

THEOREM 10. *A form of f order n is a zero form if and only if for*

$n = 2$, $-|f|$ *is a square,*
$n = 3$, $c(f) = 1$,
$n = 4$, $c(f) = 1$ *or* $|f|$ *is not a square,*
$n \geq 5$, *always.*

PROOF. $n=2$: Let $f=a_1x_1^2+a_2x_2^2$. If f is a zero form, then $(x_1x_2^{-1})^2 = -a_1^{-1}a_2$ and $-|f|$ is a square. Conversely, if $-|f|$ is a square, then $x_1=(-a_1a_2)^{1/2}$, $x_2=a_1$ is a zero of f.

$n=3$: Let $f=a_1x_1^2+a_2x_2^2+a_3x_3^2$. A short computation shows that $c(f)=(-a_1a_2, -a_1a_3)$. If f is a zero form, let ξ, η, ζ be a solution of $f=0$. We can assume $\xi\neq 0$. Then $x=\eta(a_1\xi)^{-1}$, $y=\zeta(a_1\xi)^{-1}$ is a solution of

$$-a_1a_2x^2 - a_1a_3y^2 = 1. \tag{6}$$

Conversely, if $c(f)=1$, then (6) has a solution and hence so has $a_2x_2^2+a_3x_3^2=-a_1x_1^2$.

$n=4$: Let $f=a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2$. We can assume $Va_i=0$ or 1. Suppose f is a zero form. If $Va_i=0$ for all i, then $c(f)=1$. If, for some i; $Va_i\neq 0$, then by Theorem 1 either some binary or ternary subform with coefficients of like value must represent zero. In the former case let $f=f_1+f_2$. If both the binary subforms f_1 and f_2 represent zero, then $f\cong H_4$ and $c(f)=1$. If f_1 is the only binary representing zero, then $-|f_1|$ is a square while $-|f_2|$ is not a square. Hence $|f|$ is not a square. Returning to the other possibility let g be the ternary subform representing zero. By the preceding case for $n=3$, $c(g)=1$. If $f=ax^2+g$, then, by (4), $c(f)=(a, |f|)c(g)=(a, |f|)$. If $c(f)\neq 1$, then $|f|$ cannot be a square. Conversely, suppose $c(f)=1$ or $|f|$ is not a square. We can assume that f has a standard form $f\cong f_1+tf_2$, where $f_1=b_1x_1^2+b_2x_2^2$ and $f_2=b_3x_3^2+b_4x_4^2$, $Vb_i=0$ for all i, since any other possible standard form for f would imply that f had a ternary subform with coefficients of like value which would by Theorem 2 represent zero. Using (4) we have $c(f)=(t, -b_3b_4)$. If $c(f)=1$, then, by Lemma 5, $-b_3b_4$ is a square, and so f_2 and hence f is a zero form.

$|f| \cong (-b_1b_2)(-b_3b_4)$. If $|f|$ is not a square, then either $-b_1b_2$ or $-b_3b_4$ must be a square implying again that f is a zero form.

$n \geqq 5$: Since f must contain a ternary subform with coefficients of like value, it is a zero form by Theorem 2.

THEOREM 11. *Two nonsingular forms f and g of order n are equivalent if and only if $|f| \cong |g|$ and $c(f) = c(g)$.*

PROOF. We have already shown the necessity of these conditions. Suppose now that $|f| \cong |g|$ and $c(f) = c(g)$. We use induction on n. The theorem is true for $n = 1$. Suppose it true for forms of order $n-1$. Write f as $f = ax^2 + f'$, where a is the leading coefficient of f and f' is a form of order $n-1$. Using (4) we have

$$c(f - ax^2) = (-a, -|f|)c(f) = c(g - ax^2).$$

Since $f - ax^2$ represents zero, we have by Theorem 10 that so does $g - ax^2$ for any n. g then represents a and can be written $g \cong ax^2 + g'$, where g' is a form of order $n-1$. By (4)

$$c(f) = (-1, a)c(f')(a, |f'|), \qquad c(g) = (-1, a)c(g')(a, |g'|).$$

Therefore $c(f') = c(g')$ since $|f'| \cong |g'|$. By the induction hypothesis $f' \cong g'$ and hence $f \cong g$.

Since the product of two non-square units is a square, every unit of K is equivalent either to 1 or to some arbitrary but fixed non-square unit ν. Also $\nu x^2 + \nu y^2$, since it represents 1, is equivalent by Lemma 4 to $x^2 + y^2$. Thus every form f of order n is equivalent to

$$(7) \quad x_1^2 + x_2^2 + \cdots + x_{r-1}^2 + ax_r^2 + tx_{r+1}^2 + tx_{r+2}^2 + \cdots + tx_{n-1}^2 + tbx_n^2,$$

where a and b are 1 or ν.

THEOREM 12. *Expression (7) is a canonical form for f.*

PROOF. We need to show only that two different expressions of form (7) are inequivalent. Let

$$y_1^2 + y_2^2 + \cdots + y_{r-1}^2 + a'y_r^2 + ty_{r+1}^2 + ty_{r+2}^2 + \cdots + ty_{n-1}^2 + tb'y_n^2$$

be another form of this type. If they were equivalent, we could cancel like terms and get

$$ax_r^2 + tbx_n^2 \cong a'y_r^2 + tb'y_n^2.$$

Theorem 3 now implies that $a = a'$ and $b = b'$.

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RELATIONS BETWEEN HYPERSURFACE CROSS RATIOS, AND A COMBINATORIAL FORMULA FOR PARTITIONS OF A POLYGON, FOR PERMANENT PREPONDERANCE, AND FOR NON-ASSOCIATIVE PRODUCTS

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This note improves, in two respects, the results of §3.6 of my paper *The hypersurface cross ratio.*[1] There it is shown that the number c_n of independent hypersurface cross ratios that can be formed of $2n$ forms in n variables is 2 for $n=2$, 5 for $n=3$, and 14 for $n=4$. The proof employs the relations between cross ratios obtained by some simple permutations of the forms; let R be the set of these relations. It is remarked that the cross ratios of $2n-1$ forms in n variables, and of $2n-1$ forms in $n-1$ variables, are connected by the same relations as the cross ratios of $2n$ forms in n variables, as far as these are consequences of the relations R, a "perhaps void restriction." We now prove that $c_n=C_{2n,n}/(n+1)$, and that the restriction is in fact void, so that a complete knowledge of the relations between the cross ratios of $2n-1$ forms, of $2n$ forms, and of $2n+1$ forms in n variables is obtained.[2] The corresponding theorems for generalized intersections and one more variable are established at the same time.

The same facts hold for a general class of function ratios, which includes hypersurface cross ratios and generalized intersections as very special cases. The number c_n of independent function ratios has a simple combinatorial meaning, and appears also as the number of partitions of a polygon by non-intersecting diagonals into triangles, or of a cyclically arranged set into non-interlaced subsets, as the number of possibilities of never losing majority (in an election or a game[3]), and as the number of different products of given terms in a given order, in a non-associative multiplication. For the combinatorial formula, seven proofs are given, six extended to generalizations.[4]

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[1] Bull. Amer. Math. Soc. vol. 51 (1945) pp. 976–984.

[2] For forms of a sufficiently high degree. Cf., on the other hand, for 5, 5 and 6 linear forms in 2, 3 and 3 variables respectively, §§3, 4, 5 of *The pentagon in the projective plane, with a comment on Napier's rule*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 985–989.

[3] Or for drops falling on a board one-half of which is supported, and similar physical schemes.

[4] For an eighth proof cf. P. Erdős and I. Kaplansky, *Sequences of plus and minus*, Scripta Mathematica vol. 12 (1946) pp. 73–75 (for $[f(n, n)]^2$, read $f(n, n)f(n+1, n+1)$, or permit only diagonal moves; in (4), read $m \leq n$). I have made use of oral remarks by A. Dvoretzky (in 2.3–2.5) and E. Jabotinsky (in 1.1).

1. **The main theorem.** 1.1. Let F be a function of n arguments (variables, points, polymonials) that is (1) almost symmetric (F becomes F or $-F$ for every permutation of the arguments), and (2) such that the $C_{2n,n}$ functions of any n from among $2n$ arguments $a_{11}, a_{12}, a_{21}, a_{22}, \cdots, a_{n1}, a_{n2}$ are independent. We then define the *function ratio* of the n pairs of arguments, in the given order, as $G=\prod F(a_{1r_1}, \cdots, a_{nr_n})^\rho$, where $\rho=(-1)^{\sum r_i}$, and the product extends over the 2^n possible systems $r_1, \cdots, r_n$. We shall prove:

The number of independent function ratios that can be formed of $2n$ arguments is $C_{2n,n}/(n+1)$.

Obviously, the relations between the function ratios do not depend on the particular function, except possibly for the influence of the factor -1 affixed to F and hence to G.

1.2. Further, the same number and relations are obtained in two more cases: one of the arguments is allowed to be a symbol p or q, defined as follows. For a symbol p, every F involving p is ± 1, and independence is only supposed to hold for the $C_{2n-1,n}$ functions of any n from among the remaining $2n-1$ arguments. For a symbol q, again every F involving q is ± 1, while in every other factor F of a function ratio, the argument paired with q in the definition of that function ratio is omitted (and possibly replaced by a factor -1); here F is a function of $n-1$ arguments, and the $C_{2n-1,n-1}$ functions of any $n-1$ from among the $2n-1$ non-symbolic arguments are supposed to be independent.

1.3. In particular, the function F may be the resultant of n forms in n variables, or the intersection of n general hypersurfaces in the projective n-space over an algebraically closed field, for forms and hypersurfaces of a sufficiently high degree (if a symbol q occurs, replace every n by $n-1$). That in these cases the resultants (and the intersections, considered as resultants with an additional linear form) are independent, will be shown separately.[5]

1.4. We consider the influence on G of three kinds of simple permutations of the arguments. (1) If two arguments of the same pair are interchanged, then G becomes $1/G$. (2) If two pairs are interchanged, then G becomes $\pm G$. By (1) and (2) there belongs, to every division of the arguments in pairs, essentially one function ratio. (3) If two arguments of one pair and a third argument are permuted cyclically, then the product of the three function ratios obtained is easily seen to be ± 1, even if one of the arguments is a symbol p or q. It follows that from among the ratios belonging to divisions with $n-2$

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[5] In the subsequent note *Independence of resultants*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 360–365.

fixed pairs, any one is determined by the two others.

1.5. Let the arguments be represented by $2n$ points on a circle. A division D in pairs is then symbolized by n chords. It has a number c of crossings (pairs of chords intersecting within the circle). Choose two chords P and Q forming a crossing and express the function ratio of D by the function ratios of the other divisions D' and D'' with the same remaining $n-2$ chords. A chord of D that crosses P and Q crosses either both new chords P' and Q' of D' or neither of them; a chord that crosses just one of P and Q crosses exactly one of P' and Q', and a chord that crosses neither P nor Q crosses neither P' nor Q'. Hence the function ratio of D can be expressed by the function ratios of divisions with numbers c' and c'' of crossings, less (by an odd number) than c, and ultimately by the function ratios of divisions with no crossing. It remains to be shown that the number of these divisions is $C_{2n,n}/(n+1)$, and that their function ratios are independent.

1.6. The independence of the function ratios belonging to divisions with $c=0$ follows from the independence of the functions F, if we show that these function ratios can be so arranged that each involves a function F that occurs in none of the preceding function ratios.

Let $1, \cdots, 2n$ be the points in their order on the circle. Arrange the sets of n numbers, and the corresponding functions F, lexicographically. Among the functions forming a given function ratio there is a last one involving the greater number of every pair. Now arrange the function ratios with $c=0$ according to their last function; it then remains to verify that each of them is determined by its last function.

For a given F, let k be the first point involved. In a division whose last function is F, k must be paired with a smaller number, and $k-1$ with a greater number. Hence if $c=0$, then $k-1$ and k must be paired. Omit both, and repeat the procedure.

A symbol p or q should be denoted by 1. The point paired with q remains till the end of the procedure.

1.7. *Of the* $N=(2n)!/n!2^n$ *divisions of* $2n$ *points on a circle into* n *pairs,* $N\cdot 2^n/(n+1)!=C_{2n,n}/(n+1)$ *are without crossing.*

For let the number of divisions without crossings be c_n. We wish to express c_{n+1} by $c_1, \cdots, c_n$. Let A be one of the $2n+2$ points. The point A must be paired with a point B for which the number of points on either arc AB is even. If the arcs bear $2j$ and $2k$ points, besides A and B, then $j+k=n$, and there are c_jc_k divisions without crossings containing the pair AB (put $c_0=c_1=1$). Hence we have the recursion formula $c_{n+1}=\sum c_jc_k$. It follows that the series[4] $y=\sum c_nx^n$ satisfies the equation $y^2=(y-1)/x$, whence $2xy=1-(1-4x)^{1/2}=1-\sum(1/2)\cdot(-1/2)\cdot\cdots\cdot(3/2-n)(-4x)^n/n!=2\sum C_{2n-2,n-1}x^n/n$, so that $c_n=C_{2n,n}/(n+1)=C_{2n+1,n}/(2n+1)$. For large values of n we have $c_n\sim 4^n n^{-3/2}\pi^{-1/2}$.

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[4] The series is convergent until $x'=0$, that is, for $|x|<1/4=\lim c_n/c_{n+1}$ (in 2.1 for $|x|<(l-1)^{l-1}/l^l$), but we need it only for formal operations.

1.8. The same recursion formula holds also for the number c_n of partitions of a convex polygon of $n+2$ sides, by non-intersecting diagonals, into triangles. Indeed, a fixed side of a polygon of $n+3$ sides belongs, in a partition as required, to a certain triangle; after choosing the triangle there remain two polygons of $j+2$ and $k+2$ sides, with $j+k=n$, to be divided in the same manner. Here $j=0,\cdots,n$, $c_0=1$. Hence, the value of c_n is again $C_{2n,n}/(n+1)$.[7]

If, for a regular polygon, congruent or symmetric partitions are considered as equal, then the numbers 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, $\cdots$ are reduced to 1, 1, 1, 1, 3, 4, 12, 27, 82, 228, 783, 2282, $\cdots$, which are $\sim c_n/2n$.

There is no natural one-to-one correspondence between the partitions of a polygon and the divisions with no crossings. For the latter, the numbers of types of congruent or symmetric divisions begin with 1, 1, 1, 2, 3, 6, 12, 25, $\cdots$ and are $\sim c_n/4n$. Congruence and symmetry are defined for an equally divided circle. The circle itself is determined by the set of all partitions with no crossing. Indeed, for every pair on the circumference there are $(n-1)^2$ pairs that appear together with the given pair in at least one partition, while for a pair that separates $2j$ from $2k$ points, $j+k=n-1$, the corresponding number is $j^2+k^2<(n-1)^2$.

1.9. The recursion formula holds also for the number c_n of different products of $n+1$ factors, in a given order, in a non-associative multiplication: every product of $n+2$ terms is obtained by multiplying a product of $j+1$ and a product of $k+1$ factors, with $j+k=n$, $j=0,\cdots,n$, $c_0=1$. Hence again $c_n=C_{2n,n}/(n+1)$.[8]

2. **Generalizations.** 2.1. To determine the number c_n of divisions

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[7] This and some more general results, including the formula for c_n in 2.2, were obtained in a series of papers from Euler until Cayley, *On the partitions of a polygon*, Proc. London Math. Soc. vol. 22 (1891) pp. 237–262; Collected Mathematical Papers, vol. 13, pp. 93–113. Cf. the references given in Rowe and Taylor, *Note on a geometrical theorem*, Proc. London Math. Soc. vol. 13 (1881–1882) pp. 102–106, to which the *Catalogue of scientific papers 1800–1900*, Royal Society of London, 1908, pp. 86 and 437 adds papers by Binet, Faure, Grunert, Kirkman, and Liouville.

[8] This was perhaps first remarked by Paolina Quarra, *Calcolo delle parentesi*, Torino Atti vol. 53 (1918) pp. 1044–1047. The connection with the result of Gummer (see footnote 11) was noted by J. H. M. Wedderburn, *The functional equation* $g(x^2)=2ax+[g(x)]^2$, Ann. of Math. (2) vol. 24 (1923) pp. 121–140, on p. 121. For these references I am indebted to the referee.

without crossings of nl points on a circle into n l-tuples, choose again, for a fixed point from among $(n+1)l$ points, the l-tuple containing the point. There remain arcs with $j_1l, \cdots, j_ll$ points such that $j_1+\cdots+j_l=n$. For the first arc, there are c_{j_1} divisions ($c_0=1$), and so on. The recursion formula obtained, $c_{n+1}=\sum c_{j_1}\cdot\cdots\cdot c_{j_l}$, corresponds to a function $y=\sum c_nx^n$ with $y^l=(y-1)/x$. By Lagrange's formula for implicit functions[9] we have $c_n=C_{ln,n}/((l-1)n+1)$ $=C_{ln+1,n}/(ln+1)$.[10] For $l=2$, this is a second (less elementary) proof for the evaluation of 1.7.

2.2. The same recursion formula is obtained for the number of partitions of a convex polygon of $m=(l-1)n+2$ sides, by non-intersecting diagonals, into polygons of $l+1$ sides (polygons with other values of m cannot be so divided). To see this choose, for a fixed side, the adjacent partial polygon, and proceed as before.

Yet, though it might seem so at first sight, c_n is not connected with the number of independent generalized function ratios of the form

$$G=\prod F(a_{1r_1},\cdots,a_{nr_n})^\rho,\qquad r_i=1,\cdots,l,\ \rho=\epsilon^{\sum r_i},\ \epsilon^l=1,\ l>2.$$

The latter number is, even for a fixed division into l-tuples, equal to the number of independent ordinary function ratios. Indeed, every generalized function ratio can without difficulty be expressed as a product of powers of ordinary function ratios, and vice versa.

2.3. If the ln points on the circle are denoted in order by $1, \cdots, ln$, then every division without crossing may again be uniquely represented by its last term: the set N comprising the greatest number in every l-tuple. As in the case of $l=2$, the smallest number of N forms an l-tuple together with the $k=l-1$ preceding numbers of the complementary set M, and so on.

Not every set N of n numbers from among $1, \cdots, ln$ is the last set of a division without crossing, for the procedure of omitting the smallest number of N, together with its predecessors, cannot always be repeated until no number is left. This can be done if and only if k times the number $n(t)$ of numbers belonging to N, from among $1, \cdots, t$ ($t\leq ln$), is not more than the number $m(t)$ of numbers belonging to the set M of the kn numbers outside N. The number c_n can, therefore, be interpreted as the number of possibilities for a candidate M in an election, who ultimately obtains k times as many votes as his adversary N, to have, during the whole time of the counting, at least k times as many votes as N. Since $1, \cdots, ln$ contains $C_{ln,n}$ sets of n numbers, the probability of "perpetual preponderance" is $c_n/C_{ln,n}=1/(kn+1)$. If an additional number 0 is prefixed and M (getting $kn+1$ of $ln+1$ votes) is required always to have *more* than k times as many votes as N, the probability is $c_n/C_{ln+1,n}=1/(ln+1)$.

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[9] Goursat, *Cours d'analyse*, 5th ed., vol. 1, 1927, p. 427.

[10] Pólya and Szegö, *Aufgaben und Lehrsätze aus der höheren Analysis*, vol. 1, 1925, part II, problem 211.

2.4. Generally, for M with $m=kn+r$ and N with n votes, we shall prove the probability of the last-mentioned kind of permanent preponderance to be r/s, where $s=m+n=ln+r$ is the total number of votes.[11] This includes a third proof of the special case $k=r=1$.

For any given arrangement of the votes, let u be the smallest number such that $r(t)=m(t)-kn(t)>0$ for every $t>u$. Then $r(u)\leqq 0$, and the vote $u+1$ belongs to M and adds 1 to $r(u)$. Hence $r(u)=0$. Changing the arrangement of the first u votes alters neither this fact nor the characteristic property of u. For every fixed $u>0$ there are k times as many arrangements beginning with a vote for M as there are beginning with one for N. On the other hand, for every arrangement beginning with a vote for N we have $u>0$. The latter arrangements have the probability n/s, hence there remains for $u=0$ the probability $1-n/s=r/s$.

For $k=1$, the first moment of equality of votes may be considered instead of the last moment, which leads to a fourth proof of 1.7.

2.5. The significance of the value r/s is given by the theorem:

Of every s arrangements that are obtained from each other by a cyclic permutation, exactly r show permanent preponderance.

In case not all the s arrangements are different, they fall into periods of different arrangements; every period contains the same number of arrangements with permanent preponderance, so that also r/s of the different arrangements have permanent preponderance.

To prove the theorem (and hence 1.7 for a fifth time), let the votes for M and N be arranged cyclically. Neither a vote v for N nor any of the k preceding votes is the beginning of an arrangement with permanent preponderance. From v go backwards until, on the arc covered, the number of votes for M is k times that for N, which must happen; for example go as long as the number of votes for M is at most k times that for N: when stopping it will be exactly k times. To every vote v for N there belongs such an arc A_v of forbidden beginnings; and to every vote for N on A_v there belongs an arc contained in A_v. Outside the maximal arcs of forbidden beginnings there remain obviously no votes for N and r votes for M. Each of them is good, since the first vote v disturbing the preponderance of M would be for N, and A_v would include the given beginning.[11]

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[11] For $k=1$ this is the "ballot problem," formulated and solved, by formula (3) of §2.7 of this note, by Bertrand, *Solution d'un problème*, C. R. Acad. Sci. Paris vol. 105 (1887) p. 369. Ibid. pp. 407, 435, and 440, E. Barbier gave, without proof, the probabilities r/s of 2.4, and r/s of 2.6 for the set of *all* arrangements. Barbier may have proved his first result, and the second one for $s>ln$, if it holds for $s=ln$, by the abovementioned formula (3); I do not see how he could prove the second result for $s=ln$.

For further history, and for related theorems, see A. Dvoretzky and Th. Motzkin, *A problem of arrangements*, Duke Math. J. vol. 14 (1947) pp. 305–313; to the different independent formulations of the ballot problem mentioned there, add P. Franklin, Amer. Math. Monthly vol. 25 (1918) p. 118, problem 2681, solution by C. F. Gummer, ibid. vol. 26 (1919) pp. 127–128, and H. D. Grossman, Scripta Mathematica vol. 12 (1946) pp. 223–225.

2.6. It is easy to see (still for $s \geqq ln$) that if, from an arc A_v, we delete every partial arc $A_{v'}$, $v' \neq v$, then there remains only the vote v, together with k votes for M, and that for each of these votes v is the first vote that disturbs preponderance. Since the deleted votes can be divided into groups of l consecutive votes, the probability ln/s for non-preponderance is equally distributed among the possible remainders, after division by l, of the number of the first vote that disturbs preponderance:

Of every $s \geqq ln$ arrangements that are obtained from each other by a cyclic permutation there are exactly n for which the number of the first vote disturbing preponderance has a given value mod l.

2.7. Instead of $r(t) = m(t) - kn(t)$, other functions of two or more non-negative integers m, n, $\cdots$, not all of which are 0, may be required to remain positive. Let the given function be $m - f(n)$, and let $p_{m,n}$ be the corresponding number of possibilities. Obviously (1) $p_{m,n} = 0$ for $m \leqq f(n)$, while for $m > f(n)$, (2) $p_{m,n} = 1$ if $mn = 0$ and (by classification of the possibilities according to the last vote) (3) $p_{m,n} = p_{m,n-1} + p_{m-1,n}$ if $mn > 0$. (Similar equalities hold for more than two variables.) By (1), (2), (3), $p_{m,n}$ is uniquely defined. There is $p_{m,n} = 0$ if $m \leqq f(n')$, $n' \leqq n$, so that $f(n)$ may be supposed to be an increasing function. Subtracting a constant from m, or adding unity to m, we may also suppose $f(0) = 0$.

The expression $q_{m,n} = C_{m+n,m}(m - kn)/(m+n) = C_{m+n-1,n} - kC_{m+n-1,m}$ fulfils (3) and $q_{m,0} = 1$. For an integer k, $q_{kn,n} = 0$, so that the $p_{m,n}$ belonging to $f(n) = kn$ equals $q_{m,n}$ for $m \geqq kn$, a fourth proof for 2.4 and a seventh proof for 1.7. This proof, the proof in 2.4, and the cycle theorem 2.5 with its two proofs remain valid for preponderance $m-kn-k'n'-\cdots>0$ over more than one adversary.

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[11] Another (for 1.7 sixth) proof of the "cycle theorem": The arrangement $u+1, \cdots, s, 1, \cdots, u$ shows permanent preponderance if and only if (1) $r(t) > r(u)$ for every $t > u$, and (2) $r(u) < r(t) + r$ for every $t < u$. Let $q \leqq 0$ be the minimum value of $r(t)$ for $t = 0, 1, \cdots$. Put $r(u) - q = p$; by (1) and (2) we have $p < r$, and u is the smallest value such that $r(t) > q + p$ for every $t > u$. To every non-negative $p < r$ corresponds a value of u; hence there are r such values.

For general $f(n)$, the expression $q_{m,n}=a_0C_{m+n,m}+a_1C_{m+n-1,m}+\cdots+a_n$, which fulfils (3), can be made equal to $p_{m,n}$ for $m\geqq f(n)$ by putting $a_0=1$ and, after determination of $a_1, \cdots, a_{n-1}$, choosing a_n such that $q_{m,n}=0$ for the greatest integer $m\leqq f(n)$. The law for the characteristic sequence of integers $a_0, a_1, \cdots$ is not known even for the simplest functions except kn with integral k (and related functions, as max $(kn-c, 0)$ or min (kn, c), $c>0$).

2.8. Another generalization wanted is the number of mixed partitions, that is, of partitions of a convex polygon by non-intersecting diagonals into polygons of l sides, where l is not restricted to a single value. The simplest case, partition into triangles and quadrangles, has the recursion formula $c_{n+1}=\sum c_jc_{n-j}+\sum c_jc_kc_{n-1-j-k}$ for the number c_n of partitions of an $(n+2)$-gon. The corresponding series $y=\sum c_nx^n$ fulfils the equation $y-1=xy^2+x^2y^3$. The sequence $c_0, \cdots$ begins 1, 1, 3, 10, 38, 154, 654, 2871, 12925, 59345, $\cdots$, with $c_{n+1}/c_n\to 27/5=5.4$.[13]

If l is allowed to take all values 3, 4, $\cdots$, and if PQR and PQR' are consecutive vertices of the given and of a partial polygon, then classification of the partitions according to the position of R' gives the formula $c_{n+1}=b_1c_n+b_2c_{n-1}+\cdots+b_{n+1}c_0$ with $c_0=1$, where b_{n+1} is the number of partitions with $R=R'$. By omission of Q in the latter and possibly connection of P with R we see that $b_{n+1}=2c_n$ for $n>0$, while $b_1=1$. Hence we have $c_{n+1}=c_0c_n+2(c_1c_{n-1}+\cdots+c_nc_0)$. There follows $(y-1)/x=2y^2-y$, $4xy=x-\sum_1^\infty C_{1/2,n}(x^2-6x)^n$. The values of $c_0, \cdots$ are 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, $\cdots$, with $c_{n+1}/c_n\to 3+2^{3/2}=5.83$.

The numbers of types of congruent or symmetric partitions of a regular polygon begin with 1, 1, 2, 3, 9, 20, 73, $\cdots$.

2.9. The number c_n of divisions of n points on a circle into sets of l points without crossing may also be generalized to different values of l, but with other results than for partitions of polygons. For $l=2$ or 3 we are led to $y-1=x^2y^2+x^3y^3$. The sequence c_n begins 1, 0, 1, 1, 2, 5, 8, 21, 42, 96, 222, 495, 1175, $\cdots$, with $c_{n+1}/c_n\to 2.61$. For $l=1$ or 2 we have $y-1=xy+x^2y^2$, and the sequence c_n begins 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, $\cdots$, with a ratio tending to 3.

If all values $l=1, 2, \cdots$ are allowed, then the numbers of types of congruent or symmetric divisions begin with 1, 1, 2, 3, 6, 9, 24, $\cdots$. The divisions themselves may be classified, for a fixed point, according to the next point in the same subset. We obtain once more the recursion formula leading to $c_n = C_{2n,n}/(n+1)$.

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[13] This and the following three limits are obtained as reciprocals of the (unique) absolutely least x with $x'=0$.

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INDEPENDENCE OF RESULTANTS

TH. MOTZKIN

In this note it is proved that the resultants of m forms, of a sufficiently high degree, in n variables are independent functions of the coefficients of the forms. The proof demands some lemmas on irreducible manifolds, and on monomial manifolds. A *monomial manifold* is defined by equalities between *monomials*, that is, products of powers of the coordinates.

In the evaluation of the number of independent hypersurface cross ratios and generalized intersections given in two other notes[1] I have assumed the above theorem to be true for $2n-1 \leq m \leq 2n+1$ and, in the case of intersections, where one of the forms is supposed to be linear for $2n-2 \leq m \leq 2n$.

The *resultant* $r = r(a_1, \cdots, a_n)$ of n forms a_k of positive degree d_k in n variables x_k within an (algebraically) closed field is uniquely defined as an irreducible polynomial in the coefficients of the forms such that $r = 1$ if a_k is a power of x_k and $r = 0$ if and only if values x_k, not all of them 0, exist for which all $a_k = 0$. The resultant is *almost symmetric*, that is, it becomes r or $-r$ if the forms are permuted. The resultant is *multiplicative* in the sense that if a form a_k is a product of forms, then r is the product of the resultants obtained by replacing a_k by each of its factors.[2]

THEOREM 1. *The $C_{m,n}$ resultants that can be formed of m forms $a_1, \cdots, a_m$ in n variables are independent functions of the coefficients of the forms, provided that the degree d_k of a_k exceeds a bound depending only on k and n.*

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[1] *The hypersurface cross ratio*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 976–984, §3.6, and *Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 352–360.

[2] All these properties of the resultant are well known. Cf. also §1.1 of the before mentioned note *The hypersurface cross ratio.*

We shall prove the theorem for $d_k \geqq C_{k-1,n-1}$. Thus $d_1, \cdots, d_n$ are arbitrary.

We have to show that there is no algebraic relation fulfilled by every system of $C_{m,n}$ values attained by the resultants of m forms of the given degrees. We say more shortly that almost every system is attained: generally, a subset P of a set Q will be said to include *almost every* element of Q if every algebraic relation fulfilled by all elements of P is also fulfilled by all elements of Q.[3]

As forms, we choose products of linear forms, the first n of which shall be powers. That for such products almost every system of values of the resultants is attained is trivial if $m=n$ or if $n=1$ and will be supposed true for $m-1$ if $m>n>1$.

Choose, accordingly, $a_1, \cdots, a_{m-1}$ so that the $b = C_{m-1,n}$ resultants formed by these products of linear forms have the desired values; these values $q_1, \cdots, q_b$ are restricted by the possibility of the choice and are, furthermore, required to be different from 0. Still $q_1, \cdots, q_b$ can be almost every system. There remain $c = C_{m-1,n-1} > 1$ resultants $r_1, \cdots, r_c$ involving a_m; they are the coordinates of a point r in the affine space S of c dimensions. We show that:

(1) The points r corresponding to all a_m of degree one, and to the chosen $a_1, \cdots, a_{m-1}$, form a set (r) "defining" (see below) an irreducible algebraic manifold R that contains, for every $k=1, \cdots, c$, a point with $r_k=0$ and no other zero coordinate.

(2) For every such set (r), almost every point of S is a product of c points (or any given number greater than c of points) of (r).

Here the product of two points $r_1, \cdots, r_c$ and $s_1, \cdots, s_c$ is the point $r_1s_1, \cdots, r_cs_c$. From (1) and (2) follows by the multiplicative property of the resultant the existence, for any given $d_m \geqq c$, and for almost every system of numbers $r_1, \cdots, r_c$, of a product a_m of d_m linear forms for which the values of the resultants are $r_1, \cdots, r_c$. Now if there should hold a relation between the values $q_1, \cdots, q_b$, $r_1, \cdots, r_c$ attained, then for every $q_1, \cdots, q_b$ there would be a relation between $r_1, \cdots, r_c$, except for those $q_1, \cdots, q_b$ for which the latter relation becomes $0=0$; but that would give one or more relations between $q_1, \cdots, q_b$.

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[3] Not, as usual in algebraic geometry, if the complementary set $Q-P$ has less dimensions than Q, which, for an irreducible manifold Q, is a stronger condition (presumably also fulfilled in our case). By the present definition P may even, for irreducible Q and if points with coordinates outside the given field are admitted, be a single point (a "general point" of Q). Compare the three further definitions of "almost every": except a set of zero measure, or density, or except a finite set.

Not every system of values need be attained by a system of independent polynomials, for example (x, xy) is never $(0, 1)$.

To prove (1) we remark that since no resultant of n forms from among $a_1, \cdots, a_{m-1}$ vanishes, no point $x_1, \cdots, x_n$ of the projective $(n-1)$-space annihilates more than $n-1$ of these forms. Hence $n-1$ forms cannot vanish for all the points of a straight line; therefore, being products of linear forms, they vanish for the points of a finite set S_k. Clearly no two sets S_j and S_k have a point in common. Hence there exists, for any S_k, a hyperplane $a_m=0$ through one of its points that does not pass through any point of the other sets, which is equivalent to saying that $r_k=0$ for the given k and only for that k.

The irreducibility of the algebraic manifold R defined by the set (r) follows from (3), taking Q as the affine n-space itself.[4]

The smallest algebraic manifold R containing all points of a given set (r) is called the manifold *defined* by (r); obviously, almost every point of R belongs to (r).[5] In particular, if (r) consists of all the points whose coordinates are the values of c given polynomials $f_1, \cdots, f_c$ in the coordinates $b_1, \cdots, b_n$ of a point of a given manifold Q of the affine n-space, and if a set (q) contains almost every point of Q then, to the points of (q), there still corresponds a set containing almost every point of R. For any additional relation for the points of R can be read as a relation restricting the points of Q.

(3) *If Q is irreducible, then R is irreducible too.*[6]

For assume a polynomial $F(r_1, \cdots, r_c)$ to vanish for every point of R. Then $G=F(f_1, \cdots, f_c)=0$ for every point of Q. If F is a product F_1F_2, then also $G=G_1G_2$, and either G_1 or G_2, say G_1, vanishes for every point of Q. Hence $F_1=0$ for every point of (r) and therefore also for every point of R.

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[4] It can even be shown that (r) and R are identical. Indeed, every resultant r_k is a form f_k of positive degree g_k in the coefficients $b_1, \cdots, b_n$ of the linear form a_m. Instead of $f_k=r_k$ we write homogeneously $r_0f_k=r_kb_0^{g_k}$. This is an algebraic correspondence between the whole projective n-space and an algebraic manifold of the projective c-space, the equations of which are obtained by eliminating $b_0, \cdots, b_n$ from the equations of the correspondence. This manifold consists of the $(c-1)$-dimensional part $r_0=0$, corresponding to all systems with $b_0=0$, and of a manifold R. Now all $f_k=0$ only if $b_1=\cdots=b_n=0$; otherwise there would exist a hyperplane through one point at least of each set S_k, whereas the n sets S_k defined by any $n-1$ from among the forms $a_1, \cdots, a_n$, which are powers of linear forms, are single points not on a common hyperplane. Hence to every point of R with $r_0\neq 0$ there corresponds a system $b_0, \cdots, b_n$ with $b_0\neq 0$; that is, the nonhomogeneous coordinates of the point are values attained by the resultants belonging to a linear form a_m.

[5] By two theorems of Hilbert, every algebraic manifold in a space over a closed field may be given either by a finite system of equations or by all its points with coordinates within the field.

[6] The example $f_1=b_1$, $f_2=b_1b_2$ shows again that not every point of R is necessarily a point of (r). Other examples are given in footnote 7.

We shall use the symbol (Q, R) for the set of all points $(q_1, \cdots, q_b, r_1, \cdots, r_c)$ with $(q_1, \cdots, q_b)$ in a given set Q of b-space and $(r_1, \cdots, r_c)$ in a given set R of c-space. Evidently, if Q and R are algebraic manifolds, then (Q, R) is also an algebraic manifold.

If Q is defined by (q) and R by (r), then $((q), (r))$ defines (Q, R). For if $F(q, r)=0$ for every point of (q) and every point of (r), and if q is a point of (q) (here we admit only points with coordinates in the given field), then $F(q, r)=0$ may be considered as an equation for r; since it holds for every point of (r), it holds also for every point of R. Now if r is a point of R, then $F(q, r)=0$ for every point of (q), hence also for every point of Q.

(4) *If Q and R are irreducible, then (Q, R) is also irreducible.*

For let $F(q, r)=0$ for all points of (Q, R). For every q in Q this is an equation for r; all these equations together are fulfilled by the points of an algebraic manifold E of c-space, which contains R. If $F=F_1F_2$, then every point of E must, because of the irreducibility of Q, belong to at least one of E_1 and E_2 (defined similarly). Since R is irreducible and contained in E, R is contained in at least one of E_1 and E_2, Q.E.D.

The *product* QR of two algebraic manifolds Q and R of the same affine space is the manifold "defined" by the points qr with q in Q and r in R.[7] Now QR is obtained from (Q, R) by the formation of products, that is, of special polynomials: hence by (4) and (3) we see that:

If Q and R are irreducible, then QR is irreducible,[8]

and that QR can be defined already by almost every point of Q and of R. In particular, if Q is a product, QR can be defined by those points of Q which are products of points of the factors of Q. Hence the multiplication of manifolds is associative.

The assertion (2) can now be written $R^e=S$. To see that this implies $R^{e+1}= \cdots =S$ note that already $R^e r=Sr=S$ provided that R contains a point r with no vanishing coordinate. But if it did not R

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[7] The product QR may contain exceptional points not of the form qr. The point $x=y=1$ is, for example, exceptional for the line $y=x+1$ as Q and the hyperbola $x+y=xy$ as R, or for $xy+2=4x+2y$ as $Q=R$; in both cases QR is the whole plane. If, for two points of Q and R with homogeneous coordinates $q_0, \cdots, q_a$ and $r_0, \cdots, r_a$, there is never $q_0r_0= \cdots =q_ar_a=0$, then there are no exceptional points, and the points p of QR are given by $p_0: \cdots :p_a=q_0r_0: \cdots :q_ar_a$; otherwise, however, the latter equations are fulfilled by every point of the space. An example for this case, but without exceptional points, is $xy=1$ as $Q=R=QR$ $(=\overline{R}$, as defined in the sequel).

[8] If the field of the coefficients of the equations is not closed and if points with coordinates outside the field are admitted, then (Q, R) and QR may be reducible. An example is the field of rational numbers with $x^2=2$ as $Q=R$ and $x=\pm\sqrt{2}$ as QR.

would be decomposable into its (nonvoid) intersections with $r_1 = 0, \cdots, r_c = 0$.[9]

It therefore remains to be proved that $R^c = S$. Neither this conclusion nor the suppositions of (2) specified in (1) are affected if R is replaced by R/r, where r is a point of R with no zero coordinate. Hence we may suppose that R contains the point 1 all of whose coordinates are 1.

A ***monomial relation*** is a relation $r_1^{h_1}\cdot \cdots \cdot r_c^{h_c} = 1$ with integral exponents. It may be written as an equation with non-negative exponents. If the coordinates of the points q and r satisfy the equation, the same is true for qr.

If r_k is a factor of one of its sides, then the equation is fulfilled by no point with $r_k = 0$ and all other coordinates different from 0. Hence the only monomial relation satisfied by all points of the manifold R of (7) is the identical relation $1 = 1$ ($h_1 = \cdots = h_c = 0$). We may express this by saying that the monomial closure $\bar{R}$ of R is S, where the ***monomial closure*** of a set is the manifold defined by the monomial relations fulfilled by all points of the set.

The assertion (2) is therefore, in virtue of (5), a consequence of:

THEOREM 2. *If an irreducible manifold R of c-space contains* 1, *then* $R^c = \bar{R}$.[10]

The theorem follows by remarking that $\bar{R}$ contains every power R^d and that R^{d+1} contains R^d and has more dimensions than R^d as long as $R^d \neq \bar{R}$. To see this it will be enough to show that $R^2 \neq R$ if $R \neq \bar{R}$, whence it follows that $(R^d)^2 \neq R^d$ if $R^d \neq \bar{R}$. For if the irreducible manifold R^{d+1}, which contains R^d, had the same number of dimensions, it would follow that $R^d = R^{d+1} = R^{d+2} = \cdots = R^{2d}$.

But if $R^2 = R$ and R is irreducible and contains 1, then obviously $R = R^0$, where R^0 is the ***multiplier manifold*** defined by the points r with $Rr = R$. Since we shall presently see that $R^0 = \overline{R^0}$, we have $R = \bar{R}$.[11]

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[9] For a manifold R with no such point r, no power equals S, since every power is contained in $r_1 \cdot \cdots \cdot r_c = 0$.

[10] Since every product of points of $\bar{R}$ fulfils the monomial relations defining $\bar{R}$, we have $\bar{R}^2 = \bar{R}$, $R^{c+1} = R^{c+2} = \cdots = \bar{R}$.

For a manifold Rr "proportional" to R, the powers $(Rr)^d = \bar{R}r^d$, $d \geqq c$, are disjoint and either periodic (if a power of r is a point of $\bar{R}$) or all different. In the latter case they do not exhaust $\overline{Rr}$, which must also contain $\bar{R}r^d$ (possibly multiplied by roots of unity) for fractional d.

[11] More generally, the product QR of two irreducible manifolds has more dimensions than R, except if Q is either proportional to a submanifold of R^0 or contained in $q_0 \cdot \cdots \cdot q_c = 0$. For if q and q' are points of Q without zero coordinates, then QR has certainly more dimensions than R unless $qR = q'R$, or $R(q/q') = R$.

Finally, why, for every algebraic manifold R, is there always $R^0 = \overline{R^0}$? Let R be given by a system of equations of the form $\sum c_\nu \rho_\nu = 0$, where the ρ_ν are monomials in $r_1, \cdots, r_e$ with non-negative exponents. The product Rs of R and a point $s_1, \cdots, s_e$ with no zero coordinate is then given by the corresponding equations $\sum c_\nu \rho_\nu \sigma_\nu = 0$, where σ_ν is formed from $s_1, \cdots, s_e$ in the same way as ρ_ν from $r_1, \cdots, r_e$. The equations may be written $\sum d_\mu \sigma_\mu = 0$, where all σ_μ, for the given s, have different values. If $Rs = R$, then the points of R fulfil the equations $\sum d_\mu = 0$, $\sum d_\mu \sigma_\mu = 0$, $\sum d_\mu \sigma_\mu^2 = 0, \cdots$. Since the Vandermonde determinant of the σ_ν is different from 0, this implies that every d_μ separately equals 0. If R was given from the outset by shortest equations, then the σ_μ belonging to the same equation must be equal; and the equalities between them form a system of monomial relations whose solutions without a vanishing coordinate are the points s with $Rs = R$. If $Rs = R$ for an s with some zero coordinates, then the corresponding coordinates of every point of R vanish and R^0 is given by monomial relations between the other coordinates. Being given by monomial relations, *the multiplier manifold R^0 is a monomial manifold* and equals its own monomial closure.

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ON THE SUM OF CUBES

E. ROSENTHALL

Large capital letters A, B, $\cdots$ (without or with subscripts) will represent integers of the quadratic number field $Ra(\rho)$ where $\rho = (-1+(-3)^{1/2})/2$. Small latin letters a, b, $\cdots$ represent rational integers, and the conjugate of a number X is denoted by $\bar{X}$.

The object of this paper is to give a method for obtaining the complete rational integer solution for the diophantine equations of the form

$$(1) \qquad \sum_{i=1}^{m} x_i^3 = 0, \qquad m > 3.$$

This equation with m even, $m=2n$, can be written as $\sum_{i=1}^{n}(X_i+\bar{X}_i)X_i\bar{X}_i=0$ where

$$(2) \qquad X_i = x_{2i-1} + \rho(x_{2i-1} - x_{2i})$$

and thus the problem of solving (1) in this case is reduced to that of finding all the integers x_i, X_i satisfying the equations

$$(3) \qquad \sum_{i=1}^{n} x_i X_i \bar{X}_i = 0,$$

$$(4) \qquad x_i = X_i + \bar{X}_i \qquad (i = 1, 2, \cdots, n)$$

and (2). When m is odd, $m=2n-1$, we solve the system (α) consisting of (3), $x_n = X_n = x_{2n-1}$ and (2), (4) for $i=1, 2, \cdots, n-1$.

The resolution of these two systems hinges on techniques developed by E. T. Bell [2],[1] being equivalent to the resolution of a system of multiplicative equations and a system of linear homogeneous equations in Ra in which the number of unknowns always exceeds the number of equations.

In solving (1) the following equations appear:

$$(5) \qquad x_1y_1 + x_2y_2 + \cdots + x_ny_n = 0$$

in which the x_i, y_i ($i=1, 2, \cdots, n$) are $2n$ independent variables;

$$(6) \qquad a_{i1}x_1 + \cdots + a_{in}x_n = 0 \qquad (i = 1, 2, \cdots, m \leqq n-1)$$

in which the n independent variables x_i are to be solved in terms of the coefficients a_{ij};

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[1] Numbers in brackets refer to references cited at the end of the paper.

$$A_1B_1C_1 = \cdots = A_{n-1}B_{n-1}C_{n-1} \tag{7}$$

in $3(n-1)$ independent variables A_i, B_i, C_i;

$$p_1A_1\bar{A}_1 = p_2A_2\bar{A}_2 = \cdots = p_nA_n\bar{A}_n \tag{8}$$

in which the p_i, A_i are independent variables.

The solution [5, p. 20 (13)] of (5) is

$$x_i = aa_i, \qquad y_i = -\sum_{j=1}^{i-1} a_jb_{j,i} + \sum_{j=1}^{n-i} a_{i+j}b_{i,i+j}$$

for $i=1, 2, \cdots, n$ with the convention that a sum in which the lower limit exceeds the upper is vacuous. Note that there are $(n^2-n)/2$ free parameters $b_{j,k}$ and $n+1$ free parameters a, a_i.

If the system (6) is of rank m then its complete solution [3] in determinantal form is written down as follows. Let e_j be the determinant obtained by deleting the jth column from the matrix of the coefficients of the system consisting of (6) and the equations

$$c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n = 0 \qquad (i = 1, 2, \cdots, n - m - 1)$$

in which the c_{ij} are arbitrary rational integers. Then

$$x_j = (-)^{j}te_j/e \qquad (j = 1, 2, \cdots, n) \tag{9}$$

where t is an arbitrary integer and $e = (e_1, e_2, \cdots, e_n)$.

The system (7) is recursive and can be solved completely by the algorithm of reciprocal arrays [1] since the integers of $Ra(\rho)$ form a principal ideal ring. System (7) is equivalent to the equations

$$A_iB_iC_i = A_{n-1}B_{n-1}C_{n-1} \qquad (i = 1, 2, \cdots, n - 2).$$

The solution of the typical equation is

$$\begin{aligned} A_i &= A_{i1}B_{i1}C_{i1}, & A_{n-1} &= A_{i1}H_{i1}F_{i1}, \\ B_i &= D_{i1}E_{i1}F_{i1}, & B_{n-1} &= D_{i1}B_{i1}J_{i1}, \\ C_i &= G_{i1}H_{i1}J_{i1}, & C_{n-1} &= G_{i1}E_{i1}C_{i1}. \end{aligned}$$

Then the values of A_{n-1} are equated, also those of B_{n-1}, and those of C_{n-1}. The resulting three systems are each of the type (7) with $n-2$ in place of $n-1$. By repetitions of the process the solution of (7) involving $(n-1) = 3^{n-1}$ free parameters $K_1, K_2, \cdots, K_{(n-1)}$ is obtained in the form

$$A_i = K_1\Phi_i(K_4, \cdots, K_{(n-1)}), \qquad B_i = K_2\Psi_i(K_4, \cdots, K_{(n-1)}),$$
$$C_i = K_3\Theta_i(K_4, \cdots, K_{(n-1)})$$

where each of Φ_i, Ψ_i, Θ_i is a monomial in $3^{n-2}-1$ of the parameters $K_3, K_4, \cdots, K_{(n-1)}$ each occurring only once in a particular Φ_i, Ψ_i, Θ_i and

$$K_1K_2K_3\Phi_i(K_4, \cdots, K_{(n-1)})\Psi_i(K_4, \cdots, K_{(n-1)})\Theta_i(K_4, \cdots, K_{(n-1)}) = K_1K_2 \cdots K_{(n-1)}$$

for $i=1, 2, \cdots, n-1$.

The resolution of (8) is also recursive. This system is equivalent to the $n-1$ equations $p_nA_n\bar{A}_n = p_iA_i\bar{A}_i$ $(i = 1, 2, \cdots, n-1)$. The solution [4, Theorem 1] of the typical equation is

$$p_i = t_{i1}V_{i1}\bar{V}_{i1}, \qquad p_n = t_{i1}L_{i1}\bar{L}_{i1},$$
$$A_i = S_{i1}\bar{U}_{i1}L_{i1}, \qquad A_n = S_{i1}U_{i1}V_{i1}.$$

Then the values of p_n are equated and also those of A_n which yield the two independent systems

$$(10) \qquad t_{i1}L_{i1}\bar{L}_{i1} = t_{n-1,1}L_{n-1,1}\bar{L}_{n-1,1}$$

and

$$(11) \qquad S_{i1}U_{i1}V_{i1} = S_{n-1,1}U_{n-1,1}V_{n-1,1}$$

for $i=1, 2, \cdots, n-2$.

System (10) is of the type (8) with $n-1$ in place of n; system (11) is of the type (7) and its solution is therefore

$$S_{i1} = K_1\Phi_i(K_4, \cdots, K_{(n-1)}), \qquad U_{i1} = K_2\Psi_i(K_4, \cdots, K_{(n-1)}),$$
$$V_{i1} = K_3\Theta_i(K_1, \cdots, K_{(n-1)})$$

for $i=1, 2, \cdots, n-1$.

Hence all integral solutions of (8) are given by

$$A_i = K_1\Phi_i\bar{K}_2\bar{\Psi}_iL_{i1}, \qquad A_n = K_1\Phi_iK_2\Psi_iK_3\Theta_i,$$
$$p_i = t_{i1}K_3\bar{K}_3\Theta_i\bar{\Theta}_i, \qquad p_n = t_{i1}L_{i1}\bar{L}_{i1};$$

with the condition (10).

The process just applied to (8) is now repeated on (10) which will yield parametric expressions for t_{i1}, L_{i1} similar to those for p_i, A_i respectively subject to systems of the type (10) and (11) with $n-1$ replaced by $n-2$. We note that this process must finally yield the solutions in the form

$$(12) \qquad A_i = KE_i, \qquad p_i = dM_i\bar{M}_i$$

where the E_i and also the M_i are products of integers of $Ra(\rho)$; all the E_i, and also all the M_i are of course not independent but they are each independent of K.

The resolution of (3) now follows. Applying (5) to (3) yields

$$X_i\bar{X}_i = aa_i, \qquad x_i = -\sum_{j=1}^{i-1} a_j b_{j,i} + \sum_{j=1}^{n-i} a_{i+j} b_{i,i+j}$$

for $i=1, 2, \cdots, n$.

Hence we must now solve the system of equations

$$X_i\bar{X}_i = aa_i \qquad (i = 1, 2, \cdots, n).$$

The solution of the typical equation is

$$X_i = p_i A_i B_i, \qquad a = p_i A_i \bar{A}_i, \qquad a_i = p_i B_i \bar{B}_i$$

in free parameters p_i, A_i, B_i.

Equating the value of a yields the system (8) and hence by (12) $A_i = KE_i$, $p_i = dM_i\bar{M}_i$ and therefore the complete solution of (3) is given by

$$(13) \qquad X_i = KR_i, \qquad x_i = -\sum_{j=1}^{i-1} a_j b_{j,i} + \sum_{j=1}^{n-i} a_{i+j} b_{i,i+j}$$

for $i=1, 2, \cdots, n$ where

$$(14) \qquad R_i = dM_i\bar{M}_i B_i E_i, \qquad a_i = dM_i\bar{M}_i B_i \bar{B}_i.$$

The resolution of (1) now follows. Put $K = k_1 + \rho k_2$, $R_i = r_i + \rho s_i$; then $KR_i + \bar{K}\bar{R}_i = k_1(2r_i - s_i) - k_2(r_i + s_i)$. Then all the X_i, x_i satisfying (3) and (4) simultaneously are given by (13) where values are assigned to the parameters which determine R_i, a_i in (14) and then the $(n^2-n+4)/2$ unknowns k_1, k_2, b_{ij} are determined from the n linear homogeneous equations

$$(15) \quad k_1(2r_i - s_i) - k_2(r_i + s_i) + \sum_{j=1}^{i-1} a_j b_{j,i} - \sum_{j=1}^{n-i} a_{i+j} b_{i,i+j} = 0$$

for $i=1, 2, \cdots, n$, a system of the type (6).

Substitute this value of X_i in (2). Equate real and imaginary parts and all the rational integer solutions of (1) with $m=2n$ will be obtained.

To solve (1) when m is odd, $m=2n-1$, we proceed much as above, replacing system (15) by system (α) which is equivalent to (13) and the equations

$$k_1(2r_i - s_i) - k_2(r_i + s_i) + \sum_{j=1}^{i-1} a_j b_{j,i} + \sum_{j=1}^{m-i} a_{i+j} b_{i,i+j} = 0$$

for $i=1, 2, \cdots, n-1$,

$$k_1 r_n - k_2 s_n + \sum_{j=1}^{n-1} a_j b_{j,n} = 0, \qquad k_1 s_n + k_2(r_n - s_n) = 0,$$

a linear homogeneous system of $n+1$ equations in $(n^2-n+4)/2$ unknowns k_1, k_2, b_{ij}. The corresponding X_i given by (13) are substituted in (2) for $i=1, 2, \cdots, n-1$ and we put $X_n = z_{2n-1}$.

We conclude by exhibiting the complete solution of $\sum_{i=1}^{6} z_i^3 = 0$ in terms of integers of $Ra(\rho)$.

The complete solution of $p_1 A_1 \bar{A}_1 = p_2 A_2 \bar{A}_2 = p_3 A_3 \bar{A}_3$ is given by (12) where

$$M_1 = GHJT, \qquad M_2 = GFLN, \qquad M_3 = LPST,$$
$$E_1 = C\bar{D}\bar{F}LNP\bar{Q}\bar{S}, \qquad E_2 = \bar{C}\bar{D}\bar{H}JPQST, \qquad E_3 = CDFGHJNQ,$$

and all the parameters are arbitrary. Hence from (14) we get the corresponding values of a_i, $R_i = r_i + \rho s_i$, where d, B_i are arbitrary. In this case (15) is a linear homogeneous system of 3 equations in 5 unknowns. To complete this linear system for resolution we adjoin the single equation

$$m_1 b_{23} + m_2 b_{13} + m_3 b_{12} + m_4 k_1 + m_5 k_2 = 0$$

with arbitrary coefficients m_i.

Hence with a_i, r_i, s_i as found above, (9) gives

$$ek_1 = t(a_1 m_1 - a_2 m_2 + a_3 m_3)(a_1(r_1 + s_1) + a_2(r_2 + s_2) + a_3(r_3 + s_3)),$$
$$ek_2 = t(a_1 m_1 - a_2 m_2 + a_3 m_3)(a_1(2r_1 - s_1) + a_2(2r_2 - s_2) + a_3(2r_3 - s_3)).$$

Then from (2) and (13) for $i=1, 2, 3$

$$z_{2i-1} = k_1 r_i - s_i k_2, \qquad z_{2i} = k_1(r_i - s_i) - k_2 r_i.$$

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McGILL UNIVERSITY

ON SOME NEW QUESTIONS ON THE DISTRIBUTION OF PRIME NUMBERS

P. ERDŐS AND P. TURÁN

1. Introduction. In connection with some recent unpublished investigations concerning the Riemann hypothesis one of us raised the question whether $\log p_n$ is convex for sufficiently large n, or at least whether it has few points of inflexion. (Throughout this paper $p_1=2$, $p_2=3, \cdots, p_n, \cdots$ denotes the sequence of primes.) In other words: Is it true that the inequalities

$$(1) \qquad p_{n-1}\cdot p_{n+1} > p_n^2, \qquad p_{m+1}p_{m-1} < p_m^2$$

both have infinitely many solutions? We shall show that the answer is affirmative.

A still simpler question is whether the sequence of primes itself is convex or concave from a certain n on. We shall prove that this is not so, that is, the equations

$$(2) \qquad \frac{p_{n-1}+p_{n+1}}{2} > p_n, \qquad \frac{p_{m-1}+p_{m+1}}{2} < p_m$$

have infinitely many solutions.[1]

If the well known hypothesis about prime twins is true, that is, if the equation $p_{n+1}-p_n=2$ has infinitely many solutions, (1) and (2) of course are trivially satisfied.

The first inequality of (2) is inserted only for the sake of completeness. It follows from the well known fact that $\limsup (p_{n+1}-p_n)=\infty$ (since $n!+2, n!+3, \cdots, n!+n$ are all composite). The proof of the other inequalities will be simple, but less trivial.

Clearly $p_{n-1}p_{n+1}>p_n^2$ implies $(p_{n-1}+p_{n+1})/2>p_n$ and $p_m>(p_{m-1}+p_{m+1})/2$ implies $p_m^2>p_{m-1}p_{m+1}$. The well known relations between the various mean values suggest the following questions: Is it true that for every t the inequalities

$$(3) \qquad \left(\frac{p_{n-1}^t+p_{n+1}^t}{2}\right)^{1/t} > p_n$$

and

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[1] Professor G. Pólya and Mr. P. Ungár communicated to us subsequently a proof very similar to our own.

$$\left(\frac{p_{m-1}^t + p_{m+1}^t}{2}\right)^{1/t} < p_m \tag{4}$$

have infinitely many solutions? By the well known relations between the means it follows that (1) and (2) is a consequence of (3) and (4), and that it suffices to prove (3) for $t<0$ and (4) for $t>0$. (The inequality about means states that $((a^t+b^t)/2)^{1/t}$ is an increasing function of t.)[2]

An elementary proof of (3) and (4) is given in §2. The only result we use about primes is that

$$\pi(x) > c_1\, x/\log x. \tag{5}$$

This can be found in the first pages of Ingham's book *The distribution of prime numbers*. ($\pi(x)$ denotes the number of primes not exceeding x.)

All these questions can be investigated by a method which is less elementary than that given in §2, but which perhaps can be used to attack some of the unsolved problems which can be raised here. Only (2) is treated by this method (in §3).

In §4 we state without proof some results about the number of solutions of (3) and (4). Finally we state some unsolved problems, which are natural generalizations of our theorems.

2. Elementary proofs.

THEOREM 1. *The inequalities* (3) *and* (4) *have infinitely many solutions.*

We need the following lemma.

LEMMA. *Let $A>0$ be any constant. Then the inequalities*

$$p_k - p_{k-1} < p_{k+1} - p_k, \qquad p_k - p_{k-1} < A p_k^{1/2}, \tag{6}$$

$$p_{k+1} - p_k < p_k - p_{k-1}, \qquad p_{k+1} - p_k < A p_k^{1/2} \tag{7}$$

have infinitely many solutions.

The proof of (6) is quite trivial. It follows from (5) that for infinitely many m and a suitable c_2, $p_{m+1}-p_m<c_2 \log p_m$. Determine the least $k>m$ for which $p_{k+1}-p_k>p_{m+1}-p_m$. Then clearly p_{k-1}, p_k, p_{k+1} satisfy (6).

Now we prove (7). Assume that (7) has only a finite number of solutions (that is, there are no solutions for $p>p_0$). Let m be large and $p_{m+1}-p_m<c_2 \log p_m$. Let p_r be the smallest prime greater than $p_m^{1/2}$. Then clearly

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[2] See, for example, Hardy-Littlewood-Pólya, *Inequalities*, p. 26.

$$(8)\qquad p_{r+1}-p_r \leqq p_{r+2}-p_{r+1} < \cdots \leqq p_{m+1}-p_m < c_2 \log p_m.$$

For if not let k $(r<k\leqq m)$ be the greatest index for which $p_{k+1}-p_k<p_k-p_{k-1}$. Then clearly p_{k-1}, p_k, p_{k+1} satisfy (7) (since $p_{k+1}-p_k\leqq p_{m+1}-p_m<c_2 \log p_m<c_2(p_m)^{1/4}<c_2(p_k)^{1/2}$). This proves (8). But if $p_{t+1}-p_t=p_{t+2}-p_{t+1}= \cdots =p_{t+s+1}-p_{t+s}=d$ we evidently have $s\leqq d$ (since the integers x, $x+d, \cdots, x+xd=x(d+1)$ can not all be primes). Hence we obtain from (8) that

$$m-r\leqq 1+2+\cdots+[c_2 \log p_m]<(c_2 \log p_m)^2$$

or

$$m=\pi(p_m)\leqq r+(c_2 \log p_m)^2\leqq p_m^{1/2}+(c_2 \log p_m)^2$$

which contradicts (5), and completes the proof of the lemma.

Now we can prove Theorem 1. Since, for $a>0$, $b>0$, $((a^t+b^t)/2)^{1/t}$ is an increasing function of t, it suffices to prove (3) if t is a negative integer not greater than -2, say $t=-l$. Let p_{k-1}, p_k, p_{k+1} satisfy (6) with $A<1/2l^2$. Then we show that they also satisfy (3). Put $p_k-p_{k-1}=u$; since $((a^t+b^t)/2)^{1/t}$ is an increasing function of a and b it will clearly be sufficient to show that (3) is satisfied in case $p_{k+1}-p_k=u+1$. Thus we have to show that $(t=-l\leqq -2)$

$$\left(\frac{(p_k-u)^{-l}+(p_k+u+1)^{-l}}{2}\right)^{-1/l}>p_k$$

or

$$(p_k-u)^{-l}+(p_k+u+1)^{-l}<2p_k^{-l}$$

or

$$(p_k+u+1)^l(2(p_k-u)^l-p_k^l)>p_k^l(p_k-u)^l.$$

Now clearly for $u<p_k^{1/2}/2l^2$

$$(9)\qquad \begin{aligned} p_k^l-ulp_k^{l-1}<(p_k-u)^l&<p_k^l-ulp_k^{l-1}+\binom{l}{2}u^2p_k^{l-2}+\cdots\\ &<p_k^l-(ul-1/2)p_k^{l-1}. \end{aligned}$$

Thus it suffices to show that

$$(p_k^l+(u+1)lp_k^{l-1})(p_k^l-2ulp_k^{l-1})>p_k^l(p_k^l-(ul-1/2)p_k^{l-1}),$$

or

$$(l - 1/2)p_k^{l-1} > 2l^2u(u+1)p_k^{l-2},$$

which is clearly satisfied for $u < p_k^{1/2}/2l^2$, which proves (3).

Now we prove (4). Assume that p_{k-1}, p_k, p_{k+1} satisfy (7) with $A < 1/2t^2$. Put $p_{k+1} - p_k = u$. As before it suffices to consider the case $p_k - p_{k-1} = u + 1$ and $t \geqq 2$. Then we have to show that

$$(p_k - (u+1))^t + (p_k + u)^t - 2p_k^t < 0.$$

We have as in (8) for $u < (1/2t^2)p_k^{1/2}$ $(t \geqq 2)$

$$(p_k + u)^t < p_k^t + (tu + 1/2)p_k^{t-1}. \tag{10}$$

Thus from (8) and (9)

$$\begin{aligned}(p_k - (u+1))^t + (p_k + u)^t - 2p_k^t &< 2p_k^t - ((u+1)t - 1/2)p_k^{t-1} \\ &\quad + (ut + 1/2)p_k^{t-1} - 2p_k^t < 0,\end{aligned}$$

which proves (4) and completes the proof of Theorem 1.

THEOREM 2. *Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers which do not form an arithmetic progression from a certain point on. Let $t < 1$ and $a_k < k^2/4(1-t) - ck$, for every c if k is sufficiently large. Then*

$$((a_{k-1}^t + a_{k+1}^t)/2)^{1/t} > a_k \tag{11}$$

have infinitely many solutions.

THEOREM 3. *Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers which do not form a convex sequence from a certain point on (that is, $a_k - a_{k-1} > a_{k+1} - a_k$ has infinitely many solutions). Let $t > 1$ and $a_k < k^2/4(1-t) - ck$ for every c if k is sufficiently large. Then*

$$((a_{k-1}^t + a_{k+1}^t)/2)^{1/t} < a_k \tag{12}$$

has infinitely many solutions.

The inequalities in both theorems are best possible in the following sense: For every c there exists a sequence $a_1 < a_2 < \cdots$ of integers with $a_k < k^2/4(1-t) - ck$ for all k, the a's not forming an arithmetic progression from a certain point on, and so that (11) has only a finite number of solutions. The same holds for (12).

REMARK. It follows from (5) and our lemma (in §2) that Theorem 1 is a consequence of Theorems 2 and 3.

We prove Theorem 2 only in the special case $t = 0$; the proof of the general case and that of Theorem 3 is similar but requires slightly longer calculations. It is well known that for $t=0$ the left side of (11) becomes $(a_{k-1}a_{k+1})^{1/2}$. Thus we have to prove that if $a_k<k^2/4-ck$ for every c if k is sufficiently large, then

$$a_{k-1}a_{k+1} > a_k^2 \tag{13}$$

has finitely many solutions.

Suppose this is not true. Then for $k_0<k$, $a_{k-1}\cdot a_{k+1}\leqq a_k^2$. Since the a's do not form an arithmetic progression from a certain point on, it is clear that the equation $a_{k+2}-a_{k+1}>a_{k+1}-a_k$ has infinitely many solutions. Put $a_{k+1}-a_k=x$; we have $a_{k+2}\geqq a_k+2x+1$. Thus since $a_{k+1}^2\geqq a_k\cdot a_{k+2}$ we have

$$(a_k + x)^2 \geqq a_k(a_k + 2x + 1), \quad \text{or} \quad x^2 \geqq a_k. \tag{14}$$

Assume now that, for some $k>k_0$, $(a_{k+1}-a_k)^2<a_k$. Determine the least $l>k$ for which $a_{l+1}-a_l>a_l-a_{l-1}$. Then we have from (14)

$$(a_{k+1} - a_k)^2 \geqq (a_l - a_{l-1})^2 \geqq a_{l-1} \geqq a_k$$

an evident contradiction. But this means that, for $k>k_0$, $(a_{k+1}-a_k)^2\geqq a_k$. Thus we clearly obtain that for large enough n the number of a's in the interval $(n^2, (n+1)^2)$ (where n^2 is counted in the interval but $(n+1)^2$ not) does not exceed 2. Thus we evidently have $a_k>k^2/4-ck$ for sufficiently large c, an evident contradiction. This completes the proof. The sequence n^2, $n(n+1)$ with an arbitrary finite set added to it shows that the result is best possible.

3. **Analytical proof.** Now we give an alternative proof of (2) which uses deeper tools. We use the prime number theorem for arithmetic progressions in the form given by A. Page[3]

$$\begin{aligned} &\left|\pi(x, k, l) - \frac{1}{\phi(k)}\int_2^x \frac{dy}{\log y}\right| \\ &\qquad < \alpha_1 x(\exp(-\alpha_2(\log x)^{1/2}) + \frac{x}{\phi(k)}\exp\left(-\alpha_2\frac{\log x}{k^{1/2}(\log k)^2}\right) \end{aligned} \tag{15}$$

where $\pi(x, k, l)$ denotes the number of primes not exceeding x which are congruent to l (mod k) (we assume $(k, l)=1$) and $\alpha_1, \cdots$ are independent of x, k and l. We also need the following result due to Kusmin:[4] Let $\beta_1, \beta_2, \cdots, \beta_n$ be real and $\nu\leqq\beta_2-\beta_1\leqq \cdots \leqq\beta_n-\beta_{n-1}\leqq 1-\nu$ $(0\leqq\nu\leqq 1/2)$. Then

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[3] A. Page, Proc. London Math. Soc. (2) vol. 39 (1935) pp. 116–141.

[4] R. O. Kusmin, Zhurnal Leningradskoe Fiziko-Matematicheskoe obshchestvo vol. 1 (1927) pp. 233–239.

$$\left|\sum_{\nu=1}^{n} e^{2\pi i\vartheta_\nu}\right| < \frac{2}{\pi\eta}\,. \tag{16}$$

We are going to show that for sufficiently large x there exist primes

$$x \leqq p_{k-1} < p_k < p_{k+1} \leqq 4x \quad \text{so that} \quad p_{k+1} - p_k < p_k - p_{k-1}. \tag{17}$$

Suppose (17) is not satisfied. Let x be sufficiently large, and consider the primes

$$\begin{aligned} p_j < x \leqq p_{j+1} < \cdots < p_{j+H} \leqq 2x < p_{j+H+1} < \cdots \\ < p_{j+H+E} \leqq 4x. \end{aligned} \tag{18}$$

Since we assume that (17) is false, we have

$$p_{j+2} - p_{j+1} \leqq p_{j+3} - p_{j+2} < \cdots < p_{j+H+E} - p_{j+H+E-1}. \tag{19}$$

We evidently have

$$p_{j+r+1} - p_{j+r} < (3/2)\log x \qquad \text{for } r \leqq H. \tag{20}$$

For if not, then, by (19), $\pi(4x)-\pi(2x)=E<(4x/3\log x)+1$ which contradicts the prime number theorem.

Put

$$S(\gamma) = \sum_{\nu=j+1}^{j+H} e^{2\pi i\gamma p_\nu}$$

where $(40\log x)^{-1}<\gamma<(12\log x)^{-1}$. We have by (20)

$$\frac{1}{20\log x} < \gamma(p_{\nu+1} - p_\nu) < \frac{1}{2}\,.$$

Thus from (16)

$$|S(\gamma)| < 20\log x. \tag{21}$$

Let q be any prime satisfying $12\log x<q<40\log x$ (such a q exists for sufficiently large x). We evidently have

$$\left|S\left(\frac{1}{q}\right)\right| = \sum_{l=1}^{q-1} e^{2\pi i l/q}\sum{}'1 \tag{22}$$

where the prime indicates that the summation is extended over the $p_\nu \equiv l \pmod{q}$ with $j+1 \leqq \nu \leqq j+H$. We have by (15)

$$\left|\sum{}'1 - \frac{1}{q-1}\int_{p_{j+1}}^{p_{j+H}} \frac{dy}{\log y}\right| < \alpha_3 x \exp\left(-\alpha_4\frac{(\log x)^{1/2}}{(\log\log x)^2}\right). \tag{23}$$

Thus from (23) and (22)

$$\left|S\left(\frac{1}{q}\right) - \frac{1}{q-1}\int_{p_{j+1}}^{p_{j+H}} \frac{dy}{\log y}\right| < \alpha_3 x q \exp\left(-\alpha_4 \frac{(\log x)^{1/2}}{(\log\log x)^2}\right).$$

We have from the prime number theorem $p_{j+1} < 1.01x$, $p_{j+H} > 1.99x$. Thus

$$\begin{aligned}\left|S\left(\frac{1}{q}\right)\right| &> \frac{1}{40\log x}\int_{1.01x}^{1.99x}\frac{dy}{\log y} \\ &\qquad - \alpha_5 x\log x\exp\left(-\alpha_4\frac{(\log x)^{1/2}}{(\log\log x)^2}\right) > \frac{1}{80}\frac{x}{(\log x)^2},\end{aligned} \tag{24}$$

which contradicts (20) and proves (16). Hence (2) follows immediately since, as remarked in the introduction, $p_{k+1} - p_k > p_k - p_{k-1}$ has (trivially) infinitely many solutions.

4. **Problems and conjectures.** In connection with the Riemann hypothesis the question arose how often the expression

$$p_{k-1}p_{k+1} - p_k^2$$

changes its sign. We can show by using Brun's method that for $k \leqq n$

$$\left(\frac{p_{k-1}^t + p_{k+1}^t}{2}\right)^{1/t} - p_k \tag{25}$$

changes its sign cn times (as remarked before if $t = 0$, (25) becomes $p_{k-1}p_{k+1} - p_k^2$).

The inequalities (2) can be stated as follows: The inequalities

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} > 1 \quad\text{and}\quad \frac{p_{n+1} - p_n}{p_n - p_{n-1}} < 1$$

have infinitely many solutions. By Brun's method we can show that

$$\limsup \frac{p_{n+1} - p_n}{p_n - p_{n-1}} > 1 \quad\text{and}\quad \liminf \frac{p_{n+1} - p_n}{p_n - p_{n-1}} < 1.$$

It is very probable that the lim sup is infinite and the lim inf is 0.

(2) can be generalized as follows: Let $\sum_{k=1}^{n} a_k x_k$ be any linear form. What is the necessary and sufficient condition that both inequalities

$$\sum_{k=1}^{n} a_k x_k > 0 \quad\text{and}\quad \sum_{k=1}^{n} a_k x_k < 0 \tag{26}$$

have infinitely many solutions in consecutive primes $p_{m+1}, \cdots, p_{m+n}$? From the prime number theorem we obtain the necessary condition $\sum_{k=1}^{n} a_k = 0$. But $x_2 - x_1$ shows that this condition is not sufficient. Pólya remarked that if (25) has infinitely many solutions we can not have $a_1 \geqq 0$, $a_1 + a_2 \geqq 0, \cdots, a_1 + a_2 + \cdots + a_n \geqq 0$. The characterization of the forms which satisfy (26) seems a difficult problem.

Finally we mention two more questions:

(1) Can the inequalities $p_{n+1} - p_n < p_{n+2} - p_{n+1} < \cdots < p_{n+k} - p_{n+k-1}$ have infinitely many solutions for every fixed k?

(2) Is it true that the number of solutions of $p_{k+1} - p_k > p_k - p_{k-1}$, $k \leqq n$ is $n/2 + o(n)$? As we already have stated we can show that the number of solutions in question is between $c_1 n$ and $(1 - c_1)n$.

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ON MERSENNE'S NUMBER M_{227} AND COGNATE DATA

H. S. UHLER

When p equals one of the 55 primes 2, 3, 5, $\cdots$, 257 then, strictly speaking, $M_p = 2^p - 1$ is called a Mersenne number. To obtain a clear perspective of the history of this special subject the reader may consult the interesting accurate paper by R. C. Archibald.[1] Without any superior value of p, it has been shown by E. Lucas that the prime or composite character of a number of the form $2^p - 1$ (p prime) may be investigated by employing the sequence 3, 7, 47, 2207, $\cdots$ when p is of the form $4n - 1$, and the sequence 4, 14, 194, 37634, $\cdots$ when $p = 4n + 1$. In both cases the law of formation of the terms is $s_k = s_{k-1}^2 - 2$. However, it is no longer necessary to use the $4n - 1$ Lucasian series since D. H. Lehmer[2] stated and proved the following theorem: "*The number* $N = 2^n - 1$, *where* n *is an odd prime, is a prime if, and only if,* N *divides the* $(n-1)$*st term of the series*

$$S_1 = 4, \ S_2 = 14, \ S_3 = 194, \cdots, S_k, \cdots,$$

where $S_k = S_{k-1}^2 - 2$." This justifies the use by the present writer of the second progression although 227 falls in the $4n - 1$ class.

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[1] R. C. Archibald, *Mersenne's numbers*, Scripta Mathematica vol. 3 (1935) pp. 112–119.

[2] D. H. Lehmer, *On Lucas's test for the primality of Mersenne's numbers*, J. London Math. Soc. vol. 9–10 (1934–1935) pp. 162–165.

On June 4th, 1947 the writer finished calculating the 226th remainder of the Lucasian sequence 4, 14, 194, $\cdots$ as applied to the 69-digit number $2^{227}-1=2156$ 79573 33720 51183 57336 12069 61570 45389 09715 53803 24579 84882 88819 93727. The result was $r_{226}=1071$ 24133 67508 18344 33653 76892 98433 16050 93930 31886 49512 37078 23311 35950. Since this residual is not zero and since the calculations were performed with extreme care it follows that M_{227} is composite.

During the entire course of the work each arithmetical operation was checked with the auxiliary moduli 10^5+1 and 10^8+1. After the date given above all of the work-strips of the whole set were again examined and checked with the modulus 10^6+1, with a different computing machine, and in conformity with the formula of succession $r_{k-1}^2=M_{227}q_k+r_k+2$.

In addition to the theoretical probability of error indicated by the product of the three independent moduli 10^5+1, 10^6+1 and 10^8+1, the reliability of the writer's method and work has been subjected to an a posteriori acid test in each of two instances. In a letter dated September 12th, 1946, Professor Lehmer informed me that he had discovered[3] ("this July 4th weekend") the factor 2349023 for M_{167} and the factor 1504073 for M_{229}. He then wrote: "It might be interesting now to try to verify your . . . final residue in each of these tests by computing Lucas' series modulis 2349023 and 1504073 and then see if the results are the same as those obtained from casting these numbers out of your final remainders." This friendly suggestion was followed by me with the results that the two 166th residues were congruent to 2160517 (mod 2349023), and the two 228th remainders were identical with the value 465373. The heretofore intentionally unpublished value of r_{166} is 59077 89471 97183 05021 04043 18653 76339 69475 17591 49076.

As explained in an earlier paper[4] the essential figures of each of the terms above the 8th of the specified Lucasian sequence were multiplied in order by the reciprocal of the chief modulus, M_{227}, in preference to direct division by M_{227}. The approximation to this reciprocal was computed to be $(1/M_{227})_a=0.$(68 zeros) 46365 07688 35927 67321 64669 07693 45493 91709 45597 34472 38753 06629 88236 46998 17232 30809 13430 64583 44938 95187 64723 82742 71 $\cdots$. These figures were derived at once from the writer's earlier trustworthy value of $(1/M_{229})_a$ and the obvious relation $M_{227}^{-1}=4M_{229}^{-1}+12M_{229}^{-2}+\cdots$. Because $(1/M_{227})_a$ was to be used 218 times as multiplier, the value just given was verified twice by multiplication by the exact value of M_{227}, once using octad segregation of the digits and again with nonad grouping. It was proved that the last figure (1) recorded for $(1/M_{227})_a$ is too small by about 0.04 unit.

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[3] D. H. Lehmer, *On the factors of* $2^n\pm1$, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 164–167.

[4] H. S. Uhler, *First proof that the Mersenne number* M_{157} *is composite*, Proc. Nat. Acad. Sci. U. S. A. vol. 30 (1944) pp. 314–316.

For future investigations it may be appropriate to record in this place the value of the tenth term of the sequence 4, 14, 194, $\cdots$ as carefully computed by the author. $s_{10}=s_9^2-2=$ 687 29682 40664 42772 38837 48623 17475 30924 24715 41086 46671 75219 26185 83088 48740 57909 57964 73288 30691 02561 04343 67796 63935 59517 20423 57306 59491 63446 06074 56471 28680 78287 60805 52030 24658 35943 90175 80883 91097 86661 85875 71741 55410 84494 92650 04751 67381 16850 59273 78181 89975 38392 60609 45226 53652 74850 90187 98812 03714. This term has 293 digits and it would be applicable to all odd primes less than and inclusive of $p=971$ in $2^p-1=M_p$. If the speed of present or future electronic computing machines should cause the operators to run out of problems it might be worth while to apply one of the machines to testing the character of M_{971} where $M_{971}+1=2^{971}=$ 199 58403 09534 71981 16563 72713 03683 85660 67451 26043 54575 41502 54724 24372 11891 86896 40657 84957 96549 26357 01089 34244 68441 92495 24397 24379 88393 59366 07391 71798 28483 14203 20005 67295 10856 76517 53772 14443 62987 18265 33567 44543 92399 33308 10455 12087 03888 88855 26844 80441 57507 12090 68757 56041 64235 84952 30344 00992 78848. The accuracy of this power of 2 may be inferred from the following quotation of a sentence in a very recent letter from Dr. John W. Wrench, Jr. "Your value of 2^{971} has been collated with mine on several occasions in the past two days, and agreement is perfect."

The writer is now engaged in applying the sequence 4, 14, 194, $\cdots$ to the investigation of the sole remaining doubtful M_p within Mersenne's range of surmise ($p<263$), namely $2^{193}-1$.

YALE UNIVERSITY

RECURSIVE PROPERTIES OF TRANSFORMATION GROUPS. II

W. H. GOTTSCHALK

The purpose of this note is to sharpen a previous result on the transmission of recursive properties of a transformation group to certain of its subgroups. [See *Recursive properties of transformation groups*, by W. H. Gottschalk and G. A. Hedlund, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 637–641.]

Let T be a multiplicative topological group with identity e. A subset R of T is said to be *relatively dense* provided that $T=RK$ for some compact set K in T.

LEMMA 1. *If R is a relatively dense closed semi-group* ($RR\subset R$) *in T, then R is a subgroup of T.*

PROOF. Suppose $r\in R$ and U is a neighborhood of e. It is sufficient to show that $r^{-1}U\cap R\neq\emptyset$. Let V be a neighborhood of e for which $VV^{-1}\subset U$ and let K be a compact set in T for which $T=RK$. There exists a finite collection F of right translates of V which covers K. Choose $k_0\in K$. Now $r^{-1}k_0=r_1k_1$ for some $r_1\in R$ and some $k_1\in K$. Again $r^{-1}k_1=r_2k_2$ for some $r_2\in R$ and some $k_2\in K$. This may be continued. Thus there exist sequences $k_0, k_1, \cdots$ in K and $r_1, r_2, \cdots$ in R such that $r^{-1}k_i=r_{i+1}k_{i+1}$ $(i=0, 1, \cdots)$. Select integers m and n $(0\leqq m<n)$ and an element V_0 of F such that $k_m, k_n\in V_0$. Now $r^{-1}k_mk_n^{-1}$ $=(r^{-1}k_mk_{m+1}^{-1})(k_{m+1}k_{m+2}^{-1})\cdots(k_{n-1}k_n^{-1})=r_{m+1}rr_{m+2}\cdots rr_n\in R$. Also $r^{-1}k_mk_n^{-1}\in r^{-1}V_0V_0^{-1}\subset r^{-1}VV^{-1}\subset r^{-1}U$. Hence $r^{-1}U\cap R\neq\emptyset$ and the proof is completed.

Now let T act as a transformation group on a topological space X. That is to say, suppose that to $x\in X$ and $t\in T$ is assigned a point, denoted xt, of X such that: (1) $xe=x$ $(x\in X)$; (2) $(xt)s=x(ts)$ $(x\in X;\ t, s\in T)$; (3) The function xt defines a continuous transformation of $X\times T$ into X. We assume for the remainder of the paper that x is a fixed point of X, T is locally compact and S is a relatively dense invariant subgroup of T. Let Σ denote the maximal subset of T for which $x\Sigma\subset(xS)^*$ where the star denotes the closure operator.

LEMMA 2. *The set Σ is a closed subgroup of T which contains S.*

PROOF. Obviously $\Sigma\supset S$. From $x\Sigma^*\subset(x\Sigma)^*\subset(xS)^*$ we conclude that Σ is closed. By Lemma 1 it is now enough to show that Σ is a

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semi-group. Suppose σ, $\tau \in \Sigma$. From $x\sigma \in (xS)^*$ it follows that $x\sigma\tau \in (xS)^*\tau \subset (xS\tau)^* \subset (x\tau S)^*$. From $x\tau \in (xS)^*$ it follows that $x\tau S \subset (xS)^*S \subset (xSS)^* \subset (xS)^*$. Hence $x\sigma\tau \in (xS)^*$. Thus $\sigma\tau \in \Sigma$ and the proof is completed.

LEMMA 3. *If W is a neighborhood of e, then $x \notin (x[T-\Sigma W])^*$.*

PROOF. We first show that if $t \in T-\Sigma$, then $x \notin (x\Sigma V_0)^*$ for some neighborhood V_0 of t. Suppose $t \in T-\Sigma$. Since $t^{-1} \notin \Sigma$ by Lemma 2, $xt^{-1} \notin (x\Sigma)^*$ and $x \notin (x\Sigma t)^*$. There are neighborhoods U of x and V of e such that $V=V^{-1}$ and $UV \cap x\Sigma t = \emptyset$. It follows that $U \cap x\Sigma tV = \emptyset$. Define $V_0 = tV$.

We may assume W is open. Define $N = K-\Sigma W$ where K is a compact set in T such that $T=SK$. Using Lemma 2 we conclude that $T=SK \subset S(N \cup \Sigma W) \subset SN \cup S\Sigma W \subset \Sigma N \cup \Sigma W$ and $\Sigma N \cap \Sigma W = \emptyset$. Hence $T-\Sigma W = \Sigma N$. By the preceding paragraph, to each $n \in N$ there corresponds a neighborhood V_n of n such that $x \notin (x\Sigma V_n)^*$. Since finitely many of the V_n cover N, $x \notin (x\Sigma N)^*$. The proof is completed.

LEMMA 4. *If U is a neighborhood of x, then there exists a compact set M in T such that $xM \subset U$ and $\Sigma \subset SM^{-1}$.*

PROOF. Define $N = K \cap \Sigma$ where K is a compact set in T such that $T=SK$. If $n \in N$, then $xn \in (xS)^*$ and $x \in (xSn^{-1})^*$. Thus $n \in N$ implies the existence of $s_n \in S$ such that $xs_nn^{-1} \in \mathit{int}\,U$ and hence the existence of a compact neighborhood W_n of s_nn^{-1} such that $xW_n \subset U$. Since N is compact by Lemma 2, there is a finite subset F of N for which $N \subset \bigcup_{n \in F} W_n^{-1}s_n$. Define $M = \bigcup_{n \in F} W_n$. Clearly $xM \subset U$. Using Lemma 2 we conclude that $\Sigma \subset SN \subset SM^{-1}$. The proof is completed.

Let there be distinguished in T certain sets, called *admissible*, which satisfy this condition: If A is an admissible set and if B is a set in T such that $A \subset BK$ for some compact set K in T, then B is an admissible set. A subgroup R of T is said to be *recursive* at x provided that to each neighborhood U of x there corresponds an admissible set A such that $A \subset R$ and $xA \subset U$.

LEMMA 5. *If T is recursive at x, then Σ is recursive at x.*

PROOF. Let U be a neighborhood of x. There are neighborhoods V of x and W of e such that $W=W^{-1}$, W is compact and $VW \subset U$. By Lemma 3 we may suppose that $V \cap x(T-\Sigma W) = \emptyset$. There exists an admissible set A in T such that $xA \subset V$. Clearly $A \subset \Sigma W$ and $xAW \subset U$. Define $B = \Sigma \cap AW$. Since $A \subset BW$, B is an admissible set. Also $B \subset \Sigma$ and $xB \subset U$. The proof is completed.

LEMMA 6. *If Σ is recursive at x, then S is recursive at x.*

PROOF. Let U be an open neighborhood of x. By Lemma 4 there exists a compact set M in T such that $xM\subset U$ and $\Sigma\subset SM^{-1}$. Let V be a neighborhood of x for which $VM\subset U$. There exists an admissible set A such that $A\subset\Sigma$ and $xA\subset V$. Hence $xAM\subset U$. Define $B=S\cap AM$. Since $A\subset BM^{-1}$, B is an admissible set. Also $B\subset S$ and $xB\subset U$. The proof is completed.

The following theorem is an immediate consequence of Lemmas 5 and 6.

THEOREM. *If T is recursive at x, then S is recursive at x.*

An interpretation of admissibility arises if we define an admissible subset of T to be a relatively dense subset of T. The term "recursive" is then replaced by "almost periodic." For other applications, see the paper cited above.

UNIVERSITY OF PENNSYLVANIA

FIXED POINT THEOREMS FOR INTERIOR TRANSFORMATIONS

O. H. HAMILTON

If M is a bounded continuum in a Euclidean plane E which does not separate E and T is an interior continuous transformation of M onto a subset of E which contains M, does T leave a point of M invariant? It is the purpose of this paper to answer this question in the affirmative for certain types of locally connected continua.

Using a notation introduced by Eilenberg [2, p. 168][1] a continuum M will be said to have property (b) provided every continuous transformation of M into the unit circle S in the Cartesian plane, with center at o, is homotopic to a constant mapping, that is, a transformation which transforms each point of M into a single point of S. If T is a continuous transformation of a subset A of the plane E into a subset B of E, then for each point x of A let $T'(x)$ be the point y of S such that the directed line segment oy is parallel in direction and sense to the directed line segment x, $T(x)$. Then T' will be referred to as the transformation of A into S derived from T. Such a transformation

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[1] Numbers in brackets refer to the references cited at the end of the paper.

has often been considered in the study of fixed point properties of continua as for instance by Alexandroff and Hopf [1, p. 479]. If T is a continuous transformation of a subset A of E into a subset B of E, A will be said to have property (b′) with respect to T if T leaves no point of A invariant and the transformation T' of A into S derived from T is homotopic to a constant mapping.

LEMMA 1. *There does not exist a bounded plane continuum M which has property* (b′) *with respect to a continuous interior transformation T of M onto a topological 2-cell I which contains M.*

PROOF. Suppose the lemma is not true and that there does exist such a bounded plane continuum M. Without loss of generality we may take I to be the circular 2-cell in the Cartesian plane with center at o, radius 1, and boundary S. Let T' designate the transformation of M into S derived from T. By assumption, M has property (b′) with respect to T and hence T' is homotopic to a constant mapping. By a theorem of Eilenberg [2, p. 168, Theorem 1′] there exists a continuous real valued function Φ which transforms M into a subset of R, the set of real numbers, such that for each point x of M, $T'(x)$ is the point $e^{i\Phi(x)}$. Since M is closed and bounded, $\Phi(M)$ is a bounded and closed subset of R. For each point x of I, let X designate the set consisting of all real numbers $\Phi(y)$, for y belonging to $T^{-1}(x)$, and let $\theta(x)$ be the greatest lower bound of the set X. Since T is an interior continuous transformation, it follows by a lemma of Eilenberg [2, p. 174] that T^{-1} is a continuous multi-valued transformation of I onto M, and hence that θ is a continuous transformation of I into R. For each point x of I let xs designate the directed half line with initial point at x which is parallel in direction and opposite in sense to the directed line segment oy, where y is the point $e^{i\theta(x)}$, and let $T_1(x)$ be the first point on xs distinct from x which xs has in common with S. We know $T_1(x)$ exists for each point x in I since M is a subset of I and since, from the continuity of θ and T^{-1} and the assumption that T leaves no point of M invariant, it follows that there is a point of M distinct from x lying on xs and hence a point of S distinct from x lying on xs. Thus, since θ is a continuous transformation of I into R, it follows that T_1 is a continuous transformation of I into the boundary of I which leaves no point of I invariant. But this contradicts the Brouwer fixed point theorem for 2-cells. The lemma is therefore true.

THEOREM 1. *If T is an interior continuous transformation of a locally connected unicoherent bounded plane continuum M onto a topological 2-cell which contains M, then T leaves a point of M invariant.*

PROOF. For suppose T leaves no point of M invariant. Then the continuous transformation T' of M into S derived from T exists and, by a theorem of Eilenberg [2, p. 168, Theorem 6′], since M is unicoherent and locally connected, M has property (b), and hence has property (b′) with respect to T. But this contradicts Lemma 1.

COROLLARY. *If T is an interior continuous transformation of a bounded locally connected plane continuum which does not separate the plane onto a topological 2-cell which contains M, then T leaves a point of M invariant.*

PROOF. For a plane continuum which does not separate the plane is unicoherent.

THEOREM 2. *If T is an interior continuous transformation of a topological 2-cell I onto a continuum M which contains I, then T leaves a point of I invariant.*

PROOF. Suppose T leaves no point of I invariant. Then the derived transformation T' exists and, since I is unicoherent, it follows by the previously cited theorem of Eilenberg that I has property (b) and hence has property (b′) with respect to T. But T transforms some subset N of I onto I. Then N has property (b′) with respect to T. But this contradicts Lemma 1.

It is easy to construct an example to show that Theorem 2 would not be true if the condition that T be interior were omitted from the hypothesis of the theorem.

Applications of the theorems of this paper to the theory of functions of a complex variable are suggested by the fact that a function of a complex variable, analytic in a region R, defines an interior continuous transformation of R into a subset of the complex plane.

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OKLAHOMA AGRICULTURAL AND MECHANICAL COLLEGE

AN EXTENSION OF ALEXANDROFF'S MAPPING THEOREM

C. H. DOWKER

1. **Introduction.** Alexandroff's fundamental mapping theorem (Überführungssatz) is a basic tool of combinatorial topology. By its use, many mapping theorems for rather general topological spaces can be shown to be consequences of the corresponding theorems for polytopes. It is especially useful in proving imbedding theorems and approximation theorems.

The chief purpose of this paper is to determine the precise conditions under which this fundamental theorem holds. It will be shown that the theorem holds in full generality, that is, for all coverings, if and only if the space is both paracompact and normal. If the space is normal, the theorem holds for all coverings which have locally finite refinements and for no others.

2. **Terminology.** The mapping theorem of Alexandroff concerns mappings of a space into the nerve of a covering.

By a *mapping* we mean a continuous transformation. By a *space* we mean a topological space, in general not satisfying any separation axiom. By a *covering* we mean a covering of the space by a finite or infinite collection of open sets.

The *nerve* of a covering $\mathfrak{U}$: $\{U_\alpha\}$ is a simplicial polytope, with vertices u_α in 1–1 correspondence with the nonempty sets U_α of the covering, such that $u_\alpha, u_\beta, \cdots, u_\gamma$ are vertices of a simplex of the nerve if and only if the corresponding sets $U_\alpha, U_\beta, \cdots, U_\gamma$ have a common point. We assume that the nerve is realized as a topological space in one of the following ways. The *natural nerve* $N(\mathfrak{U})$ is the nerve realized with the natural metric: $\rho(x, y) = (\sum(x_\alpha - y_\alpha)^2)^{1/2}$, where x_α, y_α are barycentric coordinates of x and y. The *geometric nerve* $G(\mathfrak{U})$ is the nerve realized with the geometric topology of Lefschetz [10, p. 9]:[1] the stars of the vertices of repeated regular subdivisions form a basis for the open sets of $G(\mathfrak{U})$. It is known [12] that $G(\mathfrak{U})$ is a metrizable space. The natural and geometric topologies coincide if and only if the nerve is locally of finite dimension [11, footnote 4].

Following Dieudonné, we call a covering $\mathfrak{U}$ of a space R *locally finite*[2]

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Locally finite = neighborhood-finite. "Locally finite" has been used by Lefschetz to mean star-finite.

if each point of R has a neighborhood meeting only a finite number of sets of $\mathfrak{U}$. A covering $\mathfrak{V}$ is called a refinement of $\mathfrak{U}$ if every set of $\mathfrak{V}$ is contained in some set of $\mathfrak{U}$. A space R is called *paracompact* (see [4]) if every covering of R has a locally finite refinement. A covering of a space R is called *point-finite* if each point of R is contained in only a finite number of sets of the covering. Clearly, every locally finite covering is point-finite.

A mapping f of a space R into the nerve of a covering $\mathfrak{U}$ is called *canonical* with respect to $\mathfrak{U}$ if the inverse image of the star[3] of each vertex of the nerve is contained in the corresponding set of $\mathfrak{U}$; in symbols: $f^{-1}(\mathrm{St}u_\alpha) \subset U_\alpha$. If $\mathfrak{U}$ is point-finite, the finite collection of sets U_α containing a point p of R correspond to the vertices u_α of a simplex in the nerve which we call the simplex $\sigma(p)$ determined by p. It can be shown [5, p. 202] that a mapping f of R into the nerve of a point-finite covering $\mathfrak{U}$ is canonical with respect to $\mathfrak{U}$ if and only if each point p of R is mapped into the closure of the simplex determined by p, that is, $f(p) \in \bar{\sigma}(p)$. Another equivalent formulation in the case of point-finite coverings is given by Lefschetz [10, p. 40].

3. **Sufficient conditions.** The mapping theorem of Alexandroff, as modified by Kuratowski and Lefschetz, states the existence, under certain conditions, of canonical mappings of a space into the nerve of a covering.

(a) *If* $\mathfrak{U}$ *is a locally finite covering of a normal space* R *there is a canonical mapping of* R *into the natural nerve of* $\mathfrak{U}$.

This form of Alexandroff's theorem is proved in [5, Theorem 1.1]. Alternatively, the proof by Lefschetz [10, pp. 45–46] for star-finite coverings can easily be extended to locally finite coverings by using Dieudonné's theorem [4, Theorem 6] that every point-finite covering $\{U_\alpha\}$ of a normal space can be shrunk to a covering $\{V_\alpha\}$ such that, for each α, the closure of V_α is contained in U_α.

(b) *If a covering* $\mathfrak{U}$ *of a normal space* R *has a locally finite refinement there is a canonical mapping of* R *into the natural nerve of* $\mathfrak{U}$.

PROOF. Let $\mathfrak{V}:\{V_\beta\}$ be a locally finite refinement of $\mathfrak{U}:\{U_\alpha\}$. For each V_β choose one of the sets U_α such that $V_\beta \subset U_\alpha$. Each V_β thus corresponds to a unique set U_α containing it. Let W_α be the union of all the sets of $\mathfrak{V}$ which correspond to U_α. Each V_β is in some W_α; hence $\mathfrak{W}:\{W_\alpha\}$ is a covering of R. Since $\mathfrak{V}$ is locally finite, so is $\mathfrak{W}$. If $(w_\alpha, w_\beta, \cdots, w_\gamma)$ is a simplex of $N(\mathfrak{W})$, there is a point

$$p \in W_\alpha W_\beta \cdots W_\gamma \subset U_\alpha U_\beta \cdots U_\gamma,$$

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[3] The star of a vertex v is the union of all simplexes having v as a vertex. The star of a vertex is an open set in either topology.

and hence $(u_\alpha, u_\beta, \cdots, u_\gamma)$ is a simplex of $N(\mathfrak{U})$. Thus there is a simplicial mapping π of $N(\mathfrak{W})$ into $N(\mathfrak{U})$ which maps each vertex w_α into the corresponding vertex u_α. Clearly π is a 1–1 mapping of $N(\mathfrak{W})$ onto a subpolytope of $N(\mathfrak{U})$. If w_α exists, $\pi^{-1}\mathrm{St}u_\alpha=\mathrm{St}w_\alpha$; otherwise $\pi^{-1}\mathrm{St}u_\alpha=0$. By (a) there is a canonical mapping g of R into $N(\mathfrak{W})$. Then, if $f=\pi g$, f is a mapping of R into $N(\mathfrak{U})$. If w_α exists,

$$f^{-1}\,\mathrm{St}\,u_\alpha=g^{-1}\pi^{-1}\,\mathrm{St}\,u_\alpha=g^{-1}\,\mathrm{St}\,w_\alpha\subset W_\alpha\subset U_\alpha.$$

Otherwise, $f^{-1}\,\mathrm{St}\,u_\alpha=0\subset U_\alpha$. Hence f is a canonical mapping of R into $N(\mathfrak{U})$.

(c) *If* $\mathfrak{U}$ *is an arbitrary covering of a paracompact normal space* R, *there is a canonical mapping of* R *into the natural nerve of* $\mathfrak{U}$.

PROOF. Since R is paracompact, $\mathfrak{U}$ has a locally finite refinement. Hence, by (b), a canonical mapping exists.

4. Necessary conditions; main theorems.

We pass now to consideration of some necessary conditions for the existence of canonical mappings.

(d) *If there exists a canonical mapping* f *of a space* R *into the natural nerve of a covering* $\mathfrak{U}$, *then* $\mathfrak{U}$ *has a locally finite refinement.*

PROOF. The natural complex $N(\mathfrak{U})$ can be shown to be a paracompact normal (metric) space [6, Theorem 2]. Hence, the covering $\{\mathrm{St}\,u_\alpha\}$ of $N(\mathfrak{U})$ by the stars of its vertices has a locally finite refinement $\{V_\beta\}$. Then $\{f^{-1}V_\beta\}$ is a locally finite covering of R. Since V_β is contained in some $\mathrm{St}\,u_\alpha$, $f^{-1}V_\beta\subset f^{-1}\,\mathrm{St}\,u_\alpha\subset U_\alpha$. Hence $\{f^{-1}V_\beta\}$ is a refinement of $\mathfrak{U}$.

(e) *Let* R *be a space such that for every covering* $\mathfrak{U}$ *of* R *there is a canonical mapping of* R *into the natural nerve of* $\mathfrak{U}$. *Then* R *is a paracompact normal space.*

PROOF. By (d), the existence of a canonical mapping into $N(\mathfrak{U})$ implies that $\mathfrak{U}$ has a locally finite refinement. Hence every covering $\mathfrak{U}$ of R has a locally finite refinement, that is, R is paracompact. Since, in particular, there is a canonical mapping of R into the nerve of each covering by two open sets, R is normal (see [10, p. 45]).

THEOREM 1. *There exists a canonical mapping of a normal space* R *into the natural nerve of a covering* $\mathfrak{U}$ *if and only if* $\mathfrak{U}$ *has a locally finite refinement.*

PROOF. This follows immediately from (b) and (d).

THEOREM 2. *There exists a canonical mapping of a space* R *into the natural nerve of an arbitrary covering if and only if* R *is paracompact and normal.*

PROOF. This follows immediately from (c) and (e).

5. **Dimension.** Alexandroff's original theorem was partly, and indeed mainly, a theorem of dimension theory. The dimension-theoretic content of the extended (and modified) theorem is indicated by Theorem 3.

The ***dimension*** of a normal space R, dim R, is defined as follows: dim $R \leqq n$ means that every finite covering of R has a finite refinement of order not greater than $n+1$.

Let K be a simplicial polytope. Let σ^n be an n-simplex of K, $\bar{\sigma}^n$ its closure, and S^{n-1} its boundary. If f maps R into K, let $A = f^{-1}S^{n-1}$ and let $f|A$ be the mapping f restricted to A. If $f|A$ cannot be extended to a mapping of $f^{-1}\bar{\sigma}^n$ into S^{n-1}, the mapping f of R into K is called ***essential*** in the closed simplex $\bar{\sigma}^n$. Thus, if f is essential in $\bar{\sigma}^n$, $f|A$ cannot be extended to a mapping of R ($\supset f^{-1}\bar{\sigma}^n$) into S^{n-1}. It can be seen that a mapping f of R into K is essential in $\bar{\sigma}^n$ if and only if, for some subset B of R, $f|B$ is essential in $\bar{\sigma}^n$ and $f(B) \subset \bar{\sigma}^n$.

THEOREM 3. *If* $\mathfrak{U}$ *is a covering of a normal space* R *and if any canonical mapping of* R *into* $N(\mathfrak{U})$ *exists, then, for some canonical mapping* f *of* R *into* $N(\mathfrak{U})$, *the image* $f(R)$ *is a subpolytope* K *of* $N(\mathfrak{U})$, dim $K \leqq$ dim R, *and* f *is essential in every closed simplex of* K.

PROOF. By (d), the existence of a canonical mapping of R into $N(\mathfrak{U})$ implies that $\mathfrak{U}$ has a locally finite refinement $\mathfrak{V}$. It can be shown [5, Lemma 3.4] that every locally finite covering $\mathfrak{V}$ of a normal space R has a locally finite refinement $\mathfrak{W}: \{W_\beta\}$ such that there is a mapping ϕ of R onto $N(\mathfrak{W})$ with the following properties: ϕ is essential in every closed simplex of $N(\mathfrak{W})$ and ϕ^{-1} St $w_\beta = W_\beta$ for every β.

Since $\mathfrak{W}$ is a refinement of $\mathfrak{U}$ there exists a simplicial transformation[4] π of $N(\mathfrak{W})$ into $N(\mathfrak{U})$ which maps w_β into u_α where U_α is a set of $\mathfrak{U}$ containing W_β. We may assume that π is continuous on each finite subpolytope of $N(\mathfrak{W})$. Since $\mathfrak{W}$ is locally finite, each point $p \in R$ has a neighborhood U meeting only a finite number of sets of $\mathfrak{W}$, and hence $\phi(U)$ is contained in a finite subpolytope of $N(\mathfrak{W})$. Then $\pi\phi$ maps a neighborhood of an arbitrary point p continuously. Hence $\pi\phi$ is continuous. Let $f = \pi\phi$.

If $\pi\phi(p) \in$ St u_α, then $\phi(p) \in$ St w_β for some w_β mapped into u_α. Hence, $p \in \phi^{-1}$ St $w_\beta = W_\beta \subset U_\alpha$. Thus f^{-1} St $u_\alpha \subset U_\alpha$. Therefore, f is a canonical mapping.

Let K be the image polytope $\pi N(\mathfrak{W}) \subset N(\mathfrak{U})$. Since $\phi(R) = N(\mathfrak{W})$, $f(R) = \pi\phi(R) = \pi N(\mathfrak{W}) = K$. Thus f is a mapping of R onto K.

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[4] See, for example [5, footnote 13].

Let σ^n be an n-simplex of K. Since $\pi N(\mathfrak{W}) = K$, there is some simplex σ^m of $N(\mathfrak{W})$ such that $\pi\sigma^m = \sigma^n$. There is some n-face σ_1^n of σ^m which is mapped onto σ^n. (If $m = n$, $\sigma_1^n = \sigma^m$.) Since ϕ is essential in every closed simplex of $N(\mathfrak{W})$, ϕ is essential in $\bar{\sigma}_1^n$ and hence there is a subset B of R such that $\phi(B) \subset \bar{\sigma}_1^n$ and $\phi | B$ is essential in $\bar{\sigma}_1^n$. Since π maps $\bar{\sigma}_1^n$ homeomorphically onto $\bar{\sigma}^n$, $\pi\phi | B$ is essential in $\bar{\sigma}^n$. Hence $\pi\phi$ is essential in $\bar{\sigma}^n$. It follows [5, Corollary 3.6] that dim $R \geqq n$; hence each simplex of K has dimension not greater than dim R. Thus $f = \pi\phi$ is essential in each closed simplex of K and dim $K \leqq$ dim R. This completes the proof.

6. Lefschetz' geometric nerve. It will now be shown that in any theorem on canonical mappings the geometric nerve can be replaced by the natural nerve, or the natural by the geometric.

THEOREM 4. *Let* $\mathfrak{U}$ *be a covering of a space* R. *There exists a canonical mapping of* R *into the geometric nerve* $G(\mathfrak{U})$ *if and only if there is a canonical mapping of* R *into the natural nerve* $N(\mathfrak{U})$.

PROOF. First, let f be a canonical mapping of R into $N(\mathfrak{U})$. Let the vertices of $N(\mathfrak{U})$ be $\{u_\alpha\}$ and let those of $G(\mathfrak{U})$ be $\{a_\alpha\}$. It is known [10, p. 21, (9.9)] that the barycentric mapping ϕ of $N(\mathfrak{U})$ onto $G(\mathfrak{U})$ which maps u_α onto a_α is continuous. Let $g = \phi f$. Then, since ϕ^{-1} St a_α = St u_α,

$$g^{-1} \operatorname{St} a_\alpha = f^{-1}\phi^{-1} \operatorname{St} a_\alpha = f^{-1} \operatorname{St} u_\alpha \subset U_\alpha.$$

Hence g is a canonical mapping of R into $G(\mathfrak{U})$.

Conversely, let g be any canonical mapping of R into $G(\mathfrak{U})$. The natural nerve of the covering $\{\operatorname{St} a_\alpha\}$ of $G(\mathfrak{U})$ by the stars of its vertices may be identified with $N(\mathfrak{U})$. It can be shown that $G(\mathfrak{U})$ is a paracompact normal (metrizable) space [6, Theorem 2]. Hence, by (c), there is a mapping ψ of $G(\mathfrak{U})$ into $N(\mathfrak{U})$ canonical with respect to the covering $\{\operatorname{St} a_\alpha\}$. Thus ψ^{-1} St $u_\alpha \subset$ St a_α. Let $f = \psi g$. Then

$$f^{-1} \operatorname{St} u_\alpha = g^{-1}\psi^{-1} \operatorname{St} u_\alpha \subset g^{-1} \operatorname{St} a_\alpha \subset U_\alpha.$$

Hence f is a canonical mapping of R into $N(\mathfrak{U})$.

Applying Theorem 4 to Theorems 1 and 2, we have:

COROLLARY 1. *There exists a canonical mapping of a normal space* R *into the geometric nerve of a covering* $\mathfrak{U}$ *if and only if* $\mathfrak{U}$ *has a locally finite refinement.*

COROLLARY 2. *There exists a canonical mapping of a space* R *into the geometric nerve of an arbitrary covering if and only if* R *is paracompact and normal.*

It likewise follows from Theorem 4 that the theorems of Lefschetz on canonical mappings into geometric nerves hold also for mappings into natural nerves. In particular, using his notion of "analytic" covering and his fundamental mapping theorem [**10**, p. 41], we have:

COROLLARY 3. *There exists a canonical mapping of a space R into the natural nerve of a point-finite covering $\mathfrak{U}$ if and only if $\mathfrak{U}$ is analytic.*

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TUFTS COLLEGE

ON MEAN VALUES

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Introduction. Historical. In 1930 Kolmogoroff and Nagumo[1] proved simultaneously a fundamental theorem on mean values. In their definition a mean value is an infinite sequence of functions: $M_1(x_1)=x_1$, $M_2(x_1,x_2)$, $M_3(x_1,x_2,x_3)$, $\cdots$, $M_n(x_1,\cdots,x_n)$, $\cdots$. Each function of this sequence has to satisfy the following conditions: $M_n(x,\cdots,x)=x$, $M_n(x_1,\cdots,x_n)$ must be a continuous, (strictly) increasing (cf. §2) and symmetric function. The terms of this sequence are connected by the "associative property": $(M_k=M_k(x_1,\cdots,x_k))$

$$M_n(x_1,\cdots,x_k,x_{k+1},\cdots,x_n)=M_n(M_k,\cdots,M_k,x_{k+1},\cdots,x_n)$$
$$(k\leqq n=1,2,3,\cdots).$$

The theorem of Kolmogoroff and Nagumo is that these conditions are necessary and sufficient for the existence of a continuous and (strictly) increasing function $f(x)$ by which the mean value can be written in the form ($f^{-1}(x)$ is the inverse function of $f(x)$):

$$(1)\quad M_n(x_1,\cdots,x_n)=f^{-1}\left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right)\ (n=1,2,3,\cdots).$$

In the next year de Finetti and Børge Jessen[2] extended this theorem for mean values of functions. De Finetti and Kitagawa[3] considered weighted means where besides the variables $x_1, x_2, \cdots, x_n$ also the weights $q_1, q_2, \cdots, q_n$ $(q_1+q_2+\cdots+q_n=1)$ were given and gave the conditions for the possibility of writing them in the form

$$(2)\quad M_n(x_1,\cdots,x_n;q_1,\cdots,q_n)=f^{-1}[q_1f(x_1)+\cdots+q_nf(x_n)]$$
$$(n=1,2,3,\cdots)$$

analogous to (1) $(q_1+\cdots+q_n=1)$.

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[1] A. Kolmogoroff, *Sur la notion de la moyenne*, Atti della R. Academia nazionale dei Lincei (6) vol. 12 (1930). M. Nagumo, *Über eine Klasse der Mittelwerte*, Jap. J. Math. vol. 7 (1930).

[2] B. de Finetti, *Sul concetto di media*, Giornale di Istituto Italiano di Attuarii vol. 2 (1931). B. Jessen, *Bemaerkninger om konvekse Funktioner og Uligheder imellem Middelvaerdier*, I; II, Matematisk Tidsskrift B (1931). Cf. also J. Horváth, *On a theorem of Børge Jessen*, Norske Videnskabers Selskabs Forhandlinger vol. 20 (1947).

[3] T. Kitagawa, *On some class of weighted means*, Proceedings of the Physico-Mathematical Society of Japan vol. 16 (1934).

Conditions by which a mean value defined for one definite number of variables can be written in the form (1) were posed first by Aumann[4] for the case when $M(x_1, \cdots, x_n)$ is an analytic function. His proof uses rather intricate considerations.

In §1 I intend to give necessary and sufficient conditions for the validity of (1) for means of one definite number of variables without supposing analyticity of $M(x_1, \cdots, x_n)$, only continuity and strict monotony as before. The property which stands instead of Kolmogoroff and Nagumo's "associativity" will be the "bisymmetry." This asserts that the function of n^2 variables

$$\psi_2(x_{11}, \cdots, x_{1n}; \cdots; x_{n1}, \cdots, x_{nn}) = M[M(x_{11}, \cdots, x_{1n}), \cdots, M(x_{n1}, \cdots, x_{nn})]$$

does not alter if we replace x_{ik} by x_{ki} and vice versa.

In §2 I show that if we drop the condition of symmetry the mean has the form (2); that is, without giving the weights we shall have the conditions by which a nonsymmetric mean is a weighted "Kolmogoroff-Nagumo mean." In §3 I try to show the importance of the condition "bisymmetry," for by dropping also the condition $M(x, \cdots, x) = x$ ("reflexivity"), that is, by considering (continuous, increasing and) bisymmetric functions $[x_1, \cdots, x_n]$ which are not means any more, we shall see that they have the form

$$[x_1, \cdots, x_n] = f^{-1}\{p_1 f(x_1) + \cdots + p_n f(x_n) + p\}. \tag{3}$$

To simplify our considerations we shall confine ourselves to mean values defined for *two* variables. This will mean no loss of generality.[5]

1. Symmetric means.

Definitions. We postulate the single-valued function of two variables $M(x, y)$, $\alpha \leqq x, y \leqq \beta$ (definitions and theorem can be extended without difficulty for open or half open intervals which can be infinite as well), to fulfill the following conditions:

(i) Strict monotony: if $x < x'$ then $M(x, y) < M(x', y)$ and the same for $y < y'$;

(ii) Continuity;

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[4] G. Aumann, *Aufbau von Mittelwerten mehrerer Argumente*, II. *Analytische Mittelwerte*, Math. Ann. vol. 110 (1935).

[5] For a detailed discussion in the general case of n variables for symmetric means, cf. J. Aczél, *The notion of mean values*, Norske Videnskabers Selskabs Forhandlinger vol. 19 (1946).—For special simplifications in the case $n = 2$ see J. Aczél, *On mean values and operations defined for two variables*, Norske Videnskabers Selskabs Forhandlinger vol. 20 (1947).

(iii) Bisymmetry: $\psi_2(x_{11}, x_{12}, x_{21}, x_{22}) = M[M(x_{11}, x_{12}), M(x_{21}, x_{22})] = M[M(x_{11}x_{21}), M(x_{12}, x_{22})]$;

(iv) Reflexivity: $M(x, x) = x$;

(v) Symmetry: $M(x, y) = M(y, x)$.

(i) and (iv) imply $x < M(x, y) < y$ if $x < y$ ("internness").

Generalizing (iii) we can define the "k-symmetry." We consider the following sequence of functions:

$$\psi_1(x_1, x_2) = M(x_1, x_2),$$
$$\psi_2(x_{11}, x_{12}, x_{21}, x_{22}) = M[M(x_{11}, x_{12}), M(x_{21}, x_{22})],$$
$$\psi_3(x_{111}, x_{112}, x_{121}, x_{122}, x_{211}, x_{212}, x_{221}, x_{222})$$
$$= M\{M[M(x_{111}, x_{112}), M(x_{121}, x_{122})], M[M(x_{211}, x_{212}), M(x_{221}, x_{222})]\},$$

. . . .

(ψ_k is a function of 2^k variables). $M(x, y)$ is "k-symmetric" if ψ_k does not alter by changing variables, the indices of which are permutations of each other. By applying (iii) repeatedly we see that *every bisymmetric function is k-symmetric.*

We call $M(x, y)$ "k-symmetric in the stronger sense" if the function ψ_k is symmetric in its 2^k variables. We can see that *every symmetric and bisymmetric function is k-symmetric in the stronger sense.*

THEOREM. *Conditions* (i), (ii), (iii), (iv), (v) *are necessary and sufficient for the existence of an increasing and continuous function* $f(x)(\alpha \leqq x \leqq \beta)$ *by which* $M(x, y)$ *has the form*

$$M(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right). \tag{4}$$

The necessity of the conditions is evident. We prove the sufficiency by constructing $f(x)$ resp. its inverse function $\phi(x) = f^{-1}(x)$. This function has to be increasing, continuous and has to satisfy the functional equation ($u = f(x)$, $v = f(y)$):

$$M[\phi(u), \phi(v)] = \phi\left(\frac{u + v}{2}\right). \tag{5}$$

We define $\phi(x)$ for the dyadic fractions as follows:

$$\begin{gathered} \phi(0) = r_0^{(0)} = \alpha, \qquad \phi(1) = r_1^{(0)} = \beta; \\ \phi(0) = r_0^{(1)} = M(\alpha, \alpha) = \alpha, \qquad \phi(1/2) = r_1^{(1)} = M(\alpha, \beta), \\ \phi(1) = r_2^{(1)} = M(\beta, \beta) = \beta; \end{gathered} \tag{6}$$

. . . . ;

$$
\begin{aligned}
\cdots,\ \phi\left(\frac{2q}{2^{k+1}}\right) &= r_{2q}^{(k+1)} = M(r_q^{(k)}, r_q^{(k)}) = r_q^{(k)}, \\
(6) \qquad \phi\left(\frac{2q+1}{2^{k+1}}\right) &= r_{2q+1}^{(k+1)} = M(r_q^{(k)}, r_{q+1}^{(k)}), \cdots;
\end{aligned}
$$

. . . .

$\phi(x)$ is increasing in consequence of (i).

Substituting the recursive formula (6) repeatedly in the expression of r_p^{k+1} we get finally a ψ_{k+1} in which only β's and α's figure as variables. We assert that the number of β's is exactly p. This can be proved by induction, because it is true for $k=0, 1$ and if we suppose that in the ψ_k representation of $r_q^{(k)}$ the number of β's is q, then, for example, in $r_{2q+1}^{(k)} = M(r_q^{(k)}, r_{q+1}^{(k)})$ this number must be $2q+1$. Similarly in the ψ_{k+1} of $M(r_{q_1}^{(k)}, r_{q_2}^{(k)})$ the number of β's is q_1+q_2. And so it follows from the "k-symmetry in the stronger sense" that $M(r_{q_1}^{(k)}, r_{q_2}^{(k)}) = M(r_{q_1'}^{(k)}, r_{q_2'}^{(k)})$ if $q_1+q_2=q_1'+q_2'$. Especially if $q_1+q_2 = 2q+s$ ($s=0$ or 1) then $M(r_{q_1}^{(k)}, r_{q_2}^{(k)}) = r_{2q+s}^{k+1}$.

This enables us to show that for our dyadic fractions, $\phi(x)$ satisfies the functional equation (5). In fact, if we consider $u=q_1/2^k$, $v=q_2/2^k$ ($q_1+q_2=2q+s$; $s=0$ or 1) $(u+v)/2=(2q+s)/2^{k+1}$,

$$
\begin{aligned}
M[\phi(u), \phi(v)] &= M\left[\phi\left(\frac{q_1}{2^k}\right), \phi\left(\frac{q_2}{2^k}\right)\right] = M(r_{q_1}^{(k)}, r_{q_2}^{(k)}) \\
(7) \qquad &= M(r_{q+s}^{(k)}, r_q^{(k)}) = r_{2q+s}^{(k+1)} = \phi\left(\frac{2q+s}{2^{k+1}}\right) = \phi\left(\frac{u+v}{2}\right).
\end{aligned}
$$

As the dyadic fractions are everywhere dense in the interval (0, 1) and $\phi(x)$ is monotonous on this set there exists a right and a left limit in every point of the interval (0, 1). We have to prove that they can not be different. For suppose on the contrary $y_1=\phi(x-0) < \phi(x+0) = y_2$ then by the "internness" $y_1 < M = M(y_1, y_2) < y_2$ and we can choose ϵ so that

$$
(8) \qquad y_1 + \epsilon < M < y_2 - \epsilon.
$$

By (ii) there exists a δ for which whenever $|y_1-y_1'| < \delta$, $|y_2-y_2'| < \delta$

$$
(9) \qquad M - \epsilon < M' = M(y_1', y_2') < M + \epsilon.
$$

We can choose $y_1' = \phi(q_1/2^k)$, $y_2' = \phi(q_2/2^k)$ with $z = (q_1+q_2)/2^{k+1} > x$,

$$
\begin{aligned}
\phi(z) &= M[\phi(q_1/2^k), \phi(q_2/2^k)] = M(y_1', y_2') \\
&= M' < M + \epsilon < y_2 = \phi(x+0),
\end{aligned}
$$

by (7), (8), (9) in contradiction with the monotony of $\phi(x)$ and thus $\phi(x)$ is continuous. It follows immediately that $\phi(x)$ satisfies (5) in every point $0 \leqq x \leqq 1$ and so we constructed $\phi(x) = f^{-1}(x)$ and thus our theorem is proved.

The function $f(x)$ just constructed is not the only one which satisfies (4). We can see immediately that every $g(x) = af(x) + b$ satisfies (4) too—but no other function. Because if

$$f^{-1}\left(\frac{f(x) + f(y)}{2}\right) = g^{-1}\left(\frac{g(x) + g(y)}{2}\right); \qquad f(x) = u, \quad f(y) = v,$$

$$gf^{-1}\left(\frac{u+v}{2}\right) = \frac{gf^{-1}(u) + gf^{-1}(v)}{2}.$$

This is Jensen's equality for the function $gf^{-1}(t)$, satisfied only by $gf^{-1}(t) = at + b$, $g(x) = af(x) + b$.[6] The $f(x)$ we constructed is determined by $f(\alpha) = 0, f(\beta) = 1$.

2. Nonsymmetric means.

THEOREM. *If the function of two variables* $M(x, y)$ *satisfies the following conditions* (cf. §1) $(\alpha \leqq x, y \leqq \beta)$:

(i′) *Strict monotony*;

(ii′) *Continuity*;

(iii′) *Bisymmetry*;

(iv′) *Reflexivity*,

then and only then there exists a continuous increasing function $f(x)$ *and a real number* $0 < p < 1$ *by which* $M(x, y)$ *has the form* $(p + q = 1)$:

$$M(x, y) = f^{-1}[pf(x) + qf(y)]. \tag{10}$$

The necessity of the conditions is evident. To prove the sufficiency we construct a symmetric mean $m(x, y)$ satisfying the conditions (i), (ii), (iii), (iv), (v) of §1 so that $m(x, y) = f^{-1}((f(x) + f(y))/2)$ and we shall show that this $f(x)$ figures in (10).

We obtain $m(x, y)$ as the limit of the following process:

$$\begin{array}{ll} \mathfrak{x}_0 = \min(x, y) & \mathfrak{y}_0 = \max(x, y), \\ \mathfrak{x}_1 = \min[M(x, y), M(y, x)], & \mathfrak{y}_1 = \max[M(x, y), M(y, x)], \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots, \\ \mathfrak{x}_{n+1} = \min[M(\mathfrak{x}_n, \mathfrak{y}_n), M(\mathfrak{y}_n, \mathfrak{x}_n)], & \mathfrak{y}_{n+1} = \max[M(\mathfrak{x}_n, \mathfrak{y}_n), M(\mathfrak{y}_n, \mathfrak{x}_n)], \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots. \end{array} \tag{11}$$

---

[6] Cf. Hardy, Littlewood, Pólya, *Inequalities*, Cambridge, 1934, p. 74.

The sequences $\bar{x}_n$ and $\bar{y}_n$ are monotonous and bounded and so they have limits $\xi=\lim_{n\to\infty}\bar{x}_n$, $\eta=\lim_{n\to\infty}\bar{y}_n$. $\xi=\eta$, for (from §1) $M(x, y)$ is k-symmetric (in the weaker sense) and thus if, for example, $\xi<\eta$

$$\begin{aligned}\xi &= \lim \bar{x}_{n+1} = \lim \min \left[M(\bar{x}_n, \bar{y}_n), M(\bar{y}_n, \bar{x}_n)\right] \\ &= \min\left[M(\xi, \eta), M(\eta, \xi)\right] > \xi\end{aligned}$$

that is, $\tau=\xi=\eta=\lim \bar{x}_n=\lim \bar{y}_n=m(x, y)$. Also we see now that

$$\begin{aligned}
&m(x, y) = \lim x_n = \lim y_n; \qquad x_0 = x, \qquad y_0 = y; \\
&x_1 = M(x, y), \qquad y_1 = M(y, x); \cdots; \\
&x_{n+1} = M(x_n, y_n), \qquad y_{n+1} = M(y_n, x_n); \cdots .
\end{aligned} \tag{12}$$

We prove that this $m(x, y)$ is a mean value satisfying the conditions (i), (ii), (iii), (iv), (v) of §1. (iv) and (v) is fulfilled evidently by (11). To prove (i), (ii), (iii) let us consider first the function $\phi(t; x, y) = M[M(t, x), M(y, t)]$; $\phi(t; t, t)=t$. $\phi(t; x, y)$ is increasing and continuous in t, x, y.

$$\begin{aligned}\phi(t; x, y) &= \phi[t; M(x, y), M(y, x)] = \phi(t; x_1, y_1) = \cdots \\ &= \phi(t; x_n, y_n) = \cdots = \phi(t; \tau, \tau) \\ &= \phi[t; m(x, y), m(x, y)]\end{aligned}$$

because $M(x, y)$ is k-symmetric (in the weaker sense). So the functional equation $t=\phi(t; x, y)$ is equivalent with

$$t = \phi(t; \tau, \tau) = \phi[t; m(x, y), m(x, y)]. \tag{13}$$

(13) is satisfied by $t=\tau=m(x, y)$ and this is its only solution, because if, for example, $t<\tau$ would be a solution too, $t=\phi(t; x, y)=\phi(t; \tau, \tau) >\phi(t; t, t)=t$. This is absurd!

Now we prove (i): if $x'>x$ then, in (12), $x_1'>x_1$, $y_1'>y_1, \cdots, x_n' >x_n$, $y_n'>y_n, \cdots, t'\geqq t$. But $t'=t$ is impossible, because from $t'=t$ and (13), $t=t'=\phi(t'; x', y)=\phi(t; x', y)>\phi(t; x, y)=t$. This is absurd, and thus $m(x, y)$ is increasing: $m(x',y)>m(x, y)$. The continuity (ii) can be proved as follows: Let $\xi^{(n)}$ be an increasing sequence converging to x, $\eta^{(n)}$ one converging to y. The sequence $\tau^{(n)}=m(\xi^{(n)}, \eta^{(n)})$ being increasing and bounded converges too: $\tau^{(n)}\to\bar{\tau}$. If we take the limit of $\tau^{(n)}=\phi(\tau^{(n)}; \xi^{(n)}, \eta^{(n)})$ we have $\bar{\tau}=\phi(\bar{\tau}; x, y)$. This is only possible if $\bar{\tau}=\tau=m(x, y)$. This and (i) gives (ii). Also the bisymmetry (iii) follows easily from (13) and the k-symmetry of $M(x, y)$. (iii) can be deduced also from the equation (15). Thus

$$m(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right) \tag{14}$$

by the Theorem of §1. Here $f(x)$ is increasing continuous, and $f(\alpha)=0$, $f(\beta)=1$.

We show that this is the $f(x)$ which figures in (1). We prove first

$$M[m(x_{11}, x_{12}), m(x_{21}, x_{22})] = m[M(x_{11}, x_{21}), M(x_{12}, x_{22})] \tag{15}$$

or with another notation: $M[m(x, y), m(u, v)] = m[M(x, u), M(y, v)]$. This is a consequence of (12) and the k-symmetry of $M(x, y)$. We write:

$$M(x_0, u_0) = M(x, u) = s = s_0, \qquad M(y_0, v_0) = M(y, v) = t = t_0,$$

$$M(x_1, u_1) = M[M(x, y), M(u, v)] = M[M(x, u), M(y, v)] = M(s, t) = s_1,$$

$$M(y_1, v_1) = M(t, s) = t_1,$$

. . . .

$$M(x_n, u_n) = M(s_{n-1}, t_{n-1}) = s_n, \qquad M(y_n, v_n) = M(t_{n-1}, s_{n-1}) = t_n,$$

. . . .

$$M[m(x, y), m(u, v)] = m(s, t) = m[M(x, u), M(y, v)]$$

$(m(x, y) = \lim x_n = \lim y_n,\ m(u, v) = \lim u_n = \lim v_n,\ m(s, t) = \lim s_n = \lim t_n)$.

Substituting (14) into (15) and writing $f\{M[f^{-1}(\xi), f^{-1}(\eta)]\} = F(\xi, \eta)$ we have:[7]

$$M\left\{f^{-1}\left(\frac{f(x_{11}) + f(x_{12})}{2}\right), f^{-1}\left(\frac{f(x_{21}) + f(x_{22})}{2}\right)\right\}$$

$$= f^{-1}\left\{\frac{f[M(x_{11}, x_{21})] + f[M(x_{12}, x_{22})]}{2}\right\},$$

$$F\left(\frac{z_{11} + z_{12}}{2}, \frac{z_{21} + z_{22}}{2}\right) = \frac{F(z_{11}, z_{21}) + F(z_{12}, z_{22})}{2}$$

$(f(x) = z)$. This is Jensen's equality[8] for functions of two variables, the only solution of which is the linear function $f\{M[f^{-1}(\xi), f^{-1}(\eta)]\} = F(\xi, \eta) = p\xi + q\eta + r$,

$$f[M(x, y)] = pf(x) + qf(y) + r$$

$(\xi = f(x),\ \eta = f(y))$. If we put here $x = y = \alpha$ we have $r = 0$; if $x = y = \beta$, $p + q = 1$. (By $x = \alpha$, $y = \beta$, $f[M(\alpha, \beta)] = q$ and by $x = \beta$, $y = \alpha$, $f[M(\beta, \alpha)]$

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[7] Cf. Aumann, *Konvexe Funktionen und die Induktion bei Ungleichungen zwischen Mittelwerten*, Bay. Akademie der Wissenschaften, Munich, Sitzungsberichte (1933). J. Aczél, *A generalization of the notion of convex functions*, Norske Videnskabers Selskabs Forhandlingen vol. 19 (1946).

[8] Cf. Hardy, Littlewood, Pólya, *Inequalities*, pp. 79–80.

$=p$.) Thus

$$M(x, y) = f^{-1}[pf(x) + qf(y)] \qquad (p + q = 1)\ Q.E.D.$$

Together with $f(x)$ evidently every $g(x) = af(x) + b$ satisfies (10) too. These are the only solutions as $m(x, y)$ is uniquely defined by $M(x, y)$ (11) and we have seen in §1 that $af(x) + b$ is the most general function satisfying (14). The weight p is uniquely defined by (10) because if $(p+q=r+s=1)$:

$$f^{-1}[pf(x) + qf(y)] = g^{-1}[rg(x) + sg(y)], \qquad w = g(z) = af(z) + b,$$

$$z = g^{-1}(w) = f^{-1}\left(\frac{w-b}{a}\right),$$

$apf(x)+aqf(y)=arf(x)+br+asf(y)+bs-b$ and thus $p=r$, $q=s$.

3. Bisymmetric functions.

THEOREM. *If a function of two variables which we shall write* $[x, y]$ is $(\alpha \leqq x, y \leqq \beta; \alpha \leqq [x, y] \leqq \beta)$:

(i″) *Increasing*;

(ii″) *Continuous*;

(iii″) *Bisymmetric*: $[[z_{11}, z_{12}], [z_{21}, z_{22}]] = [[z_{11}, z_{21}], [z_{12}, z_{22}]]$,

then and only then there exists an increasing continuous function $f(x)$ *and three real numbers* r, s, t, *by which*

$$[x, y] = f^{-1}\{rf(x) + sf(y) + t\}. \tag{16}$$

We prove the theorem by constructing a mean value (§2) $M(x, y) = f^{-1}\{pf(x)+qf(y)\}$ and by showing that this is the $f(x)$ which figures in (16).

We see from (i″) and (ii″) that the function $F(z) = [z, z]$ is continuous and increasing. The functions $F^{-1}(z)$ and $F^2(z) = F\{F(z)\} = [[z, z], [z, z]]$ are, also. We prove that the function $z = M(x, y) = F^{-1}([x, y])$ is a mean value which satisfies the conditions (i′), (ii′), (iii′), (iv′) of §2. In fact, (i′) and (ii′) follow from (i″) and (ii″); as for (iv′): $M(z, z) = F^{-1}([z, z]) = F^{-1}F(z) = z$. We have to verify yet (iii′). We write:

$$[z_{11}, z_{12}] = [z_1, z_1], z_1 = M(z_{11}, z_{12}), [z_{11}, z_{21}] = [\bar{z}_1, \bar{z}_1], \bar{z}_1 = M(z_{11}, z_{21}),$$

$$[z_{21}, z_{22}] = [z_2, z_2], z_2 = M(z_{21}, z_{22}), [z_{12}, z_{22}] = [\bar{z}_2, \bar{z}_2], \bar{z}_2 = M(z_{12}, z_{22}),$$

$$[z_1, z_2] = [z, z], \qquad z = M(z_1, z_2) = M\{M(z_{11}, z_{12}), M(z_{21}, z_{22})\},$$

$$[\bar{z}_1, \bar{z}_2] = [\bar{z}, \bar{z}], \qquad \bar{z} = M(\bar{z}_1, \bar{z}_2) = M\{M(z_{11}, z_{21}), M(z_{12}, z_{22})\}.$$

By applying (iii″) repeatedly we have $z = \bar{z}$, because

$$\begin{aligned}F^2(z) &= [[z, z], [z, z]] = [[z_1, z_2], [z_1, z_2]] = [[z_1, z_1], [z_2, z_2]] \\ &= [[z_{11}, z_{12}], [z_{21}, z_{22}]] = [[z_{11}, z_{21}], [z_{12}, z_{22}]] = [[\bar{z}_1, \bar{z}_1], [\bar{z}_2, \bar{z}_2]] \\ &= [[\bar{z}_1, \bar{z}_2], [\bar{z}_1, \bar{z}_2]] = [[\bar{z}, \bar{z}], [\bar{z}, \bar{z}]] = F^2(\bar{z}),\end{aligned}$$

$$M\{M(z_{11}, z_{12}), M(z_{21}, z_{22})\} = z = \bar{z} = M\{M(z_{11}, z_{21}), M(z_{12}, z_{22})\}.$$

Thus we have by the theorem of §2: $[x, y] = F\{M(x, y)\} = Ff^{-1}\{pf(x) + qf(y)\} = \psi\{pf(x) + qf(y)\}$ $(\psi(z) = Ff^{-1}(z))$.

We substitute this result into $[[x, y], [x, y]] = [[x, x], [y, y]]$ (this follows from (iii′′)),

$$\psi(f\psi\{pf(x) + qf(y)\}) = \psi(pf\psi f(x) + qf\psi f(y)).$$

If we write $f\psi(z) = h(z)$; $f(x) = u$, $f(y) = v$ we have $h(pu + qv) = ph(u) + qh(v)$. This is Jensen's equality[9] and hence $f\psi(z) = h(z) = \omega z + t$ (ω, t are constants)

$$f([x, y]) = \omega\{pf(x) + qf(y)\} + t = rf(x) + sf(y) + t \quad Q.E.D.$$

As the $M(x, y)$ is uniquely determined by $[x, y]$, all functions $g(x)$ satisfying (16) have the form $g(x) = af(x) + b$. The "weights" r, s are defined uniquely by $M(x, y)$ but t is not. For if $f^{-1}(rf(x) + sf(y) + t) = g^{-1}(\rho g(x) + \sigma g(y) + \tau)$, $w = g(z) = af(z) + b$, $z = g^{-1}(w) = f^{-1}((w - b)/a)$ then $arf(x) + asf(y) + at = a\rho f(x) + b\rho + a\sigma f(y) + b\sigma + \tau - b$ and thus $\rho = r$, $\sigma = s$, $\tau = at + b(1 - \rho - \sigma) = at + b(1 - r - s)$. If $r + s \neq 1$ we can choose b so that $\tau = 0$, namely $b = at/(r + s - 1)$. Hence for $\phi(x) = f(x) + t/(r + s - 1)$ and for every $\psi(x) = a\phi(x)$ (and only for them):

$$[x, y] = \phi^{-1}\{r\phi(x) + s\phi(y)\} \qquad (r + s \neq 1). \tag{17}$$

If $[x, y] = x \circ y$ satisfies instead of (iii′′) the stronger conditions $x \circ (y \circ z) = (x \circ y) \circ z$ and $x \circ y = y \circ x$, that is, if $x \circ y$ is an increasing continuous, "associative, and commutative operation"[10] we have, by putting, in (16), $r = s = 1$, from (17), the following corollary.

COROLLARY. *Every increasing, continuous, associative, and commutative operation has the form*

$$x \circ y = f^{-1}\{f(x) + f(y) + t\} = \phi^{-1}\{\phi(x) + \phi(y)\}. \tag{18}$$

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[9] Hardy, Littlewood, Pólya, *Inequalities*, p. 74.

[10] This associativity has naturally nothing to do with the associativity defined for mean values by Kolmogoroff and Nagumo quoted in §1.

ON AN INEQUALITY OF P. TURÁN CONCERNING LEGENDRE POLYNOMIALS

G. SZEGÖ

The following remarkable inequality is due to the Hungarian mathematician P. Turán: If $P_n(x)$ denotes as usual Legendre's polynomial of the nth degree, we have

$$(1)\qquad \Delta_n(x) = (P_n(x))^2 - P_{n-1}(x)P_{n+1}(x) \geqq 0, \qquad n \geqq 1; -1 \leqq x \leqq 1,$$

with equality only for $x = \pm 1$. The purpose of this note is to give several proofs for this theorem different from that of Turán.[1]

1. **Proof.** The following arrangement is somewhat similar to that of Turán. By using the classical recursion

$$(2)\qquad P_{n+1}(x) = \frac{2n+1}{n+1}\, xP_n(x) - \frac{n}{n+1}\, P_{n-1}(x)$$

we find for the polynomial $\Delta_n(x)$ the representation

$$(3)\qquad P_n^2 + \frac{n}{n+1}\, P_{n-1}^2 - \frac{2n+1}{n+1}\, xP_nP_{n-1}.$$

This is a quadratic form in P_n and P_{n-1} which is positive provided

$$(4)\qquad \frac{n}{n+1} > \left(\frac{n+1/2}{n+1}\, x\right)^2, \quad \text{or} \quad |x| < \frac{(n(n+1))^{1/2}}{n+1/2} = \cos\theta_0.$$

For these x the theorem is already proved. For the remaining $x = \cos\theta$, that is, for $0 < \theta \leqq \theta_0$, we use Mehler's formula

$$(5)\qquad P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+1/2)u}{(2(\cos u - \cos\theta))^{1/2}}\, du$$

and obtain

$$(6)\qquad \begin{aligned} \Delta_n(\cos\theta) = \pi^{-2}\int_0^\theta\int_0^\theta &(\cos u - \cos\theta)^{-1/2}(\cos v - \cos\theta)^{-1/2} \\ \cdot\{2\cos(n+1/2)u\cos(n+1/2)v &- \cos(n-1/2)u\cos(n+3/2)v \\ &- \cos(n-1/2)v\cos(n+3/2)u\}\, du dv. \end{aligned}$$

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[1] I owe Mr. Turán also some other remarkable properties of the polynomial $\Delta_n(x)$.

The expression in the braces is

$$
(7)\qquad
\begin{aligned}
&\cos (n + 1/2)(u + v) + \cos (n + 1/2)(u - v)\\
&\qquad - (1/2) \cos [(n - 1/2)u + (n + 3/2)v]\\
&\qquad - (1/2) \cos [(n - 1/2)u - (n + 3/2)v]\\
&\qquad - (1/2) \cos [(n - 1/2)v + (n + 3/2)u]\\
&\qquad - (1/2) \cos [(n - 1/2)v - (n + 3/2)u]\\
&= \cos (n + 1/2)(u + v)(1 - \cos (u - v))\\
&\qquad + \cos (n + 1/2)(u - v)(1 - \cos (u + v)),
\end{aligned}
$$

so that $\Delta_n > 0$ follows provided $(n + 1/2)|u \pm v| \leq (n + 1/2)2\theta \leq (n+1/2)2\theta_0 \leq \pi/2$. But this is obvious since

$$
(8)\qquad \theta_0 < \frac{\pi}{2} \sin \theta_0 = \frac{\pi}{2} \frac{1}{2n + 1}\cdot
$$

2. **Proof.** We expand $\Delta_n(x)$ in a finite series of Legendre polynomials which will contain only even terms:

$$
(9)\qquad \Delta_n(x) = c_0P_0(x) + c_1P_2(x) + c_2P_4(x) + \cdots + c_nP_{2n}(x).
$$

We show that the coefficients $c_1, c_2, \cdots, c_n$ are negative. Then the minimum will be reached if $P_m(x)$ is maximum, that is, for $x = 1$. But $\Delta_n(1) = 0$, thus the inequality will follow.

For this purpose we use a formula due to Adams, Ferrers and F. Neumann for the coefficients of the Legendre expansion of the product of two Legendre polynomials.[3] It is the simplest to state this formula in form of an integral:

$$
(10)\qquad
\begin{aligned}
i(a, b, c) &= \int_{-1}^{1} P_aP_bP_c\,dx\\
&= \begin{cases} 0 \text{ if } a + b + c \text{ odd},\\ 0 \text{ if } a + b + c \text{ even but no triangle with sides } a, b, c \text{ exists},\\ \dfrac{2}{2s + 1}\,\dfrac{g_{s-a}g_{s-b}g_{s-c}}{g_s} \text{ if } a + b + c = 2s,\ s \text{ integer, and a triangle with sides } a, b, c \text{ exists.}\end{cases}
\end{aligned}
$$

Here

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[3] See, for instance, E. T. Whittaker and G. N. Watson, *Modern analysis*, American ed., 1943, p. 331.

$$g_s = \frac{1\cdot 3\cdots(2s-1)}{2\cdot 4\cdots 2s}; \qquad g_0 = 1. \tag{11}$$

In our case

$$\begin{aligned}\frac{2}{4\nu+1}c_\nu &= i(n, n, 2\nu) - i(n-1, n+1, 2\nu)\\ &= \frac{2}{2(n+\nu)+1}\left(\frac{g_\nu g_\nu g_{n-\nu}}{g_{n+\nu}} - \frac{g_{\nu-1}g_{\nu+1}g_{n-\nu}}{g_{n+\nu}}\right), \qquad \nu \geqq 1.\end{aligned} \tag{12}$$

Bu $g_\nu/g_{\nu-1}$ is increasing so that (12) is negative. This proves the assertion.

The same argument shows that in the expansion

$$(P_n(x))^2 - P_{n-l}(x)P_{n+l}(x) = c_0P_0(x) + c_1P_2(x) + \cdots + c_nP_{2n}(x) \tag{13}$$

the first l coefficients $c_0, c_1, \cdots, c_{l-1}$ are positive and all the others negative.

3. **Proof.** Professor Pólya has called my attention to the fact that inequalities of the Turán type occur in the study of entire functions of the form

$$\sum_{n=0}^{\infty} \frac{u_n}{n!} z^{n+r} = e^{-\alpha z^2+\beta z}z^r \prod (1+\beta_n z)e^{-\beta_n z} \tag{14}$$

where $\alpha \geqq 0$, β and β_n real, $\sum \beta_n^2$ convergent. These functions, studied by Laguerre, Pólya and others[3] are (apart from a constant factor) the only ones which are the limits of polynomials with only real roots. A sequence of such polynomials is for instance that of the Jensen polynomials

$$u_0 + \binom{n}{1}u_1 z + \binom{n}{2}u_2 z^2 + \cdots + u_n z^n \tag{15}$$

which approach the entire function (14) as $n \to \infty$, provided we replace z by z/n. Under the conditions mentioned these polynomials have only real roots. If we denote them by $z_1, z_2, \cdots, z_n$ the inequality $u_{n-1}^2 - u_{n-2}u_n \geqq 0$ is a trivial consequence of the following inequality:

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[3] See G. Pólya and I. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math. vol. 144 (1914) pp. 89–113; cf. pp. 96–97.

$$\left(\frac{z_1+z_2+\cdots+z_n}{n}\right)^2 \geqq \frac{z_1z_2+z_1z_3+\cdots}{\binom{n}{2}}. \tag{16}$$

The generating function

$$\sum_{n=0}^{\infty}\frac{P_n(x)}{n!}z^n = e^{xz}J_0((1-x^2)^{1/2}z) \tag{17}$$

shows, in view of well known properties of the Bessel function J_0, that the conditions mentioned above are indeed satisfied. This furnishes the theorem.

Taking into account the identities

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}\frac{z^n}{n!} = 2^{\lambda-1/2}\Gamma(\lambda+1/2)e^{xz}((1-x^2)^{1/2}z)^{1/2-\lambda} \\
&\qquad\qquad \cdot J_{\lambda-1/2}((1-x^2)^{1/2}z), \qquad \lambda > -1/2, \\
&\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\frac{z^n}{n!} = \Gamma(\alpha+1)e^{z}(xz)^{-\alpha/2}J_\alpha(2(xz)^{1/2}), \qquad \alpha > -1, \\
&\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}z^n = e^{2xz-z^2}
\end{aligned} \tag{18}$$

where $P_n^{(\lambda)}$, $L_n^{(\alpha)}$, H_n denote the ultraspherical, Laguerre and Hermite polynomials, respectively, we conclude the analogous inequalities for these polynomials.[4]

4. **Proof.** Finally we can avoid the use of transcendental functions and work only with the Jensen polynomial (15). It can be written in this particular case in the following form:

$$\begin{aligned}
&P_0(x)+\binom{n}{1}P_1(x)z+\binom{n}{2}P_2(x)z^2+\cdots+P_n(x)z^n \\
&\qquad = (1+2xz+z^2)^{n/2}P_n\left(\frac{1+xz}{(1+2xz+z^2)^{1/2}}\right).
\end{aligned} \tag{19}$$

This identity can be shown by the ordinary generating function or by the generating function (17) or by the first integral of Laplace.[5] Now let x be fixed, $-1<x<1$. We obtain for the roots of the polynomial (19) in z the condition

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[4] We follow the notation of G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, pp. 80, 98, 102. The inequality for the Hermite polynomials was pointed out to me by P. Turán.

[5] See G. Szegő, loc. cit. p. 87.

$$\frac{1+xz}{(1+2xz+z^2)^{1/2}} = x_\nu \tag{20}$$

where x_ν denotes a root of P_n. Or

$$z = \frac{x(x_\nu^2-1) \pm x_\nu((1-x_\nu^2)(1-x^2))^{1/2}}{x^2-x_\nu^2}, \tag{21}$$

thus the roots in z are all real. Using the trivial inequality (16) the assertion follows.

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NOTE ON THE EIGENVALUES OF THE STURM-LIOUVILLE DIFFERENTIAL EQUATION

GERALD FREILICH

In discussing eigenvalues and eigenfunctions of the Sturm-Liouville differential equation

$$L(u) + \lambda\rho u = 0, \qquad L(u) = (pu')' - qu,$$

with

$$\left.\begin{aligned} p(x) &\geqq m > 0 \\ q(x) &\geqq 0 \\ \beta \geqq \rho(x) &\geqq \alpha > 0 \end{aligned}\right\} \quad \text{for } a \leqq x \leqq b, \text{ and for some } \alpha,\ \beta, \text{ and } m,$$

and the boundary conditions

$$u(a) = c_1u(b), \qquad u'(a) = c_2u'(b), \qquad c_1c_2p(a) = p(b),$$

we find that we can represent our eigenfunctions as unit normals in the directions of the principal axes of an ellipsoid in function space. We define our function space F as the set of all functions $v(x)$, $a \leqq x \leqq b$, which satisfy the boundary conditions of the Sturm-Liouville equation. The origin of our space will be the function $u(x)=0$. We can now metrize F by defining our inner product (u, v) for

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$u\in F$, $v\in F$, as

$$(u, v) = \int_a^b puv dx.$$

Also $|u| = ((u, u))^{1/2}$, and u is orthogonal to v if and only if $(u, v) = 0$. Let me now define

$$D(u, v) = \int_a^b (pu'v' + quv)dx.$$

Notice $D(u, u) = \int_a^b (p(u')^2 + qu^2)dx \geqq 0$.

In terms of D, I shall define an infinite-dimensional ellipsoid in our function space. Take a unit vector in F, that is, $(u, u) = 1$. On this, lay off a length $r = 1/(D(u, u))^{1/2}$. The set of points of F thus determined for all unit vectors of F constitutes the ellipsoid. It can be shown that the unit normals in the directions of the principal axes constitute a complete orthonormal set of eigenfunctions. Furthermore, if we arrange this set such that u_1 is the unit vector in the direction of the longest principal axis, u_2 the unit vector in the direction of the next longest principal axis, and so on, then $\lambda_1 = D(u_1, u_1)$, $\lambda_2 = D(u_2, u_2)$, $\cdots$ constitute an increasing set of eigenvalues, λ_i being the eigenvalue of u_i.

To show that $\lambda_n \approx n^2$ as $n \to \infty$, we must define the nth principal axis $u_n(x)$ independently of the previous $n-1$ principal axes $u_i(x)$, $i = 1, 2, \cdots, n-1$. This is customarily done in the following manner:

Take a set of $n-1$ linearly independent vectors of F, $v_1, v_2, \cdots, v_{n-1}$, through the origin and consider all unit vectors $u \perp v_i$, $i = 1, 2, 3, \cdots, n-1$. Hence $(u, u) = 1$, $(u, v_i) = 0$. Let $\min D(u, u) = f(v_i)$. Then $\max f(v_i) = \lambda_n$.

The way I define λ_n runs as follows:

Let $v_1, v_2, \cdots, v_n$ be a set of n linearly independent vectors of F. Consider

$$u = c_1v_1 + c_2v_2 + \cdots + c_nv_n.$$

Then we normalize u getting $u/|u|$. It is easy to see that $D(u/|u|, u/|u|) = D(u, u)/(u, u)$. Now let $g(v_i) = \max D(u, u)/(u, u)$ for all u as defined above. Then

$$\min g(v_i) = \lambda_n.$$

To prove this, let $v_i = u_i$, for $i = 1, 2, \cdots, n$. Then

$$(u, u) = (\sum c_iu_i, \sum c_ju_j) = \sum c_ic_j(u_i, u_j) = \sum c_i^2,$$

$$\frac{D(u, u)}{(u, u)} = \frac{D(\sum c_i u_i, \sum c_j u_j)}{\sum c_i^2} = \frac{\sum c_i^2 \lambda_i}{\sum c_i^2} \leqq \frac{\sum c_i^2 \lambda_n}{\sum c_i^2} = \lambda_n,$$

since $\lambda_i \leqq \lambda_n$ for $i = 1, 2, \cdots, n-1$. Hence $g(u_i) \leqq \lambda_n$. However, if $c_i = 0$, for $i = 1, 2, \cdots, n-1$, $c_n = 1$, then $u = u_n$, and hence $D(u, u)/(u, u) = \lambda_n$, $g(u_i) = \lambda_n$. To complete the proof I must show that for all v_i, $g(v_i) \geqq \lambda_n$.

Let $u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ be such that $u \perp u_i$ for $i = 1, \cdots, n-1$, that is, $(u, u_i) = 0 = c_1(u_i, v_1) + \cdots + c_n(u_i v_n)$. This is always possible since I have $n-1$ equations in n unknowns. Now by the definition of u_n, for all vectors $u \perp u_i$, $i = 1, 2, \cdots, n-1$,

$$\lambda_n = D(u_n, u_n) \leqq \frac{D(u, u)}{(u, u)}.$$

Hence $g(v_i) = \max D(u, u)/(u, u) \geqq \lambda_n$.

With this definition, we can now develop inequalities for the eigenvalues. Consider

$$(p_1 u')' - q_1 u + \lambda \rho_1 u = 0, \tag{1}$$

$$(p_2 u')' - q_2 u + \mu \rho_2 u = 0 \tag{2}$$

with

$$p_1(x) \leqq p_2(x), \qquad q_1(x) \leqq q_2(x), \qquad \rho_1(x) \geqq \rho_2(x) \qquad \text{for } a \leqq x \leqq b,$$

and with both equations satisfying similar boundary conditions. Denoting the eigenvalues of (1) by

$$\lambda_1, \lambda_2, \lambda_3, \cdots$$

and those of (2) by

$$\mu_1, \mu_2, \mu_3, \cdots,$$

we shall prove that $\lambda_n \leqq \mu_n$ for $n = 1, 2, \cdots$.

Let $v_1, v_2, \cdots, v_n$ be a fixed set of n independent vectors. Let $u = c_1 v_1 + \cdots + c_n v_n$. For (1) the normalized u is $u/|u|_1$ and $D_1(u/|u|_1, u/|u|_1) = D_1(u, u)/(u, u)_1$, $g_1(v_i) = \max D_1(u, u)/(u, u)_1$. For (2) the normalized u is $u/|u|_2$, and $g_2(v_i) = \max D_2(u, u)/(u, u)_2$. Since $D_1(u, u) = \int (p_1(u')^2 + q_1 u^2) dx$, $D_2(u, u) = \int (p_2(u')^2 + q_2 u^2) dx$, it follows that $D_2(u, u) \geqq D_1(u, u)$. Since $(u, u)_1 = \int \rho_1 u^2 dx$, $(u, u)_2 = \int \rho_2 u^2 dx$, it follows that $(u, u)_1 \geqq (u, u)_2$. Hence $D_2(u, u)/(u, u)_2 \geqq D_1(u, u)/(u, u)_1$, $g_2(v_i) \geqq g_1(v_i)$. Since $\min g_2(v_i) = \mu_n$, and $\min g_1(v_i)$ λ_n, then $\mu_n \geqq \lambda_n$.

With these preliminaries, the proof about the asymptotic behavior of λ_n is standard.

An example of the use of this new definition of λ_n is that it affords a method of obtaining upper bounds and approximations to λ_n.

Take $v_i = v_i(x, a_1, a_2, \cdots, a_r)$, with r parameters, such that v_i for $i = 1, 2, \cdots, n$ satisfies the boundary conditions, that is, it is a vector in function space. Then form

$$u = c_1 v_1 + \cdots + c_n v_n,$$

and normalize u, getting

$$(u, u) = \sum c_i c_j (v_i, v_j) = 1.$$

We can now calculate max $D(u, u)$, for

$$D(u, u) = \sum c_i c_j D(v_i, v_j),$$

and hence we get

$$\frac{\partial[\sum c_i c_j D(v_i, v_j)]}{\partial c_i} = \sigma \frac{\partial[\sum c_i c_j (v_i, v_j)]}{\partial c_i},$$

which together with the equation

$$\sum c_i c_j (v_i, v_j) = 1$$

determine the c_i.

Hence max $D(u, u) = h(a_1, a_2, \cdots, a_r) \geqq \lambda_n$. Hence to approximate λ_n, we minimize the function $h(a_1, a_2, \cdots, a_r)$. Good approximations depend of course on the choice of the original v_i. It is to be noticed that this is just an extension of the ordinary method of computing λ_0.

BROWN UNIVERSITY

A HOMOGENEOUS DIFFERENTIAL SYSTEM OF INFINITE ORDER WITH NONVANISHING SOLUTION

S. BOCHNER AND D. V. WIDDER

1. **Introduction.** The only solution of the homogeneous differential system

$$\frac{d^{2k-1}}{dt^{2k-1}}\left[t^k f(t)\right] = 0, \tag{1}$$

$$f(0+) = f(+\infty) = 0 \tag{2}$$

is $f(t) \equiv 0$. This may be seen by observing that none of the fundamental solutions of equation (1), t^n $(n = -k, -k+1, \cdots, k-2)$, satisfies the boundary conditions (2). We now replace equation (1) by another of infinite order, retaining the boundary conditions (2), and investigate the possibility of a nonvanishing solution. We let k become infinite in equation (1) after introduction of a factor. Set

$$L_{k,t}[f(x)] = \frac{(-t)^{k-1}}{k!(k-2)!}\frac{d^{2k-1}}{dt^{2k-1}}\left[t^k f(t)\right] \tag{3}$$

and consider the system

$$\lim_{k\to\infty} L_{k,t}[f(x)] = 0, \qquad 0 < t < \infty, \tag{4}$$

with the boundary conditions (2). Since the differential operator (3) serves to invert the Stieltjes transform,[1]

$$f(x) = \int_0^\infty \frac{\phi(t)}{x+t}\,dt, \tag{5}$$

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \phi(t),$$

it is clear that the only solution of the system (4), (2) of the form (5) must vanish identically. Accordingly if the limit (4) exists boundedly,[2] for example, there is no nonvanishing solution of the system. However, we shall show that if the limit (4) merely exists at each point of the interval then there are many nonvanishing solutions of the system. One very simple solution is $f(t) = t(1+t)^{-2}$. We shall find

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[1] See *The Laplace transform* by D. V. Widder, Princeton University Press, 1946, p. 345.

[2] D. V. Widder, loc. cit. p. 373.

it convenient to make an exponential change of variable which will change equation (4) into

$$\lim_{n\to\infty} D\prod_{k=1}^{n}\left(1-\frac{D^2}{k^2}\right)y(x) = 0,$$

where D indicates differentiation with respect to x. The system in question becomes

$$(6)\qquad (\sin \pi D)y(x) = 0, \qquad -\infty < x < \infty,$$

$$(7)\qquad y(-\infty) = y(+\infty) = 0,$$

and a nonvanishing solution is

$$y(x) = e^{-x}(1+e^{-x})^{-2}.$$

2. Existence of a nonvanishing solution.

THEOREM 1. *The function*

$$k(x) = \frac{e^{-x}}{(1+e^{-x})^2}$$

satisfies the system (6), (7).

To prove this let us introduce the following notation:

$$E_n(D) = D\prod_{k=1}^{n}\left(1-\frac{D^2}{k^2}\right),$$

$$g(x) = \frac{1}{1+e^{-x}},$$

$$c_n = \frac{(2n+1)!}{n!\,n!}.$$

We shall show by induction that

$$(8)\qquad E_n(D)g(x) = c_n[k(x)]^{n+1}.$$

This is true for $n=0$ since $g'(x)=k(x)$. Assume equation (8) true when n is replaced by $n-1$ and differentiate both sides twice with respect to x,

$$\frac{d^2}{dx^2}E_{n-1}(D)g(x) = -c_{n-1}\left\{\frac{2n(2n+1)e^{-(n+1)x}}{(1+e^{-x})^{2n+2}} + \frac{n^2e^{-nx}}{(1+e^{-x})^{2n}}\right\}.$$

If this equation is divided by n^2 and subtracted from equation (8) for

$n-1$ we obtain

$$\begin{aligned} E_n(D)g(x) &= \frac{c_{n-1}}{n^2}\,\frac{2n(2n+1)e^{-(n+1)x}}{(1+e^{-x})^{2n+2}} \\ &= c_n h(x)^{n+1}. \end{aligned} \tag{9}$$

By one more differentiation we have

$$E_n(D)h(x) = c_n(n+1)h(x)^n h'(x). \tag{10}$$

Since $h(x)$ is an even function it is clear that $E_n(D)h(x)$ vanishes at $x=0$. For each $x\neq 0$ the right-hand side of equation (10) is the general term of a convergent series and hence tends to zero with n. The test ratio of the series is

$$\lim_{n\to\infty}\frac{c_n}{c_{n-1}}\left(\frac{n+1}{n}\right)h(x) = 4h(x),$$

and this is clearly less than unity. Thus

$$(\sin \pi D)h(x) = 0, \qquad -\infty < x < \infty.$$

It is clear by inspection that $h(\pm\infty)=0$. This proves the theorem.

3. Derivatives of the solution.

THEOREM 2. *Every even derivative of the function $h(x)$ of Theorem* 1 *satisfies the system* (6), (7).

By equation (9) we have $E_n(D)g^{(p)}(x)=c_nD^ph(x)^{n+1}$, $p=0, 1, 2, \cdots$. Without computing the pth derivative of $h(x)^{n+1}$ completely we can obtain an upper bound for it as follows. It is clear that

$$D^p h(x)^{n+1} = D^p\frac{1}{(e^{x/2}+e^{-x/2})^{2n+2}} = \frac{a_p(x)}{(e^{x/2}+e^{-x/2})^{2n+p+2}}, \tag{11}$$

where

$$a_p(x) = \sum_{j=0}^{p} c_{p,j}e^{(j-p/2)x}. \tag{12}$$

Here the coefficients $c_{p,j}$ need not be determined. If

$$|c_{p,j}| \leq A_pC_{p,j}, \qquad j = 0, 1, \cdots, p,$$

then

$$|a_p(x)| \leq A_p(e^{x/2}+e^{-x/2})^p, \qquad -\infty < x < \infty. \tag{13}$$

By differentiation of equation (12) we have

$$|a_p'(x)| \leq A_p \frac{p}{2}(e^{x/2}+e^{-x/2})^p. \tag{14}$$

By induction on p we obtain by differentiating equation (11) that

$$a_{p+1}(x) = a_p'(x)(e^{x/2}+e^{-x/2}) - 2^{-1}(2n+p+2)a_p(x)(e^{x/2}-e^{-x/2}).$$

Then inequalities (13) and (14) give

$$|a_{p+1}(x)| \leq A_p(n+p+1)(e^{x/2}+e^{-x/2})^{p+1}$$

or

$$A_{p+1} = (n+p+1)A_p.$$

For $p=0$, $a_p(x)=1$, and $A_0=1$. Hence

$$A_p = \frac{(n+p)!}{n!}$$

and

$$\begin{aligned} |D^p[k(x)]^{n+1}| &\leq \frac{(n+p)!}{n!}[k(x)]^{n+1} \\ |E_n(D)h^{(p)}(x)| &\leq c_n\frac{(n+p+1)!}{n!}[k(x)]^{n+1}. \end{aligned} \tag{15}$$

As in §2 the right-hand side of inequality (15) tends to zero with $1/n$ for each $x\neq 0$. Moreover if p is even, the left-hand side is an odd function and consequently vanishes at the origin. Hence $h^{(p)}(x)$ satisfies equation (6) for even p.

Since

$$\begin{aligned} h^{(p)}(x) &= E_0(D)g^{(p)}(x) = c_0D^p[k(x)], \\ |h^{(p)}(x)| &\leq c_0p!k(x), \end{aligned} \tag{16}$$

it follows that $h^{(p)}(x)$ vanishes at $\pm\infty$, and the theorem is proved.

4. Further solutions. We can now obtain a very large class of solutions of the system (6), (7).

THEOREM 3. *If the function*

$$f(w) = \sum_{k=0}^{\infty} a_k w^k$$

is an even function of order one and minimal type and if $k(x)$ is the *function of Theorem* 1, *then the function*

$$F(x) = \sum_{k=0}^{\infty} a_k h^{(k)}(x)$$

satisfies the system (6), (7).

Let $\epsilon_0, \epsilon_1, \cdots$ be a sequence of complex numbers tending to zero. Then the most general function of order one and minimal type takes the form[3]

$$f(w) = \sum_{k=0}^{\infty} \frac{(\epsilon_k w)^k}{k!} w^k.$$

Since $f(w)$ is even, $\epsilon_k = 0$ for even k. Clearly $F(x)$ is also even, and $E_n(D)F(x)$ is odd for each n, and vanishes at the origin. That is, $F(x)$ satisfies equation (6) at $x=0$.

If $x \neq 0$ we may determine a positive number ϵ less than unity and such that

$$4e^{-x} < (1 + e^{-x})^2(1 - \epsilon). \tag{17}$$

Now determine an integer N so large that $|\epsilon_k| \leqq \epsilon$ when $k \geqq N$. We have at once that

$$E_n(D)F(x) = \sum_{k=0}^{\infty} \frac{\epsilon_k^k}{k!} E_n(D) h^{(k)}(x). \tag{18}$$

Inequality (15) is sufficient to establish the uniform convergence of the series (18) and thus to validate the term-by-term differentiation. By Theorem 2 it is clear that

$$\lim_{n\to\infty} \sum_{k=0}^{N} \frac{\epsilon_k^k}{k!} E_n(D) h^{(k)}(x) = 0.$$

Moreover, by inequality (15)

$$\left| \sum_{k=N}^{\infty} \frac{\epsilon_k^k}{k!} E_n(D) h^{(k)}(x) \right| \leqq c_n [h(x)]^{n+1} \sum_{k=N}^{\infty} \frac{(n+k+1)!}{n!} \frac{\epsilon^k}{k!}$$
$$\leqq \frac{(n+1)\, c_n}{(1-\epsilon)^{n+2}} [h(x)]^{n+1}.$$

But this dominating function is again the general term of a series with test ratio

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[3] See, for example, L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 1931, p. 235.

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} \frac{k(x)}{1-\epsilon} = \frac{4k(x)}{1-\epsilon}.$$

This ratio is less than 1 by (17), so that $F(x)$ satisfies (6) for all x. Finally, from inequality (16)

$$|F(x)| \leqq c_0 k(x) \sum_{k=0}^{\infty} |\epsilon_k|^k,$$

and $F(x)$ also satisfies equations (7). This completes the proof.

5. A counter-example. We can now show that the result of Theorem 3 is best possible in a certain sense.

THEOREM 4. *The word "minimal" in Theorem 3 cannot be replaced by the word "normal".*

To prove this it will be sufficient to exhibit an even entire function

$$f(w) = \sum_{k=0}^{\infty} a_k w^k$$

of order 1 and type t (arbitrarily small) such that the function

$$F(x) = \sum_{k=0}^{\infty} a_k k^{(k)}(x)$$

will fail to satisfy equation (6) for some x. We choose $a_k = (it)^k/k!$ for even k and $a_k = 0$ for odd k. Then $f(w) = \cos(tw)$ is a function with the properties desired. Moreover the function $F(x)$ becomes

$$F(x) = \frac{k(x+it) + k(x-it)}{2} = \operatorname{Re} k(x+it).$$

Simple computations give

$$\begin{aligned} &-\frac{E_n(D)F(x)}{(n+1)c_n} \\ (19) \qquad &= \operatorname{Re}\left\{\frac{(e^{x/2} - e^{-x/2})\cos(t/2) + i(e^{x/2} + e^{-x/2})\sin(t/2)}{[(e^{x/2} + e^{-x/2})\cos(t/2) + i(e^{x/2} - e^{-x/2})\sin(t/2)]^{2n+2}}\right\}. \end{aligned}$$

If we define an argument α and a modulus λ as follows,

$$\tan\alpha = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} \tan\frac{t}{2}, \qquad \lambda^2 = e^x + e^{-x} + 2\cos t$$

equation (19) becomes

$$
\begin{aligned}
&-\frac{\lambda^{2n+3}E_n(D)F(x)}{(n+1)c_n}\\
(20)\qquad &= \operatorname{Re}\left\{\frac{(e^{x/2}-e^{-x/2})\cos(t/2)+i(e^{x/2}+e^{-x/2})\sin(t/2)}{e^{i\alpha(2n+3)}}\right\}.
\end{aligned}
$$

If we choose x so near zero that $\lambda<2$, it is clear that $\lambda^{2n+3}/[(n+1)c_n]$ approaches zero with $1/n$. If in addition we choose x so near zero that $\alpha=(2\pi)/N$, where N is an odd integer (which is seen to be possible from the definition of α), then the right-hand side of equation (20) reduces to

$$(e^{x/2}-e^{-x/2})\cos(t/2)\neq 0$$

for infinitely many values of n. Hence for such x

$$\lim_{n\to\infty} E_n(D)F(x)\neq 0,$$

and our result is established.

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THE RANGE OF A VECTOR MEASURE

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The purpose of this note is to prove that the range of a countably additive finite measure with values in a finite-dimensional real vector space is closed and, in the non-atomic case, convex. These results were first proved (in 1940) by A. Liapounoff.[1] In 1945 K. R. Buch (independently) proved the part of the statement concerning closure for non-negative measures of dimension one or two.[2] In 1947 I offered a proof of Buch's results which, however, was correct in the one-dimensional case only.[3] In this paper I present a simplified proof of Liapounoff's results.

Let X be any set and let S be a σ-field of subsets of X (called the measurable set of X). A ***measure*** μ (or, more precisely, an N-dimensional measure $(\mu_1, \cdots, \mu_N)$) is a (bounded) countably additive function of the sets of S with values in N-dimensional, real vector space (in which the "length" $|\xi_1| + \cdots + |\xi_N|$ of a vector $\xi = (\xi_1, \cdots, \xi_N)$ is denoted by $|\xi|$). The measure $(\mu_1, \cdots \mu_N)$ is ***non-negative*** if $\mu_i(E) \geqq 0$ for every $E \in S$ and $i = 1, \cdots, N$.

For a numerical (one-dimensional) measure μ_0, $\mu_0^*(E)$ will denote the total variation of μ_0 on E; in general, if $\mu = (\mu_1, \cdots, \mu_N)$, μ^* will denote the non-negative measure $(\mu_1^*, \cdots, \mu_N^*)$. The length $|\mu^*| = \mu_1^* + \cdots + \mu_N^*$ is always a non-negative numerical measure.[4]

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[1] *Sur les fonctions-vecteurs complètement additives*, Bull. Acad. Sci. URSS. Sér. Math. vol. 4 (1940) pp. 465–478. In a subsequent paper with the same title, in the same journal (vol. 10 (1946) pp. 277–279), Liapounoff gave a very elegant example to show that neither convexity nor closure can always be asserted in the infinite-dimensional case. Related questions for finitely additive measures have been discussed by A. Sobczyk and P. C. Hammer, *The ranges of additive set functions*, Duke Math. J. vol. 11 (1944) pp. 847–851.

[2] *Some investigations of the set of values of measures in abstract space*. K. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser vol. 21.

[3] *On the set of values of a finite measure*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 138–141. I am indebted to Professors Børge Jessen and Jean Dieudonné for calling my attention to the fact that the statement and proof of Lemma 5 of that paper are wrong. The error in the proof is the implicit assumption that if S is covered by sets of the form $U \cap V$ then the intersection of any U that occurs with any V that occurs is also a set of the covering. It is not difficult to show that the conclusion of Lemma 5 holds if and only if every closed set of either of the two given topologies is compact with respect to the other. It follows that if, for instance, the given topologies are Hausdorff then the conclusion holds if and only if they are identical.

[4] For the notion of total variation, as well as all other concepts and results of elementary measure theory, see S. Saks, *Theory of the integral*, Warsaw, 1937, chaps. 1 and 2.

A measurable set E is an *atom* of μ if $\mu(E)\neq 0$ and if for every measurable set $F\subset E$ either $\mu(F)=0$ or $\mu(F)=\mu(E)$. The measure μ is *purely atomic* if there exists a sequence $\{E_n\}$ of pairwise disjoint measurable sets such that $X=\bigcup_n E_n$ and every E_n is an atom of each μ_i, $i=1, \cdots, N$; μ is *purely non-atomic* if none of its coordinates μ_i ($i=1, \cdots, N$) has any atoms.

The measure μ is *absolutely continuous* with respect to the measure ν if $|\mu^*(E)|=0$ for every measurable set E for which $|\nu^*(E)|=0$ (that is, if the numerical measure $|\mu^*|$ is absolutely continuous with respect to the numerical measure $|\nu^*|$). A necessary and sufficient condition that μ be absolute continuous with respect to ν is that for every $\epsilon>0$ there exist a $\delta>0$ such that $|\mu^*(E)|<\epsilon$ for every measurable set E for which $|\nu^*(E)|<\delta$. For any two measures μ and ν there exists a measurable set E_0 such that $|\nu^*(E_0)|=0$ and $|\mu^*(E-E_0)|=0$ for every measurable set E for which $|\nu^*(E)|=0$, that is, the measure μ', defined by $\mu'(E)=\mu(E-E_0)$, is absolutely continuous with respect to ν.

A measure μ is *semi-convex* if every measurable set E contains a measurable set F for which $\mu(F)=\mu(E)/2$. For any measure μ and measurable set E let $K(\mu, E)$ be the class of all real-valued measurable functions ϕ on E for which $0\leqq\phi(x)<1$ and $\mu(\{x:\phi(x)<\lambda\})=\lambda\mu(E)$ for $0\leqq\lambda\leqq 1$. A measure μ is *convex* if for every measurable set E the class $K(\mu, E)$ is not empty.

LEMMA 1. *If μ is a semi-convex measure and E is any measurable set then there exists a sequence $\{E_n\}$ of measurable subsets of E such that for every $k=1, 2, \cdots$ and any k distinct positive integers $n_1, \cdots, n_k$*

$$\mu(E_{n_1}\cap\cdots\cap E_{n_k}) = \frac{1}{2^k}\mu(E). \tag{1}$$

PROOF. Let $E_1\subset E$ be any measurable set with $\mu(E_1)=\mu(E)/2$. Suppose that measurable sets $E_1, \cdots, E_n$ have already been constructed so that (1) holds for any k distinct integers $n_1, \cdots, n_k$, $1\leqq n_i\leqq n$, $i=1, \cdots, k$. Consider each of the 2^n sets of the form

$$E(\epsilon_1, \cdots, \epsilon_n) = E_1^{\epsilon_1}\cap\cdots\cap E_n^{\epsilon_n}, \tag{2}$$

where $\epsilon_i=0$ or 1 ($i=1, \cdots, n$) and, for any set F, $F^1=F$ and $F^0=X-F$. The value of μ for each such set is $\mu(E)/2^n$. From each set of the form (2) select a measurable subset $F(\epsilon_1, \cdots, \epsilon_n)$ so that $\mu(F(\epsilon_1, \cdots, \epsilon_n))=\mu(E(\epsilon_1, \cdots, \epsilon_n))/2$; write E_{n+1} for the union of the selected subsets. The inductive procedure so described proves

Lemma 1.

Lemma 2. *A semi-convex measure is convex.*

Proof. There is no loss of generality in considering non-negative measures only. Let μ be a non-negative semi-convex measure, let E be any measurable set, and let $\{E_n\}$ be a sequence of sets with the property described in Lemma 1. If

$$x \in E_* = \liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n$$

then write $\phi(x)=0$; otherwise write $\phi(x)=\sum_{n=1}^{\infty} \epsilon_n(x)/2^n$, where ϵ_n is the characteristic function of E_n. Clearly ϕ is measurable and $0 \leqq \phi(x) < 1$. For any dyadically rational number $\lambda = k/2^n$, with $n=0, 1, 2, \cdots$ and $k=1, \cdots, 2^n$, it is easy to verify that

$$\{x:\phi(x) < \lambda\} = E_* \cup \left\{x: \sum_{i=1}^{n} \epsilon_i(x)/2^i < \lambda\right\}.$$

Since the set $\{x: \sum_{i=1}^{n} \epsilon_i(x)/2^i < \lambda\}$ is the union of exactly k pairwise disjoint sets of the form $E_1^{\epsilon_1} \cap \cdots \cap E_n^{\epsilon_n}$ ($\epsilon_i = 0$ or 1, $i=1, \cdots, n$) and since $\mu(E_*)=0$, it follows that

$$\mu(\{x:\phi(x) < \lambda\}) = \lambda\mu(E). \tag{3}$$

The extension of (3) to values of λ which are not dyadically rational is an immediate consequence of the countable additivity of μ.

Lemma 3. *Let μ be a non-negative convex measure and let E be any measurable set. If $\phi \in K(\mu, E)$ and if ν is a measure absolutely continuous with respect to μ then $\nu(\{x:\phi(x) < \lambda\})$ is a continuous function of λ, $0 \leqq \lambda \leqq 1$.*

Proof. For any $\epsilon > 0$ select a $\delta > 0$ so that $|\nu(F)| \leqq |\nu^*(F)| < \epsilon$ for every measurable set F for which $|\mu(F)| = |\mu^*(F)| < \delta$. If $0 \leqq \lambda_1 < \lambda_2 \leqq 1$ and $(\lambda_2 - \lambda_1)|\mu(E)| < \delta$ then

$$|\mu(\{x:\lambda_1 \leqq \phi(x) < \lambda_2\})| = (\lambda_2 - \lambda_1)|\mu(E)| < \delta$$

and therefore $|\nu(\{x:\phi(x) < \lambda_2\}) - \nu(\{x:\phi(x) < \lambda_1\})| < \epsilon$.

Lemma 4. *If μ is a convex measure and E and F are any two measurable sets then for every λ, $0 \leqq \lambda \leqq 1$, there exists a measurable set $C(\lambda)$ with the following properties.*

(i) $C(0)=E$ *and* $C(1)=F$.

(ii) $\mu(C(\lambda)) = (1-\lambda)\mu(E) + \lambda\mu(F)$.

(iii) *If μ is also non-negative and if ν is absolutely continuous with respect to μ then $\nu(C(\lambda))$ is continuous function of λ.*

COROLLARY. *The range of a convex measure is convex.*

PROOF. Let ϕ and ψ be functions in $K(\mu, E-F)$ and $K(\mu, F-E)$ respectively; write $C(\lambda)=(E\cap F)\cup\{x:\phi(x)<1-\lambda\}\cup\{x:\psi(x)<\lambda\}$. The relations (i) are an immediate consequence of the definition of $C(\lambda)$ and (iii) follows from Lemma 3. The relation (ii) is proved by the following chain of equations.

$$\begin{aligned}\mu(C(\lambda)) &= \mu(E\cap F) + \mu(\{x:\phi(x) < 1-\lambda\}) + \mu(\{x:\psi(x) < \lambda\})\\ &= \mu(E\cap F) + (1-\lambda)\mu(E-F) + \lambda\mu(F-E)\\ &= (1-\lambda)[\mu(E\cap F) + \mu(E-F)] + \lambda[\mu(E\cap F) + \mu(F-E)].\end{aligned}$$

LEMMA 5. *If $\mu=(\mu_1, \cdots, \mu_N)$ is non-negative and purely non-atomic and if each μ_i $(1<i\leqq N)$ is absolutely continuous with respect to its predecessor then μ is convex.*

PROOF. In view of Lemma 2 it is sufficient to prove that μ is semi-convex. For $N=1$ this is clear.[5] Suppose now (in case $N>1$) that the $(N-1)$-dimensional measure $\mu'=(\mu_1, \cdots, \mu_{N-1})$ is convex; it follows from the hypotheses that the (one-dimensional) measure $\nu'=\mu_N$ is absolutely continuous with respect to it. Any measurable set E contains a measurable set E_0 such that $\mu'(E_0)=\mu'(E)/2$; write $F_0=E-E_0$. If $\nu'(E_0)=\nu'(E)/2$ there is nothing to prove; in the contrary case it may be assumed that $\nu'(E_0)<\nu'(E)/2$ and $\nu'(F_0)>\nu'(E)/2$. Lemma 4 applied to μ', ν', E_0, F_0 in place of μ, ν, E, F yields the existence of a number λ, $0\leqq\lambda\leqq 1$, such that $\nu'(C(\lambda))=\nu'(E)/2$ and

$$\begin{aligned}\mu'(C(\lambda)) &= (1-\lambda)\mu'(E_0) + \lambda\mu'(F_0)\\ &= (1-\lambda)\frac{\mu'(E)}{2} + \lambda\frac{\mu'(E)}{2} = \frac{\mu'(E)}{2}.\end{aligned}$$

It follows that $\mu(C(\lambda))=\mu(E)/2$, and consequently the inductive proof of Lemma 5 is complete.

LEMMA 6. *A purely non-atomic non-negative measure is convex.*

PROOF. If $\mu=(\mu_1, \cdots, \mu_N)$ is purely non-atomic, write $\mu_i'=\sum_{n=i}^{N}\mu_n$. Since $\mu'=(\mu_1', \cdots, \mu_N')$ satisfies the conditions of Lemma 5, μ' is convex. Let T be the inverse of the linear transformation which carried μ into μ', $\mu=T\mu'$. From the linearity of T and the semi-convexity of μ' it follows that μ is semi-convex and hence convex.

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[5] Cf. Lemma 2 of the paper cited in footnote 2.

LEMMA 7. *A purely non-atomic measure is convex.*

PROOF. For each $i=1, \cdots, N$ there exists a measurable set E_i such that $\mu_i(E\cap E_i)\geqq 0$ and $\mu_i(E-E_i)\leqq 0$ for every measurable set E. On all measurable subsets of any one of the 2^N sets of the form $E_1^{\epsilon_1}\cap \cdots \cap E_N^{\epsilon_N}$ ($\epsilon_i=0$ or 1, $i=1, \cdots, N$) each of the measures $\mu_1, \cdots, \mu_N$ is of constant sign. Lemma 6 may then be applied to the set function $\mu_{\epsilon_1,\cdots,\epsilon_N}=(\pm\mu_1, \cdots, \pm\mu_N)$, where the ambiguous sign is chosen in each case so that $\mu_{\epsilon_1,\cdots,\epsilon_N}$ is a non-negative measure. The convexity of $\mu_{\epsilon_1,\cdots,\epsilon_N}$ and the disjointness of any two distinct ones of the sets $E_1^{\epsilon_1}\cap \cdots \cap_N^{\epsilon_N}$ imply the convexity of μ.

LEMMA 8. *If μ is a measure with convex range R and Π is a supporting plane of the closure $\bar{R}$ then $R\cap\Pi\neq 0$.*

PROOF. Let L be a real linear function on the value space of μ such that $\Pi=\{\xi : L(\xi)=\lambda\}$ for some real number λ. The fact that Π supports $\bar{R}$ can be expressed (after a suitable choice of signs) by the relation $\lambda=\inf \{L(\mu(E)) : E\in S\}$. Since a numerical measure always assumes its minimum there exists a measurable set E for which $L(\mu(E))=\lambda$; for this E, clearly, $\mu(E)\in R\cap\Pi$.[6]

LEMMA 9. *If μ is a measure with range R and $\xi\in R$ then there exists a measure μ' whose range R' is obtained from R by a translation by $-\xi$; if μ is convex then μ' is also.*

PROOF. If $\xi=\mu(A)$, write $\mu'(E)=\mu(E-A)-\mu(E\cap A)$. A trivial calculation shows that for every measurable set E, $\mu'(E)=\mu((E-A)\cup(A-E))-\xi$ (so that $R'\subset R-\xi$), and $\mu(E)-\xi=\mu'((E-A)\cup(A-E))$ (so that $R-\xi\subset R'$). If μ is convex and E is any measurable set, let F_- and F_+ be measurable subsets of $E-A$ and $E\cap A$ respectively such that $\mu(F_-)=\mu(E-A)/2$ and $\mu(F_+)=\mu(E\cap A)/2$. The equation $\mu'(F_-\cup F_+)=\mu'(E)/2$ implies that μ' is semi-convex and hence convex.

LEMMA 10. *The range of a convex measure is closed.*

PROOF. Let $\mu=(\mu_1, \cdots, \mu_N)$ be a convex measure with range R. If R is one-dimensional then $\mu(E)=\mu_0(E)\xi$, where μ_0 is a numerical measure and ξ is a vector, and the result is clear. Assume next, inductively, that $N>1$ and that the result is known whenever the dimension of R is not greater than $N-1$. The proof of closure will be carried out by showing that $\bar{R}\cap\Pi\subset R$ for every supporting plane Π of $\bar{R}$.

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[6] This lemma and its proof are given in Liapounoff's first paper, cited in footnote 1.

Since, by Lemma 8, Π meets R in at least one point ξ, and since, by Lemma 9, $R-\xi$ is also the range of a convex measure (and is supported at the origin by $\Pi-\xi$), there is no loss of generality in assuming that Π contains the origin. Let L be a linear function such that $\Pi=\{\xi:L(\xi)=0\}$ and $\inf\{L(\mu(E)):E\in S\}=0$; then the function ν, defined by $\nu(E)=L(\mu(E))$, is a non-negative numerical measure. Let E_0 be a measurable set such that $\nu(E_0)=0$ and that the measure μ^-, defined by $\mu^-(E)=\mu(E-E_0)$, is absolutely continuous with respect to ν. Let R^+ be the range of the measure μ^+, defined by $\mu^+(E)=\mu(E\cap E_0)$. Since $L(\mu^+(E))=L(\mu(E\cap E_0))=\nu(E\cap E_0)=0$, $R^+\subset\Pi$, and therefore, by the induction hypothesis, R^+ is closed.

If $\xi\in\bar{R}\cap\Pi$ then there exists a sequence $\{E_n\}$ of measurable sets such that $\mu(E_n)\rightarrow\xi$ and therefore $\nu(E_n)=L(\mu(E_n))\rightarrow L(\xi)=0$. It follows that $\mu(E_n-E_0)=\mu^-(E_n)\rightarrow 0$ and therefore that $\mu^+(E_n)=\mu(E_n\cap E_0)\rightarrow\xi$. Since $\mu^+(E_n)\in R^+$ and R^+ is closed it follows, finally, that $\xi\in R^+\subset R$, and therefore that $\bar{R}\cap\Pi\subset R$.

LEMMA 11. *The range of a non-negative measure is closed.*

PROOF. If $\mu=(\mu_1,\cdots,\mu_N)$ is a non-negative measure let $T=(t_{ij})$ be a nonsingular $N\times N$ matrix with positive elements and write $\mu'=(\mu_1',\cdots,\mu_N')=T\mu$. Each of the non-negative measures $\mu_1',\cdots,\mu_N'$ is absolutely continuous with respect to every other one. Since the range of μ is obtained from this range of μ' by applying T^{-1} it may be assumed that μ itself has the absolute continuity property. It follows that any atom of each coordinate is an atom of all others. If Y is the union of the (at most countably many) common atoms and $Z=X-Y$, and if $\mu'(E)=\mu(E\cap Y)$ and $\mu''(E)=\mu(E\cap Z)$ then μ' is purely atomic and μ'' is purely non-atomic. The range R of μ is the vector sum of the respective ranges R' and R'' of μ' and μ''. The range R'' is closed by Lemmas 7 and 10; the proof of Lemma 3 of my preceding paper (cited in footnote 3) applies without any change to prove that R' is closed. Since the vector sum of two compact sets is compact the proof of Lemma 11 is complete.

The proof of Lemma 7 shows that the range of an arbitrary (not necessarily non-negative) N-dimensional measure is the vector sum of 2^N sets each of which may be shown to be closed by Lemma 11. The preceding results may, accordingly, be summed up as follows.

THEOREM. *The range of every measure is closed; the range of every purely non-atomic measure is convex.*

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ON THE DISTRIBUTION OF THE MAXIMUM OF SUCCESSIVE CUMULATIVE SUMS OF INDEPENDENTLY BUT NOT IDENTICALLY DISTRIBUTED CHANCE VARIABLES

ABRAHAM WALD

1. **Introduction.** Let $X_1, X_2, \cdots$, and so on be a sequence of chance variables and let S_i denote the sum of the first i X's, that is,

$$S_i = X_1 + \cdots + X_i \quad (i = 1, 2, \cdots, \text{ad inf}). \tag{1.1}$$

Let M_N denote the maximum of the first N cumulative sums $S_1, \cdots, S_N$, that is,

$$M_N = \max (S_1, \cdots, S_N). \tag{1.2}$$

The distribution of M_N, in particular the limiting distribution of a suitably normalized form of M_N, has been studied by Erdös and Kac [1][1] and by the author [2] in the special case when the X's are independently distributed with identical distributions.

In this note we shall be concerned with the distribution of M_N when the X's are independent but not necessarily identically distributed. In particular, the mean and variance of X_i may be any functions of i.

In §2 lower and upper limits for M_N are obtained which yield particularly simple limits for the distribution of M_N when the X's are symmetrically distributed around zero.

In §3 the special case is considered when X_i can take only the values 1 and -1 but the probability p_i that $X_i = 1$ may be any function of i. The exact probability distribution of M_N for this case is derived and expressed as the first row of a product of N matrices.

The limiting distribution of $M_N/N^{1/2}$ is treated in §4. Since the interesting limiting case arises when the mean of X_i ($i \leq N$) is not only a function of i but also a function of N, we have to introduce a double sequence of chance variables. That is, for any N we consider a sequence of N chance variables $X_{N1}, \cdots, X_{NN}$. Let μ_{Ni} denote the mean and σ_{Ni} the standard deviation of X_{Ni}. Let, furthermore, S_{Ni} denote the sum $X_{N1} + \cdots + X_{Ni}$ and M_N the maximum of $S_{N1}, \cdots, S_{NN}$. With the help of a method used by Erdös and Kac [1], the following theorem is established in §4:

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[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 1.1 *Let* $\{X_{Ni}\}$ *and* $\{X^*_{Ni}\}$ $(i=1, \cdots, N; N=1, 2, \cdots,$ *ad inf.*) *be two sequences of chance variables such that the following conditions are fulfilled*:

(a) *The X's are independently distributed.*

(b) *The sequence* $\{\sigma_{Ni}\}$ $(i=1, \cdots, N;\ N=1,\ 2, \cdots,$ *ad inf.*) *has a positive lower bound and a finite upper bound.*

(c) $\mu_{Ni}N^{1/2}$ *is a bounded function of i and N.*

(d) *The third absolute moment of* X_{Ni} *is a bounded function of i and N.*

(e) *The conditions* (a)–(d) *remain valid if we replace* X_{Ni} *by* X^*_{Ni}.

(f) *The equation*

$$\lim_{N\to\infty}\left[\frac{\mu^*_{N1}+\cdots+\mu^*_{Nj_i}}{\sigma^{*2}_{N1}+\cdots+\sigma^{*2}_{Nj_i}} - \frac{\mu_{N1}+\cdots+\mu_{Ni}}{\sigma^2_{N1}+\cdots+\sigma^2_{Ni}}\left(\frac{\sigma^2_{N1}+\cdots+\sigma^2_{NN}}{\sigma^{*2}_{N1}+\cdots+\sigma^{*2}_{NN}}\right)^{1/2}\right] = 0 \tag{1.3}$$

holds for all i and N where μ^*_{Ni} *is the mean and* σ^*_{Ni} *is the standard deviation* X^*_{Ni} *and* j_i *is the smallest positive integer for which*

$$\frac{\sigma^{*2}_{N1}+\cdots+\sigma^{*2}_{Nj_i}}{\sigma^{*2}_{N1}+\cdots+\sigma^{*2}_{NN}} \geqq \frac{\sigma^2_{N1}+\cdots+\sigma^2_{Ni}}{\sigma^2_{N1}+\cdots+\sigma^2_{NN}}.$$

Let

$$\bar{M}_N = M^*_N\left(\frac{\sigma^2_{N1}+\cdots+\sigma^2_{NN}}{\sigma^{*2}_{N1}+\cdots+\sigma^{*2}_{NN}}\right)^{1/2} \tag{1.4}$$

where M^*_N *is the same function of the* X^**'s as* M_N *is of the X's. Then for any positive* ϵ *we have*

$$\liminf_{N\to\infty}\left[\text{prob}\left\{M_N < cN^{1/2}\right\} - \text{prob}\left\{\bar{M}_N < (c-\epsilon)N^{1/2}\right\}\right] \geqq 0 \tag{1.5}$$

and

$$\liminf_{N\to\infty}\left[\text{prob}\left\{\bar{M}_N < (c+\epsilon)N^{1/2}\right\} - \text{prob}\left\{M_N < cN^{1/2}\right\}\right] \geqq 0. \tag{1.6}$$

The following corollary is a simple consequence of Theorem 1.1:

COROLLARY 1.1. *Let* N' *be any positive integral valued and strictly increasing function of N for which prob* $\{\bar{M}_{N'} < cN'^{1/2}\}$ *converges to a limit function* $P(c)$ *at all continuity points c of* $P(c)$ *as* $N\to\infty$. *Then also*

$$(1.7) \qquad \lim_{N\to\infty} \text{prob}\,\{M_{N'} < cN'^{1/2}\} = P(c)$$

at all continuity points c of $P(c)$.

The validity of Corollary 1.1 can be derived from that of Theorem 1.1 as follows: Let $c=c_0$ be a continuity point of $P(c)$ and substitute N' for N in (1.5) and (1.6). For any positive ρ all limit points of prob $\{\bar{M}_{N'}<(c_0-\epsilon)N'^{1/2}\}$ and prob $\{\bar{M}_{N'}<(c_0+\epsilon)N'^{1/2}\}$ will lie in the interval $[P(c_0)-\rho, P(c_0)+\rho]$ for sufficiently small ϵ. Hence, equations (1.5) and (1.6) imply that

$$(1.8) \qquad \begin{aligned} P(c_0) - \rho &\leq \liminf_{N\to\infty} \text{prob}\,\{M_{N'} < c_0N'^{1/2}\} \\ &\leq \limsup_{N\to\infty} \text{prob}\,\{M_{N'} < c_0N'^{1/2}\} \leq P(c_0) + \rho. \end{aligned}$$

Since (1.8) is true for any positive number ρ, Corollary 1.1 is proved.

The result in Corollary 1.1 can be expressed also by saying that for any subsequence $\{N'\}$ of $\{N\}$ for which $\bar{M}_{N'}/N'^{1/2}$ has a limiting distribution as $N\to\infty$, also $M_{N'}/N'^{1/2}$ has a limiting distribution which is equal to that of $\bar{M}_{N'}/N'^{1/2}$.

It can easily be verified that the conditions (e) and (f) can always be satisfied for chance variables X^*_{Ni} which take only the values 1 and -1 with properly chosen probabilities. Thus, the results of §3 may be used to compute

$$\text{prob}\left\{M^*_N < N^{1/2}c\left(\frac{\sigma^{*2}_{N1} + \cdots + \sigma^{*2}_{NN}}{\sigma^2_{N1} + \cdots + \sigma^2_{NN}}\right)^{1/2}\right\}.$$

2. **Derivation of upper and lower bounds for M_N.** Let $X_1, \cdots, X_N$ be a set of N variables and let

$$(2.1) \qquad \tilde{X}_i = X_{N-i+1} \qquad (i = 1, 2, \cdots, N).$$

Let, furthermore,

$$(2.2) \qquad \tilde{M}_i = \max(\tilde{X}_i, \tilde{X}_i + \tilde{X}_{i-1}, \cdots, \tilde{X}_i + \cdots + \tilde{X}_1), \qquad (i = 1, \cdots, N).$$

Clearly

$$(2.3) \qquad \tilde{M}_N = M_N = \max(X_1, X_1 + X_2, \cdots, X_1 + \cdots + X_N).$$

If $X_1, \cdots, X_N$ are independent chance variables, the chance variables $\tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_N$ form a simple Markoff chain, that is, the conditional distribution of $\tilde{M}_{i+1}$, given $\tilde{M}_1, \cdots, \tilde{M}_i$, depends only on $\bar{M}_i$. This is an immediate consequence of the relations:

$$\bar{M}_{i+1} = \bar{M}_i + \bar{X}_{i+1} \qquad \text{if } \bar{M}_i > 0 \tag{2.4}$$

and

$$\bar{M}_{i+1} = \bar{X}_{i+1} \qquad \text{if } \bar{M}_i \leqq 0. \tag{2.5}$$

We shall now prove the following theorem:

Theorem 2.1. *The inequality*

$$\bar{M}_i \leqq | \epsilon_1 \bar{X}_1 + \cdots + \epsilon_i \bar{X}_i | \qquad (i = 1, \cdots, N) \tag{2.6}$$

holds where $\epsilon_1 = 1$, $\epsilon_i = 1$ *if* $\epsilon_1 \bar{X}_1 + \cdots + \epsilon_{i-1} \bar{X}_{i-1} > 0$ *and* $\epsilon_i = -1$, *if* $\epsilon_1 \bar{X}_1 + \cdots + \epsilon_{i-1} \bar{X}_{i-1} \leqq 0$.

Proof. Clearly, (2.6) holds for $i = 1$. We shall prove (2.6) for $i+1$ assuming that it holds for i. For this purpose it is sufficient to show, because of (2.4) and (2.5), that

$$| \epsilon_1 \bar{X}_1 + \cdots + \epsilon_{i+1} \bar{X}_{i+1} | - | \epsilon_1 \bar{X}_1 + \cdots + \epsilon_i \bar{X}_i | \geqq \bar{X}_{i+1}. \tag{2.7}$$

Denote $| \epsilon_1 \bar{X}_1 + \cdots + \epsilon_i \bar{X}_i |$ by c_i. If $c_i > 0$, then $\epsilon_{i+1} = 1$ and inequality (2.7) goes over into

$$| c_i + \bar{X}_{i+1} | - c_i \geqq \bar{X}_{i+1}, \tag{2.8}$$

which is obviously true. If $c_i \leqq 0$, $\epsilon_{i+1} = -1$ and inequality (2.7) is equivalent with

$$| | c_i | + \bar{X}_{i+1} | - | c_i | \geqq \bar{X}_{i+1}, \tag{2.9}$$

which is obviously true. Hence, Theorem 2.1 is proved.

We shall now prove a theorem giving a lower bound for $\bar{M}_i$.

Theorem 2.2. *The inequality*

$$\bar{K}_i = | \epsilon_1 \bar{X}_1 + \cdots + \epsilon_i \bar{X}_i | - 2 \max_{j \leqq i} | \bar{X}_j | \leqq \bar{M}_i \qquad (i = 1, \cdots, N) \tag{2.10}$$

holds where the ϵ*'s are defined as in Theorem* 2.1.

Proof. Theorem 2.2 is obviously true for $i = 1$. We shall assume that it is valid for i and we shall prove it for $i+1$. It follows from (2.4) and (2.5) that

$$\bar{M}_{i+1} - \bar{M}_i \geqq \bar{X}_{i+1}, \tag{2.11}$$

$$\bar{M}_{i+1} \geqq \bar{X}_{i+1}. \tag{2.12}$$

Hence, to prove (2.10) for $i+1$ assuming that it is true for i, it is sufficient to show that at least one of the following two inequalities holds:

$$\bar{K}_{i+1} - \bar{K}_i \leqq \bar{X}_{i+1}, \tag{2.13}$$

$$\bar{K}_{i+1} \leqq \bar{X}_{i+1}. \tag{2.14}$$

Consider first the case when $|\bar{X}_{i+1}| \leqq |\epsilon_1\bar{X}_1 + \cdots + \epsilon_i\bar{X}_i|$. In this case (2.13) always holds, as can easily be verified. If $|\bar{X}_{i+1}| > |\epsilon_1\bar{X}_1 + \cdots + \epsilon_i\bar{X}_i|$ and $\bar{X}_{i+1} \geqq 0$, then (2.13) holds again. If $|\bar{X}_{i+1}| > |\epsilon_1\bar{X}_1 + \cdots + \epsilon_i\bar{X}_i|$ and $\bar{X}_{i+1} < 0$, then $|\epsilon_1\bar{X}_1 + \cdots + \epsilon_i\bar{X}_i + \epsilon_{i+1}\bar{X}_{i+1}| \leqq |\bar{X}_{i+1}|$ and, therefore, $\bar{K}_{i+1} \leqq |\bar{X}_{i+1}| - 2 \max_{j \leqq i+1} |X_j| \leqq -|\bar{X}_{i+1}| = \bar{X}_{i+1}$. Thus, in this case the inequality (2.14) holds. This completes the proof of Theorem 2.2.

Since $\bar{M}_N = M_N$, Theorems 2.1 and 2.2 yield the following limits for M_N

$$\begin{aligned} &|\epsilon_1\bar{X}_1 + \cdots + \epsilon_N\bar{X}_N| - 2 \max_{i \leqq N} |\bar{X}_i| \\ &\qquad\qquad \leqq M_N \leqq |\epsilon_1\bar{X}_1 + \cdots + \epsilon_N\bar{X}_N|. \end{aligned} \tag{2.15}$$

Suppose now that $X_1, \cdots, X_N$ are chance variables such that the conditional distribution of X_i $(i = 1, \cdots, N)$ for any given values of $X_{i+1}, \cdots, X_N$ is symmetric around the origin. Then the probability distribution of $|\epsilon_1\bar{X}_1 + \cdots + \epsilon_N\bar{X}_N|$ is the same as that of $|X_1 + \cdots + X_N|$, and the distribution of $|\epsilon_1\bar{X}_1 + \cdots + \epsilon_N\bar{X}_N| - 2 \max_{i \leqq N} |\bar{X}_i|$ equals that of $|X_1 + \cdots + X_N - 2\max_{i \leqq N}|X_i||$. It then follows from (2.15) that the following theorem holds:

THEOREM 2.3. *If the conditional distribution of X_i $(i = 1, 2, \cdots, N)$, for any given value of $X_{i+1}, \cdots, X_N$ is symmetric around the origin, the inequality*

$$\begin{aligned} &\text{prob}\,\{|X_1 + \cdots + X_N| < c\} \leqq \text{prob}\,\{M_N < c\} \\ &\qquad\qquad \leqq \text{prob}\,\{|X_1 + \cdots + X_N| - 2 \max_{i \leqq N} |X_i| < c\} \end{aligned} \tag{2.16}$$

holds for any value c.

Inequality (2.15) has also some interesting implications for the asymptotic distribution theory of M_N. In most cases we shall be concerned with the limiting distribution of $M_N/N^{1/2}$ as $N \to \infty$ (this is the case discussed in §4). If $(1/N^{1/2}) \max_{i \leqq N} |X_i|$ converges stochastically to zero, as will usually be the case, inequality (2.15) implies that the limiting distribution of $M_N/N^{1/2}$ is the same as that of $(1/N^{1/2})|\epsilon_1\bar{X}_1 + \cdots + \epsilon_N\bar{X}_N|$.

3. The distribution of M_N when X_i can take only the values 1 and -1. Let $X_1, \cdots, X_N$ be independently distributed chance variables such that X_i can take only the values 1 and -1. Let p_i denote the probability that $X_i = 1$. The probability that $X_i = -1$ is then equal to $1 - p_i = q_i$.

Let $\bar{X}_i$ and $\bar{M}_i$ $(i = 1, \cdots, N)$ be defined by (2.1) and (2.2), respectively. One can easily verify that $\bar{M}_i$ can take only the values $-1, 0, 1, 2, \cdots, i$. Let c_{ij} denote the probability that $\bar{M}_i = j$ for $j = 1, \cdots, i$, and let c_{i0} be the probability that $\bar{M}_i \leq 0$. It follows from the definition of the $\bar{M}$'s that the following recursion formulas hold:

$$c_{i+1,0} = q_{i+1}c_{i0} + q_{i+1}c_{i1}, \tag{3.1}$$

$$c_{i+1,j} = p_{i+1}c_{i,j-1} + q_{i+1}c_{i,j+1} \qquad (j = 1, 2, \cdots, i+1). \tag{3.2}$$

Since $\bar{M}_N = M_N$, we have

$$\text{prob}\,\{M_N = j\} = c_{Nj} \qquad \text{for } j = 1, \cdots, N, \tag{3.3}$$

$$\text{prob}\,\{M_N \leq 0\} = c_{N0}. \tag{3.4}$$

We shall now construct N square matrices $A_1, \cdots, A_N$, each having $N+1$ rows and $N+1$ columns, such that the first row of the product matrix $A_1A_2 \cdots A_N$ is equal to $(c_{N0}, c_{N1}, \cdots, c_{NN})$. Let a^k_{ij} denote the element in the ith row and jth column of the matrix A_k $(i, j = 1, \cdots, N+1;\ k = 1, \cdots, N)$. We put

$$\begin{aligned} &a^k_{11} = q_k; \qquad a^k_{i,i+1} = p_k \qquad (i = 1, 2, \cdots, N); \\ &\qquad\qquad a^k_{i,i-1} = q_k \qquad (i = 2, 3, \cdots, N+1) \end{aligned} \tag{3.5}$$

and all other elements a^k_{ij} equal to zero. It then follows easily from the recursion formulas (3.1) and (3.2) that the first row of the product $A_1A_2 \cdots A_N$ is equal to $(c_{N0}, c_{N1}, \cdots, c_{NN})$. Thus, the first row of the product $A_1A_2 \cdots A_N$ yields the exact probability distribution of M_N.

Starting with the initial values $c_{10} = q_1$, $c_{11} = p_1$, $c_{1j} = 0$ for $j > 1$, the final values $c_{N0}, c_{N1}, \cdots, c_{NN}$ can be best computed by repeated application of the recursion formulas (3.1) and (3.2).

4. Proof of Theorem 1.1. Let $\{X_{Ni}\}$ and $\{X^*_{Ni}\}$ be two double sequences of chance variables for which conditions (a)–(f) of Theorem 1.1 are fulfilled. Let k be a positive integer and $N_1, \cdots, N_k$ a set of positive integers such that $N_1 < N_2 < \cdots < N_k = N$. Let, furthermore,

$$(4.1)\qquad P_{N,k}(c) = \text{prob}\left\{\max\left(S_{NN_1}, S_{NN_2}, \cdots, S_{NN_k}\right) < cN^{1/2}\right\}.$$

Because of conditions (b) and (c) of Theorem 1.1, there exist two finite values A and B such that $A \geqq N\mu_{Ni}^2$ and $B \geqq \sigma_{Ni}^2$ for all N and i. Let $\phi(k)$ be an upper bound of the values

$$(4.2)\qquad \frac{N_1}{N}, \quad \frac{N_2 - N_1}{N}, \cdots, \quad \frac{N_k - N_{k-1}}{N}.$$

For any positive ϵ the following inequality holds:

$$(4.3)\qquad P_{N,k}(c - \epsilon) - \frac{\phi(k)}{\epsilon^2}\left[B + A\phi(k)\right] \leqq P_N(c) \leqq P_{N,k}(c),$$

where $P_N(c) = \text{prob}\left\{M_N < cN^{1/2}\right\}$. Using a method given by Erdős and Kac [1], the author [2] has proved the above inequality when $\mu_{Ni} = \mu_N$, $\sigma_{Ni} = 1$ and $N_j = [jN/k]$. To adapt the proof given in [2] to the more general case treated here, it is sufficient to replace the right-hand member of (2.6) in [2] by

$$(4.4)\qquad \frac{(N_{i+1} - N_i)B + (N_{i+1} - N_i)^2\mu_N^2}{\epsilon^2 N},$$

where $\mu_N^2 = \max\left(\mu_{N1}^2, \cdots, \mu_{NN}^2\right)$.

For the purpose of proving Theorem 1.1, we shall choose N_j to be the smallest positive integer for which

$$(4.5)\qquad \sigma_{N1}^2 + \cdots + \sigma_{NN_j}^2 \geqq \frac{j(\sigma_{N1}^2 + \cdots + \sigma_{NN}^2)}{k}.$$

Since σ_{Ni}^2 has a positive lower bound and a finite upper bound, there exists a positive constant h, independent of k, such that h/k is an upper bound of the values (4.2). It then follows from (4.3) that

$$(4.6)\qquad P_{N,k}(c - \epsilon) - \frac{1}{\epsilon^2 k}(a - b/k) \leqq P_N(c) \leqq P_{N,k}(c)$$

when a and b are positive constants independent of N, k, c and ϵ.

Clearly, if Theorem 1.1 is true for the special case when $\sigma_{N1}^2 + \cdots + \sigma_{NN}^2 = \sigma_{N1}^{*2} + \cdots + \sigma_{NN}^{*2}$, it must be true also in the general case. Hence, it is sufficient to prove Theorem 1.1 when $\sigma_{N1}^2 + \cdots + \sigma_{NN}^2 = \sigma_{N1}^{*2} + \cdots + \sigma_{NN}^{*2}$. In what follows we shall therefore restrict ourselves to this special case.

Let N_j^*, $P_{N,k}^*(c)$, and $P_N^*(c)$ have the same meaning with reference to the X^*'s as N_j, $P_{N,k}(c)$, and $P_N(c)$ with reference to the X's. Then we have

$$(4.7)\qquad P^*_{N,k}(c-\epsilon) - \frac{1}{\epsilon^2 k}(a^* + b^*/k) \leq P^*_N(c) \leq P^*_{N,k}(c),$$

where a^* and b^* are positive constants independent of N, k, c and ϵ.

Let $G^N_{k1}, G^N_{k2}, \cdots, G^N_{kk}$ be independently and normally distributed chance variables and let the mean and standard deviation of G^N_{ki} be equal to the mean and standard deviation of $(k/N)^{1/2}(S_{N\nu_i} - S_{N\nu_{i-1}})$, respectively. Let, furthermore,

$$(4.8)\qquad \begin{aligned} &Q_{N,k}(c) \\ &\quad = \text{prob}\,\{\max\,(G^N_{k1}, G^N_{k1} + G^N_{k2}, \cdots, G^N_{k1} + \cdots + G^N_{kk}) < ck^{1/2}\}. \end{aligned}$$

Clearly, the mean and standard deviation of G^N_{ki} are bounded functions of N, k and i. Furthermore, the standard deviation of G^N_{ki} has a positive lower bound. It then follows from condition (d) and the central limit theorem that

$$(4.9)\qquad \lim_{N\to\infty}\,[Q_{N,k}(c) - P_{N,k}(c)] = 0.$$

Let G^{*N}_{ki} and $Q^*_{N,k}(c)$ have the same meaning with reference to the X^*'s as G^N_{ki} and $Q_{N,k}(c)$ with reference to the X's. We then have

$$(4.10)\qquad \lim_{N\to\infty}\,[Q^*_{N,k}(c) - P^*_{N,k}(c)] = 0.$$

It follows from condition (f) of Theorem 1.1 that

$$(4.11)\qquad \lim_{N\to\infty} E(G^N_{ki} - G^{*N}_{ki}) = 0,$$

$$(4.12)\qquad \lim_{N\to\infty} E[(G^N_{ki})^2 - (G^{*N}_{ki})^2] = 0.$$

Hence

$$(4.13)\qquad \lim_{N\to\infty}\,[Q_{N,k}(c) - Q^*_{N,k}(c)] = 0.$$

From (4.9) and (4.10) and (4.13) we obtain

$$(4.14)\qquad \lim_{N\to\infty}\,[P_{N,k}(c) - P^*_{N,k}(c)] = 0.$$

Equations (4.6) and (4.14) give

$$(4.15)\qquad \liminf_{N\to\infty}\left[P_N(c) - P^*_{N,k}(c-\epsilon) + \frac{1}{\epsilon^2 k}\left(a + \frac{b}{k}\right)\right] \geq 0$$

and

$$\liminf_{N\to\infty} [P^*_{N,k}(c) - P_N(c)] \geqq 0. \tag{4.16}$$

Since

$$P^*_{N,k}(c - \epsilon) \geqq P^*_N(c - \epsilon) \tag{4.17}$$

and since, because of (4.7),

$$P^*_{N,k}(c) - \frac{1}{\epsilon^2 k}(a^* + b^*/k) \leqq P^*_N(c + \epsilon), \tag{4.18}$$

we obtain from (4.15) and (4.16)

$$\liminf_{N\to\infty} \left[P_N(c) - P^*_N(c - \epsilon) + \frac{1}{\epsilon^2 k}\left(a + \frac{b}{k}\right)\right] \geqq 0 \tag{4.19}$$

and

$$\liminf_{N\to\infty} \left[P^*_N(c + \epsilon) + \frac{1}{\epsilon^2 k}\left(a^* + \frac{b^*}{k}\right) - P_N(c)\right] \geqq 0. \tag{4.20}$$

Hence, since k can be chosen arbitrarily large, we obtain

$$\liminf_{N\to\infty} [P_N(c) - P^*_N(c - \epsilon)] \geqq 0 \tag{4.21}$$

and

$$\liminf_{N\to\infty} [P^*_N(c + \epsilon) - P_N(c)] \geqq 0. \tag{4.22}$$

This concludes the proof of Theorem 1.1. It may be of interest to note that (4.21) and (4.22) imply that for any subsequence $\{N'\}$ of the sequence $\{N\}$ we have

$$\begin{aligned} \liminf_{N\to\infty} P^*_{N'}(c - \epsilon) \leqq \liminf_{N\to\infty} P_{N'}(c) &\leqq \limsup_{N\to\infty} P_{N'}(c) \\ &\leqq \limsup_{N\to\infty} P^*_{N'}(c + \epsilon). \end{aligned} \tag{4.23}$$

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1. P. Erdös and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 292–302.

2. A. Wald, *Limit distribution of the maximum and minimum of successive cumulative sums of random variables*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 142–153.

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APPROXIMATION BY POLYGONS AND POLYHEDRA

LÁSZLÓ FEJES TÓTH

In order to investigate as to what order of magnitude a plane (or a skew) curve can be approximated by n-sided polygons, we have to start from a definition of the *deviation*. One of the usual definitions is as follows:[1] the deviation $\eta(H, K)$ of two curves H and K is the smallest number η for which H is contained in the neighbourhood of K with radius η and conversely K is contained in the neighbourhood of H with radius η. By a neighbourhood with radius η of a plane (skew) curve we understand the set of the points of all the circles (spheres) with radius η whose centres are the points of the curve.

This notion of deviation satisfies the usual requirements; for example, the triangle inequality is satisfied, that is, if H, K and L are three curves, then $\eta(H, K)+\eta(K, L)\geqq\eta(H, L)$.

Corresponding to a rectifiable Jordan-curve J, let us construct the polygon $\bar{P}_n$ whose deviation from J is least for all inscribed n-sided polygons, P_n: $\eta(\bar{P}_n, J)\leqq\eta(P_n, J)$. If we impose further restrictions upon J, the sequence $\eta(\bar{P}_1, J)$, $\eta(\bar{P}_2, J)$, $\cdots$ will tend towards zero with order $1/n^2$, that is, the limit value

$$1/a = \liminf n^2\eta(P_n, J) = \lim n^2\eta(\bar{P}_n, J)$$

exists. We shall call a the *approximability of J by inscribed*[2] *polygons.* a is a constant which depends only on J. The approximability of a circle of radius r is, for example, $a=2/\pi^2r$.

Interesting problems arise if we consider which curve can be approximated the worst among a certain class of curves, that is, whose approximability is the smallest. We easily obtain,[3] for example, that among all closed convex curves with continuous curvature of given

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[1] See, for example, Bonnesen-Fenchel, *Theorie der konvexen Körper*, Berlin, 1934.

[2] Similarly in the case of a convex arc we can define the approximability by circumscribed polygons, which equals a. The approximability by arbitrary polygons equals $2a$. In all that follows we restrict ourselves to the case of inscribed polygons and polyhedra.

[3] Cf. L. Fejes, *Einige Extremaleigenschaften des Kreisbogens bezüglich der Annäherung durch Polygone*, Acta Univ. Szeged. vol. 10 (1943) pp. 164–173. Starting from a different notion of deviation—the so-called deviation of area—among all convex arcs AB lying in the interior of the triangle AOB and enclosing a given area with the side AB, the conic arc touching AO and BO can be approximated worst. See L. Fejes, *Über eine Extremaleigenschaft der Kegelschnittbogen*, Monatshefte für Mathematik und Physik vol. 50 (1943) pp. 317–326.

perimeter the circle, more generally that among all continuously curved convex arcs of given length l and total curvature ω the arc of a circle, can be approximated worst.

This is an immediate consequence of the explicit formula to be proved below, which expresses a by the natural equation of J. We shall see that *if the curvature G of a rectifiable Jordan-curve J of length l is a continuous function of the length of arc s, then*

$$\frac{1}{a} = \frac{1}{8}\left(\int_0^l |G(s)|^{1/2} ds\right)^2,$$

where a is the approximability of J by inscribed polygons.[4]

Hence by Schwarz's inequality

$$\frac{1}{a} \leq \frac{1}{8}\int_0^l ds \cdot \int_0^l |G(s)|\, ds = \frac{1}{8}\, l\omega,$$

where equality holds only in the case $G(s) = \text{const.}$, that is, when J is an arc of a circle.

We shall prove our formula under the more general condition that J is a *skew curve*.[5] It is worthwhile to note that the torsion of the skew curve does not enter into our formula and hence the approximability is invariant under all transformations which do not alter the curvature as a function of the length of arc.[6]

First we estimate the deviation $\eta(p, j)$ of a chord p from the corresponding arc j. For this purpose let us write the equations of j in a coordinate system which coincides with the moving trihedron in the center of the arc j of length σ:[7]

$$x_1 = s + s^2\epsilon_1(s),$$

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[4] We give two more analogous results without proof, and for the sake of simplicity only for the case of a closed convex curve J: $\lim n^2(t-t_n) = (1/12)(\int_0^l G^{1/3}ds)^3$, $\lim n^2(l-l_n) = (1/24)(\int_0^l G^{1/3}ds)^3$ where t denotes the area of the oval surrounded by J, t_n the area of the inscribed n-sided polygon having a maximal area, and l_n the perimeter of the inscribed n-sided polygon having a maximal perimeter.

Let us observe that $\int_0^l G^{1/3}ds$ is the "affine length-of-arc" of J. (See, for example, W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. 2, Berlin, 1923.)

We mention also without a proof that if we approximate a curve lying on a surface by polygons consisting of arcs of geodesic lines, we have to put into our formulas the geodetic curvature instead of G.

[5] This extension of my formula was first found by my pupil J. Horváth but his proof was not quite satisfactory.

[6] Such deformations of a curve can be illustrated on a simple model. See Hilbert-Cohn-Vossen, *Anschauliche Geometrie*, Berlin 1932, footnote 2, p. 182.

[7] See, for example, A. R. Forsyth, *Lectures on the differential geometry of curves and surfaces*, Cambridge 1912, pp. 10–11.

$$\begin{aligned} x_2 &= 2^{-1}Gs^2 + s^2\epsilon_2(s), \qquad -\sigma/2 \leqq s \leqq \sigma/2,\\ x_3 &= s^2\epsilon_3(s), \end{aligned}$$

where G is the curvature at the considered point and $\epsilon_1(s)$, $\epsilon_2(s)$, $\epsilon_3(s)$ are continuous functions of s which vanish for $s=0$.

Let us consider at the same time the arc j' whose equations are

$$\begin{aligned} x_1 &= s,\\ x_2 &= 2^{-1}Gs^2, \qquad -\sigma/2 \leqq s \leqq \sigma/2,\\ x_3 &= 0. \end{aligned}$$

Obviously $\eta(j, j')$ can not be greater than the maximal distance of two points belonging to the same parameter:

$$\begin{aligned} \eta(j, j') &\leqq \max s^2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)^{1/2}\\ &\leqq (\sigma^2/4) \max (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)^{1/2} = (\bar{\sigma}^2/4)\epsilon(\bar{\sigma}); \qquad |\bar{\sigma}| \leqq \sigma/2. \end{aligned}$$

The deviation of the chord p' connecting both ends of j' from p has the same upper bound: $\eta(p, p') \leqq (\sigma^2/4)\epsilon(\bar{\sigma})$ and the deviation of j' from p' is obviously $\eta(p', j') = 2^{-1}G\sigma^2/4$.

Hence from the triangle inequalities

$$\eta(j, p) \leqq \eta(j, j') + \eta(j', p') + \eta(p', p)$$

and

$$\eta(j', p') \leqq \eta(j', j) + \eta(j, p) + \eta(p, p')$$

we get $8^{-1}G\sigma^2 - 2^{-1}\sigma^2\epsilon(\bar{\sigma}) \leqq \eta(p, j) \leqq 8^{-1}G\sigma^2 + 2^{-1}\sigma^2\epsilon(\bar{\sigma})$.

Let us consider now a sequence $P_1, P_2, \cdots$ of polygons inscribed into J so that the greatest among the sides $p_1^n, p_2^n, \cdots, p_n^n$ of P_n shall tend to zero with an increasing n. By the above inequality for any such sequence

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\eta(j_i^n, p_i^n))^{1/2} = \frac{1}{8^{1/2}}\int_0^l G^{1/2} ds.$$

Let us investigate the sequence of polygons for which

$$\eta(p_i^n, j_i^n) = \eta(J, P_n) \qquad i = 1, 2, \cdots, n;\ n = 1, 2, \cdots.$$

In this case

$$\lim n(\eta(J, P_n))^{1/2} = \frac{1}{8^{1/2}}\int_0^l G^{1/2} ds.$$

As $\eta(J, P_n)$ is the maximal among the deviations $\eta(p_i^n, j_i^n)$, we get for every other sequence

$$\lim n(\eta(J, P_n))^{1/2} \geqq \frac{1}{8^{1/2}} \int_0^l G^{1/2} ds,$$

assuming that the limit value on the left side exists. Therefore

$$\liminf n(\eta(J, P_n))^{1/2} = \frac{1}{8^{1/2}} \int_0^l G^{1/2} ds \qquad \text{q.e.d.}$$

We have seen that among all ovals in the plane with given perimeter the circle can be approximated worst. The question arises as to what can be said about the approximation by polygons consisting of arcs of circles.

Let us restrict ourselves to sketch only the main lines of the proof. For the detailed exposition the proof given in the preceding section can serve as an example.

We define the *approximability α of a curve J by inscribed circular arc-polygons* by the limit value $1/\alpha = \liminf n^2\eta(C_n, J)$ where C_n denotes a polygon consisting of n arcs of circles whose vertices lie on J.

Let $\overline{C}_n$ be the polygon whose deviation from J is least for all polygons C_n. We may suppose that for the arcs of circles $c_1, c_2, \cdots, c_n$ of $\overline{C}_n$ and the corresponding arcs, $j_1, j_2, \cdots, j_n$ of J

$$\eta(c_i, j_i) = \eta(\overline{C}_n, J), \qquad i = 1, 2, \cdots, n,$$

holds.

To estimate the deviation $\eta(c, j)$ we must also consider the members of third order in the canonical equations. Supposing $G'(s)$ to be continuous, we have

$$\begin{aligned} x_1 &= s - 6^{-1}G^2s^3 + \cdots, \\ x_2 &= 2^{-1}Gs^2 + 6^{-1}G's^3 + \cdots. \end{aligned} \qquad -\sigma/2 \leqq s \leqq \sigma/2,$$

The equations of the circle of curvature at the same point are

$$x_1 = s - 6^{-1}G^2s^3 + \cdots, \quad x_2 = 2^{-1}Gs^2 + \cdots$$

If $G' \neq 0$, the circle of curvature intersects the curve and the shortest distances of the circles of curvature from the points of j with parameter values $s = \sigma/2$ and $s = -\sigma/2$ are $6^{-1}|G'|(\sigma/2)^3 + \cdots$. Let us turn the circle of curvature around the point of intersection so that the point with parameter value $s = \sigma/2$ lies on J. In this case a point of the circle with parameter value $-\sigma/2 + \cdots$ lies on J too, and two points of the arc $-\sigma/2 \leqq s \leqq \sigma/2$ of the circle of curvature with the parameter values $s = \pm(1/3^{1/2})(\sigma/2) + \cdots$ become the most remote ones from J, the maximal distance being $(3^{1/2}/216)|G'|\sigma^3 + \cdots$.

Consequently the arc j can be approximated by a circular arc c whose ends are coincident with the ends of j so that

$$\eta(c, j) = \frac{3^{1/2}}{216}|G'|\sigma^3 + \cdots .$$

On the other hand, it can be seen easily that this constant is exact.

Let us take the third root of these equalities involving the arcs c_i of the polygon $\overline{C}_n$. By summing up and passing to the limit we get the theorem: *Let J be a plane curve with the natural equation $G=G(s)$, $0 \leqq s \leqq l$, where $G'(s)$ is a continuous function. The reciprocal value of the approximability α of J by inscribed circular arc polygons is*

$$\frac{1}{\alpha} = \frac{3^{1/2}}{216}\left(\int_0^l |G'|^{1/3} ds\right)^3 .$$

Similarly it can be seen that this same α is the approximability of a skew curve by inscribed polygons consisting of arcs of screw lines. The approximability of a skew curve by inscribed circular arc polygons is

$$\frac{1}{\alpha} = \frac{3^{1/2}}{216}\left(\int_0^l (|G'|^2 + G^2T^2)^{1/6} ds\right)^3 ,$$

where T denotes the torsion.

Making use of Hölder's integral inequality[8]

$$\left(\int_a^b \phi^{1/3} dx\right)^3 \leqq (b-a)^2 \int_a^b \phi dx; \qquad \phi(x) \geqq 0,\ a \leqq x \leqq b$$

we get

$$\frac{1}{\alpha} \leqq \frac{3^{1/2}}{216} l^2 \int_0^l |G'| ds,$$

which means that *among the plane curves of monotone curvature with the length l and maximal curvature M the curve with the natural equation $G=(M/l)s$ can be approximated worst by polygons consisting of arcs of circles*. This curve is the so-called *clothoid*[9] (Cornu's spiral) which plays an important role in the theory of the interference of light.

Contrary to the approximability by common polygons, there is no oval of given perimeter which can be approximated worst by polygons of circular arcs; for, we can easily construct an oval with given perimeter out of clothoid arcs whose approximability α is arbitrarily small.

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[8] See, for example, Hardy-Littlewood-Pólya, *Inequalities*, Cambridge, 1934.

[9] See, for example, G. Kowalewski, *Natürliche Geometrie*, Berlin, 1931.

In the following we make no claim for rigour.

Let us consider an ovaloid S. We define the *approximability A of S by inscribed polyhedra* by the limit $1/A = \liminf n\eta(Q_n, S)$, where Q_n denotes an inscribed polyhedron having n vertices.[10]

We assert that *the approximability A by inscribed polyhedra of an ovaloid S with continuous Gauss measure of curvature G can be expressed by the surface integral of $G^{1/2}$:*

$$\frac{1}{A} = \frac{3^{1/2}}{9} \int_S G^{1/2} df.$$

The same formula holds for the approximability by polyhedric surfaces of a part of an elliptically curved surface bordered by a continuously curved closed curve which does not cut itself. Thus *the approximability is invariant under all deformations of the surface,* that is, under the transformations which do not alter the line element at any point.

According to Schwarz's inequality

$$\frac{1}{A} \leq \frac{3^{1/2}}{9} \left(\int_S df \cdot \int_S G df \right)^{1/2} = \frac{3^{1/2}}{9} (F\Omega)^{1/2}$$

where $\Omega = \int_S G df$ is the total curvature of S, F its area. Equality holds only for a surface which can be deformed into a part of a sphere of radius $(F/\Omega)^{1/2}$. Particularly: *among all closed ovaloids with given area the sphere can be approximated worst.*[11]

For the sake of simplicity we restrict ourselves to closed ovaloids. Let us consider a small triangle Δ whose vertices lie on S and let Δ' be the small part of S lying near Δ whose normals intersect Δ. Let us place Δ in the neighbourhood of a point M of S, so that, fixing the deviation $\eta(\Delta, \Delta')$, the area of Δ—which we denote by Δ too—shall be as great as possible. Let the plane drawn parallel to the tangential plane at M at a distance η meet S in the Dupin's curve D. The position of Δ in question will be given by the maximal triangle inscribable in D.

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[10] We arrive at the same approximability A by replacing the word "vertices" by "faces" or "inscribed" by "circumscribed" or both. The approximability by arbitrary polyhedra is $2A$.

[11] Starting from the notion of deviation of volume, the reciprocal value of the approximability is proportional to the square of the "affine surface." (L. Fejes, *Über die isoperimetrische bzw. isepiphane Eigenschaft der Ellipse bzw. des Ellipsoids*, Matematikai és Természettudományi Értesítő vol. 62 (1943) pp. 93–94; L. Fejes, *Extremális pontrendszerek a síkban, a gömbfelületen és a térben*, Acta Sci. Math. et Nat. vol. 23 (Kolozsvár 1944). Hence by the isepiphane property of the ellipsoid found by Blaschke (*Vorlesungen über Differentialgeometrie*, vol. 2, pp. 198–201) the ellipsoid can be approximated worst among all ovaloids of given volume.

In the first approximation D can be considered as an ellipse with axes $(2\eta R_1)^{1/2}$ and $(2\eta R_2)^{1/2}$, where R_1 and R_2 are the two main radii of curvature at the point M. Thus the area of the maximal triangle inscribable in D is

$$\Delta_{\max} = \frac{3\cdot 3^{1/2}}{4}(4\eta^2 R_1 R_2)^{1/2} + \cdots ,$$

that is, for the triangle with area Δ

$$\eta(\Delta, \Delta') > \frac{2\cdot 3^{1/2}}{9} G^{1/2}\Delta + \cdots .$$

We can suppose that every face of the polyhedron Q_n inscribed in S is a triangle, since otherwise we could divide them into triangles. But a polyhedron consisting of triangles with n vertices has $2n-4$ faces. Thus summing up the above inequalities concerning the faces of Q_n and increasing n indefinitely we get

$$2 \liminf n\eta(Q_n, S) \geq \frac{2\cdot 3^{1/2}}{9}\int_S G^{1/2} df.$$

On the other hand the bound on the right side can be attained by a suitably-chosen sequence of polyhedra; this completes the proof.

Let us note that the equality just proved does not hold for hyperbolically-curved surfaces. Consider, for example, the part H of a hyperboloid of revolution of one sheet which lies between two planes parallel to its minimal circular section at the same distances on both sides. Let $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ be two regular polygons inscribable in the circles bordering H, so that A_iB_i $(i=1, 2, \cdots, n)$ shall be generators of H. The deviation $\eta(H, Q_{2n})$ between H and the polyhedric surface Q_{2n} having the faces $A_1A_2B_1, \cdots, A_nA_1B_n$, $B_1B_2A_2, \cdots, B_nB_1A_1$ is identical with the deviation between $A_1A_2 \cdots A_n$ and the circle around it; hence its order of magnitude is $1/n^2$ and not $1/n$.

Finally we draw a conclusion from our estimation concerning $\eta(Q_n, S)$. Let S be a sphere of radius R and Q_n a system of n points on S. Write the greatest possible spherical calotte s_n on S which does not contain any point of Q_n.

If we denote the radius of the circle at the base of s_n by r_n and the least convex covering of Q_n by Q_n too, then[12]

$$\eta(Q_n, S) = R - (R^2 - r_n^2)^{1/2}.$$

For small values of r_n, this equals $r_n^2/2R + \cdots$ and thus by our estimation concerning $\eta(Q_n, S)$

$$r_n^2 > \frac{2\cdot 3^{1/2}}{9}\,\frac{4\pi R^2}{n} + \cdots$$

or, supposing the system Q_n lies on a part of S with area F,

$$r_n^2 > \frac{2\cdot 3^{1/2}}{9}\,\frac{F}{n} + \cdots ;$$

It follows immediately that *the upper bound of the radii of circles which does not contain any point of a plane system of points with density d is*

$$r \geqq \left(\frac{2\cdot 3^{1/2}}{9d}\right)^{1/2}.$$

This inequality is exact; equality holds for a lattice of equilateral triangles.

By the density of a system of points we understand here—provided that it exists—the limit value $d = \lim n(F)/F$, where $n(F)$ means the number of points in the region F of area F and the limit refers to the case when F increases beyond all bounds by transformations of similitude.

Let us consider a system of circles of radius ρ covering the plane. For the system of centres of circles obviously $r \leqq \rho$, hence *the density of any system of congruent circles covering the plane is given by*

$$\pi\rho^2 d \geqq \frac{2\cdot 3^{1/2}\pi}{9} \qquad (= 1.209\cdots).$$

This result is due to R. Kershner.[13]

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[12] If the centre of the sphere S lies outside Q_n the square root in our equality must be taken with negative sign.

[13] R. Kershner, *The number of circles covering a set*, Amer. J. Math. vol. 61 (1939) pp. 665–671. For different generalizations of the above inequality see the second paper quoted in footnote 11 and further L. Fejes, *Einige Bemerkungen über die dichteste Lagerung inkongruenter Kreise*, Comment. Math. Helv. vol. 17 (1944–1945) pp. 256–261.

CONVEX FUNCTIONS

E. F. BECKENBACH

1. A problem of Cauchy. In 1821, Cauchy [19][1] proposed and solved the problem of determining the class of continuous real functions $f(x)$ which satisfy the equation

$$f(x_1) + f(x_2) = f(x_1 + x_2) \tag{1}$$

for all real x_1 and x_2. I think you would enjoy reading Cauchy's elegant treatment of this simple problem. But possibly you would enjoy more solving it yourself, or seeing to what considerations you are led if you omit the hypothesis of continuity.

The discontinuous solutions of (1) have been studied extensively. I have mentioned this equation because *any* solution of it satisfies

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) &= f\left(\frac{x_1}{2}\right) + f\left(\frac{x_2}{2}\right) \\ &= \frac{1}{2}\left\{\left[f\left(\frac{x_1}{2}\right) + f\left(\frac{x_1}{2}\right)\right] + \left[f\left(\frac{x_2}{2}\right) + f\left(\frac{x_2}{2}\right)\right]\right\} \\ &= \frac{1}{2}\left[f(x_1) + f(x_2)\right], \end{aligned}$$

and therefore satisfies

$$f\left(\frac{x_1 + x_2}{2}\right) \leqq \frac{1}{2}\left[f(x_1) + f(x_2)\right], \tag{2}$$

an inequality with which we shall be especially concerned.

2. Definition of convex function. A real function $f(x)$, defined in the interval $a<x<b$, is said to be *convex* provided that for all x_1 and x_2, with $a<x_1<x_2<b$, and for all positive q_1 and q_2 satisfying $q_1+q_2=1$, we have

$$f(q_1x_1 + q_2x_2) \leqq q_1f(x_1) + q_2f(x_2). \tag{3}$$

A convex function necessarily is continuous for $a<x<b$.

Geometrically, the condition of convexity is that each arc of the

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[1] Numbers in brackets refer to the references cited at the end of the paper.

curve $y=f(x)$ lie nowhere above the chord joining the end points of the arc.

If $f''(x)$ exists at each point of the interval, then a necessary and sufficient condition that $f(x)$ be convex there is that we have

$$f''(x) \geqq 0 \qquad (a < x < b).$$

If the strict inequality in (3) holds throughout, we say that $f(x)$ is *strictly convex*. The terminology varies, however, and some authors use the terms "non-concave" and "convex" in place of "convex" and "strictly convex," respectively.

Similar definitions hold for concave functions. Briefly, $f(x)$ is concave if and only if $-f(x)$ is convex.

3. **Elementary properties and examples.** Clearly if $f(x)$ and $g(x)$ are convex functions in the interval $a<x<b$, then $f(x)+g(x)$ and $\max[f(x), g(x)]$ are convex there, as is $cf(x)$ for non-negative constants c.

The limit of a convergent sequence of convex functions is convex; also, if it is finite, so is the upper envelope of a family of convex functions.

A convex function has a left-hand derivative and a right-hand derivative at each point of (a, b). The right-hand derivative is not less than the left-hand derivative, and both are nondecreasing functions of x. It follows that these derivatives are equal except at most at the point of a denumerable set of points.

If for fixed x_1 and x_2 in (a, b) the sign of equality in (3) holds at a single interior point of the subinterval (x_1, x_2), then the sign of equality holds throughout (x_1, x_2).

In the part of (a, b) outside the subinterval (x_1, x_2), the graph of $y=f(x)$ lies nowhere below the line through $[x_1, f(x_1)]$ and $[x_2, f(x_2)]$.

The function $|x-a|$ is convex, its graph being V-shaped; therefore, for instance, so is the function

$$g(x) = 2|x| + |x-1| + |x-2|,$$

the continuous graph of which consists of a succession of line-segments. Again, for the above function $g(x)$, the function $\max[x^2, g(x)]$ is convex.

The differentiation test shows that for $x>0$ the functions $x \log x$ and $\log 1/x$ are convex.

4. **Early history of convex functions.** Convex functions were first defined and systematically studied by J. L. W. V. Jensen [34, 35] in 1905, who adopted (2) as defining inequality. We shall say that a

function satisfying (2) for all x_1 and x_2 in (a, b) is *convex in the sense of Jensen*, or briefly *convex* (*J*). For recognizing the importance of the class of convex functions and signalizing it, Jensen deserves great credit.

Of course mathematicians were able before 1905 to recognize and utilize conditions under which the graph of a function turns its convexity downward.

Thus, as Jensen noted in an addition to his cited paper [35], in 1889 Hölder [32] had obtained the fundamental inequality

$$f\left(\sum_{j=1}^{m} q_j x_j\right) \leqq \sum_{j=1}^{m} q_j f(x_j) \qquad \left(a < x_j < b,\, q_j > 0,\ \sum_{j=1}^{m} q_j = 1\right), \tag{4}$$

for the subclass of convex functions $f(x)$ for which $f''(x)$ is continuous in (a, b). The inequality (4) expresses the relation that the function value at the weighted average of the x_j's is not greater than the weighted average of the function values at the x_j's.

Stolz [68] in 1893 showed that if $f(x)$ is continuous and satisfies (2), then $f(x)$ has a left-hand derivative and a right-hand derivative at each point of (a, b).

Also in 1893, Hadamard [24] showed that if $f(x)$ has an increasing derivative (so that $f(x)$ is convex), then for $a<x_1<x_2<b$ we have

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)dx.$$

In 1896, Hadamard [25] announced the result which Landau [40] later glamorized by designating it the Three Circles Theorem: *If $f(z)$ is an analytic function of the complex variable z in the annulus $a<|z|<b$, and $M(r; |f|)$ denotes the maximum of $|f(z)|$ on the circle $|z| = r$, then the graph of* $\log M(r; |f|)$ *as a function of* $\log r$ *turns its convexity downward.* That is, $\log M$ is a convex function of $\log r$.

In 1897 and 1898, Hadamard [26, 27] observed that for a surface S given in geodesic representation,

$$ds^2 = du^2 + \mu^2 dv^2 \qquad (\mu \geqq 0),$$

the Gaussian, or total, curvature K is given at points where $\mu \neq 0$ by

$$K = -\frac{1}{\mu}\frac{\partial^2 \mu}{\partial u^2}, \tag{5}$$

so that if K is of one sign over all of S, then $\partial^2\mu/\partial u^2$ is of the opposite sign. Thus if S is a surface of negative curvature, then $\mu(u, v_0)$ is a convex function of u.

Let

$$(6)\qquad \sum_{j,k=0}^{\infty} a_{j,k}z^j w^k$$

be a power series in two complex variables, which is convergent for a pair of values z_0, w_0 neither of which is zero. That is, every simple series formed from the terms of (6) converges for $z=z_0$, $w=w_0$. Then (6) converges absolutely for all z, w satisfying $|z|<|z_0|$, $|w|<|w_0|$. A pair of positive numbers r and ρ such that the series converges for $|z|<r$ and $|w|<\rho$ simultaneously, but diverges for $|z|>r$ and $|w|>\rho$ simultaneously, is called *a pair of associated radii of convergence*. Thus for

$$\sum_{j=0}^{\infty} z^j w^j = \frac{1}{1-zw},$$

clearly $r\rho=1$, so that $1/\rho=r$ and $\log 1/\rho=\log r$.

In 1902, Fabry [22] showed that if r and ρ are associated radii of convergence, then $\log 1/\rho$ is a convex function of $\log r$, as illustrated in the above example. Faber [21] and Hartogs [31] showed conversely that if r and ρ are positive variables such that $\log 1/\rho$ is a convex function of $\log r$, then there exist series of the form (6) for which r and ρ are associated radii of convergence.

Minkowski [44] in 1903 studied convex bodies by means of *Stützfunktionen*, which are particular convex functions of more than one independent variable.

Possibly the convexity of some significant convex functions which were studied before 1905 was recognized though not mentioned. Let $f(z)$ be analytic for $|z|\leqq r$, with a zero of order $m\geqq 0$ at $z=0$, so that

$$f(z)=z^m g(z),$$

where $g(0)\neq 0$, and let the moduli of the zeros of $f(z)$ in $0<|z|<r$ be $r_1, r_2, \cdots, r_n$, with $0<r_1\leqq r_2\leqq \cdots \leqq r_n$. Then we have the familiar formula

$$(7)\qquad \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta=\log\frac{|g(0)|\,r^{m+n}}{r_1r_2\cdots r_n},$$

which can be written as

$$(8)\qquad \mathfrak{M}_0(r;\,|f|)=\exp\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta=\frac{|g(0)|\,r^{m+n}}{r_1r_2\cdots r_n}.$$

Thus as r increases from $r=0$, the exponent $m+n$ increases by integral amounts at the moduli of the zeros of $f(z)$, and the ***geometric mean*** $\mathfrak{M}_0(r;\ |f|)$ is, by successive intervals some of which might be degenerate, constant out to the modulus of the first zero of $f(z)$, then a linear function of r, then a quadratic, and so on.

Clearly it follows from (7) and (8) that $\log \mathfrak{M}_0(r;\ |f|)$ is a convex function of $\log r$; this is a limting case of a theorem established by Hardy [29] in 1914. Actually it follows also from (8) that $\mathfrak{M}_0(r;\ |f|)$ is a convex function of r [6]. However, the discoverer of (7) remarked in this connection only that the left-hand member of (7) is a non-decreasing function of r. Established in 1898, (7) is known as *Jensen's formula* [33].

5. **Functions convex in the sense of Jensen.** If $f(x)$ is convex, then $f(x)$ satisfies (2) as special case of (3); that is, if $f(x)$ is convex, then $f(x)$ is convex in the sense of Jensen, or convex (J). Conversely, if $f(x)$ is *continuous* and convex (J), then $f(x)$ is convex. But, though a convex function must be continuous, a function which is convex (J) is not necessarily continuous.

As Hamel [28] showed in 1905, the existence of discontinuous functions $f(x)$ which satisfy (1) is readily established by means of a Hamel basis B for the real numbers.

A Hamel basis B for the real numbers is a set of real numbers b, the *elements* of B, such that each real $x \neq 0$ has a unique representation of the form

$$x = \sum_{j=1}^{n(x)} r_j b_j,$$

where $n(x)$ is finite, the r_j are rational numbers with $r_j \neq 0$, and the b_j are elements of B.

The general solution of (1) is given by

$$f(x) = f\left(\sum_{j=1}^{n(x)} r_j b_j\right) = \sum_{j=1}^{n(x)} r_j f(b_j) \qquad (x \neq 0;\ f(0) = 0),$$

where $f(b)$ denotes an arbitrary function on B.

Since the set of real numbers is non-denumerable, and the r_j are rational, the set of elements of B must be non-denumerable, and therefore there must be an accumulation point p_0 of B in B. If, for instance, we define $f(b)$ to be 0 on B except at the accumulation point p_0, and take $f(p_0) = 1$, then $f(x)$ necessarily is discontinuous at p_0.

To obtain a discontinuous function which is strictly convex (J), we might then, for instance, add x^2 to the above function $f(x)$; or we might take the function

$$f^*(x) = \max\,[x^2, f(x) + x^2],$$

which is bounded from below by $y = x^2$.

6. **Generalizations of convex functions.** In 1908, Phragmén and Lindelöf [51] showed that if $f(z)$ is an entire function of finite order ρ, then the function

$$h(\theta) = \limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r^\rho}$$

has the following property: If $0<\theta_2-\theta_1<\pi/\rho$, and $H(\theta)$ is the function of the form

$$A \cos \rho\theta + B \sin \rho\theta$$

which coincides with $h(\theta)$ at θ_1 and at θ_2, then for $\theta_1<\theta<\theta_2$ we have

$$h(\theta) \leqq H(\theta).$$

Accordingly, $h(\theta)$ is said to be a ***sub-trigonometric function.*** Pólya [53] showed that sub-trigonometric functions have certain differential properties in common with convex functions, and Valiron [69] extended the analysis to functions $f(x)$ dominated by functions of the form

$$A\phi(x) + B\psi(x)$$

for suitably restricted functions $\phi(x)$ and $\psi(x)$.

More generally, let $\{F(x)\}$ be a family of continuous functions $F(x)$ defined in an interval $a<x<b$, such that for given P_1: (x_1, y_1) and P_2: (x_2, y_2), with $a<x_1<x_2<b$, there is a unique member of $\{F(x)\}$ through P_1 and P_2. Functions $f(x)$ dominated by $\{F(x)\}$ might be said [1] to be *convex relative to* $\{F(x)\}$. With Bing [5], we have defined functions which are *convex* (J) *relative to* $\{F(x)\}$.

There are families $\{F(x)\}$ which are not topologically equivalent to the family $\{L(x)\}$ of all non-vertical line-segments terminating on $x=a$, $x=b$. And there are families $\{F(x)\}$ such that $f(x)$ might be convex relative to $\{F(x)\}$ yet nowhere differentiable. But many properties of convex functions, particularly those concerning measure, do hold for these general functions.

In terms of divided differences of order n, Popoviciu [55] has defined convex functions of order n. If a convex function $f(x)$ of order n and a polynomial $p(x)$ of degree n (or less) have equal values for $n+1$ values of x, then alternately $f(x)\leqq p(x)$ and $f(x)\geqq p(x)$ in the successive subintervals bounded by these $n+1$ values of x.

My colleague Drandell [20] is combining the notions of functions convex relative to $\{F(x)\}$ and of convex functions of order n. *Added in proof*: Shortly before the delivery of this address, results in this same program were announced by Tornheim in Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1119–1120.

A function $f(x, y)$ is said to be ***doubly convex*** provided that, in its domain of definition, $f(x, y)$ is a convex function of x for each y, and a convex function of y for each x. Thus $f(x, y) \equiv xy$ is doubly convex Of several additional generalizations of convex functions $f(x)$, we shall later consider two additional generalizations to functions of two (or more) independent variables; these are (i) convex functions of P: (x, y), and (ii) subharmonic functions.

To indicate the possibility of yet another generalization, we recall the inequality of Steiner [42]: For two continuous surfaces

$$S_1:\ z = z_1(x, y), \qquad S_2:\ z = z_2(x, y), \qquad (x, y) \text{ in } R,$$

where R is a Jordan region, let S denote the surface

$$S:\ z = [z_1(x, y) + z_2(x, y)]/2.$$

If $A(S)$ denotes the area of S, then we have

$$A(S) \leqq [A(S_1) + A(S_2)]/2.$$

7. **Measure and connectedness.** Investigations of the discontinuous solutions of (1) and (2) have been concerned particularly with properties of measure and connectedness.

Jensen [35] proved that if $f(x)$ satisfies (2) and is bounded from above, then $f(x)$ is continuous. This result has been extended by Bernstein and Doetsch [9], Blumberg [11], Sierpenski [67], and Ostrowski [49].

If $f(x)$ is convex (J) and is not continuous, then $f(x)$ cannot be bounded on any subinterval or even on a set of positive measure; $f(x)$ cannot be a measurable function; and either the set M of points of the graph of $y=f(x)$ is dense in the entire strip $a<x<b$, $-\infty<y<+\infty$, or there is a continuous convex function $\phi(x)$ such that the set M lies in $a<x<b$, $\phi(x)<y<+\infty$, and is dense there. For example, the graph of the above function $f^*(x)$ is dense in the part of the plane above $y=x^2$. For any circular disc D such that M is dense in D, the x-projection of the part of M in D has zero interior measure and positive exterior measure.

It has been shown by Jones [36] that the graphs $y=f(x)$ of the discontinuous solutions of (1) serve to illustrate relatively easily certain weird topological properties of connected sets. Thus there are discontinuous solutions of (1) for which the graphs are connected, and others for which the graphs are totally disconnected. But even though the graph might be connected, it must be punctiform; that is, it can contain no nondegenerate continuum.

We shall return to considerations of measure and connectedness in discussing convex sets.

8. **Convex sets.** A set S of points in the plane or in space is *convex* provided that for each pair of points P_1 and P_2 in S, the entire line-segment P_1P_2 is contained in S [44, 13].

A function $f(P)$, defined on a convex set S, is said to be *convex* provided that for each pair of points P_1 and P_2 in S, and for all positive q_1 and q_2 satisfying $q_1+q_2=1$, we have

$$f(q_1P_1 + q_2P_2) \leqq q_1f(P_1) + q_2f(P_2),$$

where $q_1P_1+q_2P_2$ has its obvious meaning.

To illustrate the connection between convex functions and convex sets, I shall state a simple theorem suggested to me by Newburgh, one of my colleagues; I think you might enjoy supplying the few lines necessary for its proof.

THEOREM. *For a closed set S and a variable point P, let*

$$d(P; S) = \min_{Q \text{ on } S} \text{distance } (P, Q).$$

A necessary and sufficient condition that S be a convex set is that $d(P; S)$ be a convex function of P.

Two of my colleagues, Green and Gustin [23], have defined sub-convex sets S of points as follows. Let λ denote a non-null set of points in the interval $0<x<1$. Then the set S is *sub-convex*, or *convex relative to* λ, provided that for each pair of points P_1 and P_2 of S, all the points of the image of λ under the homothetic mapping of the line-segment $0<x<1$ on the line-segment P_1P_2 are contained in S.

If S is convex, then S is convex relative to any λ.

Green and Gustin have found that many properties concerning measure and connectedness of solutions of (1) have analogues in the theory of sub-convex sets. Thus if a set S, convex relative to λ, contains a nonlinear continuum, or is of positive interior measure, then S is nearly convex; that is, there is a convex set T such that S is contained in T, and S coincides with T except for the possible omission of part of the boundary of T. Again, if a set S, convex relative to λ, contains a nonlinear connected set, then S is connected.

9. **Inequalities.** Jensen gave the study of (algebraic) inequalities as

principal object of his investigation of convex functions. He showed that the basic inequality (4) holds provided $f(x)$ is convex (J) and the q_j are rational, whence it follows that if $f(x)$ is continuous and convex (J) then the hypothesis that the q_j are rational can be dropped.

The inequality (4) has been generalized by McShane [43].

Applications of (4) to certain convex functions yield familiar algebraic inequalities, including the three which are usually taken to be fundamental: the inequality between the geometric and arithmetic means, the inequality of Hölder, and the inequality of Minkowski.

Thus from the convexity of $-\log x$ for $x>0$ we obtain [54]

$$-\log\left(\sum_{j=1}^{n} q_j a_j\right) \leqq -\sum_{j=1}^{n} q_j \log a_j = -\log \prod_{j=1}^{n} a_j^{q_j} \qquad (a_j > 0)$$

or

$$\prod_{j=1}^{n} a_j^{q_j} \leqq \sum_{j=1}^{n} q_j a_j, \tag{9}$$

which is the inequality between the geometric and arithmetic means.

Hölder's inequality can be obtained from (9), and Minkowski's inequality follows from Hölder's [65].

The inequality between the geometric and arithmetic means is a special case of a more general inequality [10]. We define the mean of order t for positive values $(a)\equiv(a_1, a_2, \cdots, a_n)$, $n\geqq 2$, and positive weights $(q)\equiv(q_1, q_2, \cdots, q_n)$ with $\sum_{j=1}^{n} q_j=1$, by

$$\mathfrak{M}_t(a; q) \equiv \left(\sum_{j=1}^{n} q_j a_j^t\right)^{1/t} \qquad (-\infty < t < 0 \text{ or } 0 < t < +\infty),$$

and

$$\mathfrak{M}_0(a; q) \equiv \prod_{j=1}^{n} a_j^{q_j}, \qquad \mathfrak{M}_{-\infty}(a; q) \equiv \min (a), \qquad \mathfrak{M}_{+\infty}(a; q) \equiv \max (a).$$

Thus for $t=-1, 0, 1$, and 2, $\mathfrak{M}_t$ is respectively the harmonic mean, the geometric mean, the arithmetic mean, and the "root-mean-square."

For positive continuous functions $f(x)$ and $q(x)$ in (a, b), with $\int_a^b q(x)dx=1$, the integral analogue of $\mathfrak{M}_t(a; q)$ is given by

$$\mathfrak{M}_t(f; q) \equiv \left\{\int_a^b q(x)[f(x)]^t dx\right\}^{1/t}$$
$$(-\infty < t < 0 \text{ or } 0 < t < +\infty),$$

and

$$\mathfrak{M}_{-\infty}(f; q) = \min f(x), \qquad \mathfrak{M}_{+\infty}(f; q) = \max f(x),$$

$$\mathfrak{M}_0(f; q) = \exp \int_a^b q(x) \log f(x) dx.$$

The function $\mathfrak{M}_t(a; q)$ is a continuous function of t for $-\infty \leqq t \leqq +\infty$.

The general inequality to which we have referred is

$$\mathfrak{M}_s(a; q) \leqq \mathfrak{M}_t(a; q) \qquad (-\infty \leqq s < t \leqq +\infty),$$

where the sign of equality holds if and only if all the a_j are equal.

The *sums* of order t, defined by

$$\mathfrak{S}_t(a) = \left(\sum_{j=1}^{n} a_j^t \right)^{1/t} \qquad (-\infty < t < 0 \text{ or } 0 < t < +\infty),$$

behave quite differently [56, 35], with

$$\mathfrak{S}_s(a) > \mathfrak{S}_t(a) \qquad (s < t), \tag{10}$$

provided s and t are both positive or both negative. Further,

$$\lim_{t\to-\infty} \mathfrak{S}_t(a) = \min (a), \qquad \lim_{t\to 0,\, t<0} \mathfrak{S}_t(a) = 0,$$

$$\lim_{t\to 0,\, t>0} \mathfrak{S}_t(a) = +\infty, \qquad \lim_{t\to+\infty} \mathfrak{S}_t(a) = \max (a).$$

The inequality (10) is sometimes [30] called *Jensen's inequality*, though I prefer to reserve this term for (4). Actually, neither inequality originated with Jensen.

10. Further study of known inequalities. The inequalities which we have been discussing express the fact that certain functions are nondecreasing or nonincreasing functions of certain parameters. The inequalities do not, however, tell *how* the functions increase or decrease.

Thus since the graph of $y = \mathfrak{M}_t(a; q)$ has two horizontal asymptotes, it must have at least one inflection point [48]. Does it necessarily have exactly one inflection point, so that $\mathfrak{M}_t(a; q)$ must be a convexo-concave function, or might there be several inflection points? There are examples in the literature [48] showing that at $t = 0$, $d^2\mathfrak{M}_t/dt^2$ might be positive, negative, or zero, so that if there were necessarily only one inflection point it would seem to be an elusive one.

It is known [30] that $\log \{ [\mathfrak{M}_t(a; q)]^t \}$ is a convex function of t, as is $\log \{ [\mathfrak{S}_t(a)]^t \}$; it follows that $[\mathfrak{M}_t(a; q)]^t$ and $[\mathfrak{S}_t(a)]^t$ are convex functions of t, since the convexity of the logarithm of a function implies the convexity of the function itself.

It is known also [39] that $\log \mathfrak{M}_t(a; q)$ and $\log \mathfrak{S}_t(a)$ are convex functions of $1/t$ for $t>0$, and concave functions of $1/t$ for $t<0$.

Recently Shniad [66], one of my colleagues, has shown that $\mathfrak{M}_t(a; q)$ is *not* necessarily a convexo-concave function of t. Explicitly, for the function

$$\mathfrak{M}_t(a; q) = \left(\frac{1}{10}e^t + \frac{8}{10}e^{2t} + \frac{1}{10}e^{3t}\right)^{1/t},$$

the second derivative is positive at $t=-2$, negative at $t=-1$, positive at $t=0$, and negative at $t=4$.

On the other hand, the ingenious analysis which led Shniad to consider the above example yielded the positive result that *for any given* $(a; q)$ *there exist (finite) values* $t_j = t_j(a; q)$, $j=1, 2$, *such that* $\log \mathfrak{M}_t(a; q)$ *is a convex function of* t *for* $t<t_1$, *and a concave function of* t *for* $t>t_2$. *Consequently,* $\mathfrak{M}_t(a; q)$ *also must be a convex function of* t *for* $t<t_1$.

As for $\mathfrak{S}_t(a)$, Bonnesen [12] has shown that this is a convex function of t for $t>1$. Later, by a different method, I obtained [3] the same result for $t>0$.

Several results which I shall mention later also involve the improvement of known inequalities by the establishment of convexity properties.

11. A hierarchy of convexity conditions. I have remarked that the convexity of the logarithm of a positive function implies the convexity of the function itself. This result holds as an instance of the following continuous hierarchy of convexity conditions.

Let C_α denote the class of positive functions $p(x)$ defined in (a, b) such that the function

$$\operatorname{sg}(\alpha)[p(x)]^\alpha \qquad (\alpha \neq 0),$$

or

$$\log p(x) \qquad (\alpha = 0),$$

is a convex function of x in (a, b), where $\operatorname{sg}(\alpha) = -1$ for $\alpha<0$ and $\operatorname{sg}(\alpha) = +1$ for $\alpha>0$.

Then $p(x)$ *is a member of* C_α *if and only if* $p(x)$ *is a member of* C_β *for all* $\beta>\alpha$ [57].

The class C_0 of functions whose logarithms are convex is particularly amenable to analysis, in that the class is closed both under addition and under multiplication [7]. The class also is particularly important, in that several physically significant functions are members of C_0.

Thus in complex variable theory there are several functions, some of which we have mentioned already, which are members of C_0 relative to $\log r$ as independent variable:

For associated radii of convergence, $\log 1/\rho$ is a convex function of $\log r$.

In the theory of meromorphic functions, the Nevanlinna function

$$T(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta + \log \frac{r^n}{r_1 r_2 \cdots r_n}$$

involves logarithms in such a way that it seems natural to consider the function

$$U(r) = \exp T(r).$$

A fundamental result of Nevanlinna [47] is that $\log U(r)$ is a convex function of $\log r$.

The Hadamard Three Circles Theorem expresses the convexity of

$$\log M(r; |f|) = \log \mathfrak{M}_{+\infty}(r; |f|)$$

as function of $\log r$.

For the geometric mean $\mathfrak{M}_0(r; |f|)$ we have pointed out the result that $\log \mathfrak{M}_0(r; |f|)$ is a convex function of $\log r$.

But we have indicated also that $\mathfrak{M}_0(r; |f|)$ satisfies the additional condition that $\mathfrak{M}_0(r; |f|)$ itself is a convex function of r itself.

Which is the stronger convexity condition on a positive function $p(r)$, the condition that $\log p(r)$ be a convex function of $\log r$, or the condition that $p(r)$ be a convex function of r? The answer is that neither implies the other, for each can hold in the absence of the other [6].

We shall pursue this matter further in the next section, in which we discuss a continuum of functions whose logarithms are convex functions of $\log r$.

12. **Hardy's theorem.** In $\mathfrak{M}_t(r; |f|)$, $0 \leqq t \leqq +\infty$, we have a continuum of functions of class C_0, in accordance with the following theorem of Hardy [29] to which we referred in §4.

HARDY'S THEOREM. *Let $f(z)$ be an analytic function of the complex variable z in $|z| < 1$. Then* $\log \mathfrak{M}_t(r; |f|)$ *is a (nondecreasing) convex function of r for any non-negative value t.*

We already have considered the limiting cases $t=0$ and $t=+\infty$.

The question arises as to whether or not $\mathfrak{M}_t(r;|f|)$ is necessarily a convex function of r for $0\leqq t\leqq +\infty$.

Gustin, Shniad, and I currently are investigating the above question, and have obtained the following results.

For any function $f(z)$ analytic in $|z|<1$, and for any t satisfying $0\leqq t\leqq +\infty$, let $\rho(t;|f|)$ denote the least upper bound of values ρ such that $\mathfrak{M}_t(r;|f|)$ is convex for $0<r<\rho$. Let $\rho(t)$ denote the greatest lower bound of $\rho(t;|f|)$ for f ranging over the class of functions analytic in $|z|<1$. Then $0\leqq\rho(t)\leqq\rho(t;|f|)\leqq 1$, and we have indicated that $\rho(0)=1$.

For the function

$$F(z)=\frac{z+a}{1+az}\qquad(0<a<1,\ |z|<1),$$

the maximum-value function

$$M(r;|F|)=\mathfrak{M}_{+\infty}(r;|F|)$$

is *strictly concave* for $0<r<1$. Accordingly, we have $\rho(+\infty;|F|)=0$, whence $\rho(+\infty)=0$.

Also, since $\mathfrak{M}_t\to\mathfrak{M}_{+\infty}$ as $t\to+\infty$, and since the limit of a convergent sequence of convex functions is convex, it follows that

$$\lim_{t\to+\infty}\rho(t)=0.$$

If $f(z)$ has at most one zero in $|z|<1$, then we have the result that

$$\rho(t;|f|)=1\qquad(0\leqq t\leqq 2);$$

if $f(z)$ has at most two zeros, then

$$\rho(t;|f|)=1\qquad(0\leqq t\leqq 1);$$

and there are similar results for any number of zeros.

Irrespective of the number of zeros of $f(z)$, we have

$$\rho(t)=1,\qquad(t=2/k;\ k=1,2,\cdots,n,\cdots).$$

The previously noted result $\rho(0)=1$ follows by a limiting process, with $k\to+\infty$.

Whether or not there are other values of t for which $\rho(t)=1$, or any values of t for which $0<\rho(t)<1$, we do not now know.

Added in proof. Shortly after the delivery of this address, Shniad showed that we have $\rho(t)<1$ for all $t>8$.

As an application, we recall the known result that the length $l(r)$

of the image of $|z| = r$ under the transformation $w = f(z)$ is a nondecreasing function of r: if $0 < r_1 < r_2 < 1$, then $l(r_1) \leqq l(r_2)$. It now can be shown that $l(r)$ *is a convex function of* r.

One of the most attractive results concerning $\rho(t; |f|)$ was obtained by Shniad as a consequence of the theorem of Hardy: namely, we have

$$\rho(t; |f(z) - f(0)|) = 1 \qquad (0 \leqq t \leqq +\infty).$$

This result, which is an instance of a theorem of Nehari [46], involves an interesting implication relative to the lemma of Schwarz, as we shall see in the next section.

13. The lemma of Schwarz and convexity.

Recalling that

$$\mathfrak{M}_{+\infty}(r; |f|) = \max_{0 \leqq \theta < 2\pi} |f(re^{i\theta})|,$$

we may state the Lemma of Schwarz as follows.

Lemma of Schwarz. *Let* $f(z)$ *be an analytic function of the complex variable* z *in* $|z| < 1$, *with* $f(0) = 0$. *If for all* r, $0 < r < 1$, *we have*

$$\mathfrak{M}_{+\infty}(r; |f|) \leqq 1,$$

then we have

$$\mathfrak{M}_{+\infty}(r; |f|) \leqq r \qquad (0 < r < 1)$$

and

$$|f'(0)| \leqq 1,$$

the signs of equality holding if and only if $f(z) = e^{i\alpha}z$, *where* α *is a real constant.*

Thus if the origin is mapped on the origin by the analytic function $w = f(z)$, and the map of the unit circle $|z| < 1$ lies in the unit circle $|w| < 1$, then the map of any smaller concentric circle of radius r lies in the concentric circle of radius r, and reaches the boundary $|w| = r$ if and only if the map is a rotation.

It is known [38] that the Lemma of Schwarz extends to means of other orders. In the statement of the lemma, we have only to replace $\mathfrak{M}_{+\infty}$ throughout by $\mathfrak{M}_{t_0}$, $0 \leqq t_0 \leqq +\infty$. Indeed, for $0 < r_1 < r_2 < 1$, we have [38], more precisely,

$$\frac{\mathfrak{M}_{t_0}(r_1; |f|)}{\mathfrak{M}_{t_0}(r_2; |f|)} \leqq \frac{r_1}{r_2} \qquad (0 \leqq t_0 \leqq +\infty). \tag{11}$$

Now (11) does not imply that the curve $y = \mathfrak{M}_{t_0}(r; |f|)$ is convex, but only that arcs of the curve having one end point at the origin lie nowhere above the corresponding chords. However, as we pointed out in §12, since $f(0)=0$ the curve $y=\mathfrak{M}_{t_0}(r; |f|)$ actually is convex.

14. **Subharmonic functions.** In studying associated radii of convergence, Hartogs [31] in 1906 used as tool a real function $R(\zeta)$ of the complex variable ζ, defined as follows.

Let $f(z, w)$ be analytic at the point $(\zeta, 0)$,

$$f(z, w) = \sum_{j,k=0}^{\infty} a_{j,k}(z-\zeta)^j w^k,$$

and let r and $\rho=\phi(r)$ be associated radii of convergence of the series. Then by definition we have

$$R(\zeta) = \lim_{r\to 0} \phi(r).$$

Hartogs showed that the function

$$g(x, y) = \log \frac{1}{R(\zeta)} \qquad (\zeta = x + iy)$$

is upper semi-continuous in its domain of definition D, and has the following property. If D' is a domain lying together with its boundary B' in D, and $h(x, y)$ is harmonic in D' and continuous in $D'+B'$, and we have $g(x, y) \leqq h(x, y)$ on B', then necessarily we have also $g(x, y) \leqq h(x, y)$ throughout D'.

Since a harmonic function $h(x, y)$ is by definition a solution of the partial differential equation

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0,$$

the class of harmonic functions is a generalization, to functions of two (or more) independent variables, of the class of linear functions of one variable. Accordingly, the above property of domination by harmonic functions is a generalization of the defining property of convex functions of one variable.

Harmonic functions $h(x, y)$ are characterized by the mean-value property that if the circular disc $(x-x_0)^2+(y-y_0)^2 \leqq r^2$ is in the domain of definition, then

$$h(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} h(x_0 + r\cos\theta, y_0 + r\sin\theta)d\theta.$$

If the above function $g(x, y)$ is $\not\equiv -\infty$, then $g(x, y)$ satisfies the mean-value inequality

$$(12) \qquad g(x_0, y_0) \leqq \frac{1}{2\pi}\int_0^{2\pi} g(x_0 + r\cos\theta,\ y_0 + r\sin\theta)\,d\theta,$$

though Hartogs did not explicitly give this result.

Hartogs showed that if the above function $g(x, y)$ is continuous together with its partial derivatives of the first and second orders, then $g(x, y)$ satisfies the differential inequality

$$\Delta g \equiv \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \geqq 0.$$

Later Levi [41] in 1910 and Julia [37] in 1925 used functions analogous to $g(x, y)$ in the study of poles and essential singularities of analytic functions of several complex variables.

In 1922, Riesz [61] made the remarkable discovery that the theorem of Hardy, to which we have referred, and which previously had been proved only with considerable difficulty, can be established very simply from the fact that $|f(z)|^p$ satisfies the mean-value inequality (12).

In a series of papers starting in 1923, by Perron [50], Remak [60], Radó and Riesz [59], Wiener [72], Whitney [71], and Carathéodory [15], the solution of the Dirichlet problem in potential theory was put in very elegant form by means of functions having the property to which we have referred. Actually, as Riesz [64] pointed out in 1926, the idea of domination by harmonic functions was involved in potential theory as early as 1887 in the sweeping-out process of Poincaré [52].

It now appeared that the class of functions involved was of intrinsic interest, because of its various applications and because of its relation to convex functions. Riesz [62, 63, 64] defined a ***subharmonic function*** to be an upper semi-continuous function $f(x, y)$ which satisfies $-\infty \leqq f(x, y) < +\infty$, $f(x, y) \not\equiv -\infty$, and which is dominated by harmonic functions as described above.

To be exact, Riesz assumed the apparently but (as Evans [58] pointed out) not actually stronger condition that $f(x, y)$ is not equal to $-\infty$ on a set of points dense in the domain of definition D, in place of the condition that $f(x, y)$ is not identically equal to $-\infty$. Personally, for several reasons, I prefer to include the function $f(x, y) \equiv -\infty$ in the class of subharmonic functions.

Essentially, subharmonic functions were involved in the study of differential geometry by Weil [70] in 1926, when he extended the isoperimetric inequality

$$a \leq \frac{1}{4\pi} l^2 \tag{13}$$

to surfaces of negative curvature, after Carleman [17] had shown in 1921 that (13) holds for Jordan regions of area a and length of perimeter l on minimal surfaces. The inequality (13) characterizes [8] surfaces of non-positive Gaussian curvature.

Thus properties of subharmonic functions now are investigated from four points of view: the study of properties of subharmonic functions for their intrinsic interest, in particular as they relate to convex functions [45]; and the study of applications in potential theory, complex variable theory, and differential geometry. The four studies are mutually stimulating.

To illustrate the first point of view, we might ask the following question: If $h(x, y)$ is harmonic, and $s(x, y)$ subharmonic, in the unit circle $x^2+y^2<1$, and $h(x, y) = s(x, y)$ on $x^2+y^2=r_0^2$ for some r_0 with $0<r_0<1$, do we necessarily have $h(x, y) \leq s(x, y)$ for $r_0^2<x^2+y^2<1$?

We shall conclude our remarks with some observations concerning differential geometry as related to convex functions, subharmonic functions, and functions of complex variables.

15. **Differential geometry.** Both convex functions and subharmonic functions serve as tools in the study of differential geometry.

The use of convex functions in differential geometry, which, as we mentioned, was initiated by Hadamard, has been extended to Riemannian spaces by Cartan [18].

In our local peripatetic seminar, Professor Busemann [14] recently has shown how, without use of differentiability hypotheses, the same results can be carried over to general metric spaces S of nonpositive curvature: S is said to be of nonpositive curvature provided each point of S has a neighborhood N such that the side bc of any geodesic triangle abc in N is at least twice as long as the (shortest) geodesic arc connecting the midpoints b', c' of the other two sides:

$$\overline{b'c'} \leq \overline{bc}/2.$$

On the other hand, elementary calculus can be used [4] to obtain, from (5), various inequalities and convexity conditions involving length and area on surfaces of nonpositive curvature and on surfaces of non-negative curvature.

Subharmonic functions are related to differential geometry especially in accordance with the following two theorems [7, 8].

THEOREM A. *Three real functions $x(u, v)$, $y(u, v)$, $z(u, v)$, continuous in a domain D, are coordinate functions of a minimal surface in conformal representation, that is, the functions are harmonic and satisfy*

$$E = G = \lambda(u, v), \qquad F = 0,$$

where

$$E = x_u^2 + y_u^2 + z_u^2, \quad F = x_u x_v + y_u y_v + z_u z_v, \quad G = x_v^2 + y_v^2 + z_v^2,$$

if and only if the distance function

$$\rho(u, v; a, b, c) = \{[x(u, v) - a]^2 + [y(u, v) - b]^2 + [z(u, v) - c]^2\}^{1/2}$$

satisfies the condition that $\log \rho$ *is a subharmonic function of* (u, v) *for every choice of the real constants* a, b, c.

THEOREM B. *A necessary and sufficient condition that a surface* S,

$$S: \quad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

given in conformal representation,

$$E = G = \lambda(u, v), \qquad F = 0,$$

be a surface of nonpositive Gaussian curvature is that $\log \lambda$ *be a subharmonic function of* (u, v).

Briefly, in accordance with Theorem A the principle of the maximum for the moduli $|f(z)|$ of analytic functions $f(z)$ of the complex variable z carries over essentially intact to minimal surfaces. And, insofar as this principle applies to $|f'(z)|$, in accordance with Theorem B the principle largely carries over to the class of surfaces of nonpositive curvature. Since minimal surfaces are special surfaces of nonpositive curvature, the latter results hold in particular on minimal surfaces.

For example, we shall consider space analogues of the Lemma of Schwarz. These analogues might suggest various problems relative to the many ramifications and generalizations of the Lemma of Schwarz in complex variable theory. In particular, we note the result of Carathéodory [16] that the hypotheses of the Lemma of Schwarz imply $|f'(z)| \leqq 1$ for $|z| \leqq 2^{1/2}-1$.

You will note that the Lemma of Schwarz applies to distances *in the containing space*, not to distances on the map $w = f(z)$. From Theorem A we can obtain the following result [7].

Let S,

$$S: \quad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \qquad u^2 + v^2 < 1,$$

be a minimal surface given in conformal representation, such that (0, 0) *is carried into* (0, 0, 0). *If S is comprised in the unit sphere,* $x^2+y^2+z^2 \leqq 1$, *then for* $u^2+v^2 \leqq r^2$, $0<r<1$, *we have*

$$[x(u, v)]^2 + [y(u, v)]^2 + [z(u, v)]^2 \leqq r^2;$$

further, we have $\lambda(0, 0) \leqq 1$. *The signs of equality hold if and only if S is a simply-covered circular disc with unit radius.*

But now Theorem B suggests the possibility of an analogue of the Lemma of Schwarz for surfaces of nonpositive curvature involving distances *on the surface S* itself [2]. We shall state the result only for the special case of a plane map.

Let $w=f(z)$ *be analytic for* $|z|<1$. *If the length function*

$$l(r, \theta) = \int_0^r |f'(\rho e^{i\theta})| \, d\rho$$

satisfies

$$l(r, \theta) \leqq 1$$

for all (r, θ) *with* $0<r<1$, *then we have*

$$l(r, \theta) \leqq r \qquad (0 < r < 1)$$

and

$$|f'(0)| \leqq 1,$$

the signs of equality holding if and only if $f(z) \equiv e^{i\alpha}z$, *where* α *is a real constant.*

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UNIVERSITY OF CALIFORNIA AT LOS ANGELES

THE FEBRUARY MEETING IN NEW YORK

The four hundred thirty-second meeting of the American Mathematical Society was held at The City College, New York, on Saturday, February 28, 1948. The attendance was about two hundred fifty, including the following two hundred fifteen members of the Society:

Milton Abramowitz, C. R. Adams, E. B. Allen, C. B. Allendoerfer, R. L. Anderson, H. A. Arnold, L. A. Aroian, R. N. Ascher, W. A. Asprey, F. E. Baker, S. F. Barber, Joshua Barlaz, F. S. Beckman, F. P. Beer, E. G. Begle, Richard Bellman, Stefan Bergman, P. G. Bergmann, Lipman Bers, W. A. Blankinship, J. H. Blau, R. P. Boas, H. W. Bode, G. L. Bolton, Enrico Bompiani, Paul Brock, A. B. Brown, R. C. Buck, W. R. Callahan, H. H. Campaigne, P. G. Carlson, K. Chandrasekharan, K. K. Chen, K. T. Chen, Herman Chernoff, W. L. Chow, Randolph Church, A. B. Clarke, E. A. Coddington, I. S. Cohen, L. W. Cohen, R. M. Cohn, W. H. H. Cowles, D. A. Darling, F. H. Davidson, M. D. Davis, C. H. Dowker, Y. N. Dowker, Nelson Dunford, Aryeh Dvoretzky, W. F. Eberlein, Samuel Eilenberg, Bernard Epstein, Paul Erdős, R. M. Exner, A. L. Fass, Herbert Federer, J. M. Feld, C. D. Firestone, Irwin Fischer, W. B. Fite, D. A. Flanders, Edward Fleisher, L. R. Ford, R. M. Foster, R. H. Fox, Bernard Friedman, C. S. Gardner, L. L. Gavurin, Abe Gelbart, J. H. Giese, H. E. Goheen, V. D. Gokhale, A. W. Goodman, M. J. Gottlieb, F. G. Gravalos, E. J. Gumbel, M. M. Gutterman, P. R. Halmos, G. J. Haltiner, R. W. Hamming, Gerald Harrison, Philip Hartman, K. E. Hazard, M. H. Heins, G. C. Helme, L. H. Herbach, T. H. Hildebrandt, L. S. Hill, Einar Hille, Joseph Hilsenrath, I. I. Hirschman, A. J. Hoffman, Banesh Hoffmann, E. M. Hull, Witold Hurewicz, Solomon Hurwitz, R. E. Huston, L. C. Hutchinson, E. T. Jabotinsky, R. A. Johnson, R. E. Johnson, F. E. Johnston, A. W. Jones, Mark Kac, Samuel Kaplan, M. E. Kellar, J. R. Kline, E. R. Kolchin, B. O. Koopman, M. S. Kramer, S. H. Lachenbruck, Solomon Lefschetz, R. A. Leibler, Howard Levene, Howard Levi, Samuel Linial, M. A. Lipschutz, E. R. Lorch, Lee Lorch, R. B. Lowe, Myles McConnon, N. H. McCoy, S. W. McCuskey, Janet McDonald, L. A. MacColl, G. R. MacLane, Dis Maly, Murray Mannos, A. J. Maria, M. H. Maria, F. I. Mautner, D. G. Mead, A. N. Milgram, S. Minakshisundaram, W. L. Mitchell, Don Mittleman, C. R. Morris, I. R. Moses, F. J. Murray, D. S. Nathan, C. A. Nelson, Paolo Nesbeda, John von Neumann, Morris Newman, A. V. Newton, P. B. Norman, Paul Olum, A. F. O'Neill, J. C. Oxtoby, Anna Pell-Wheeler, B. J. Pettis, C. R. Phelps, E. L. Post, M. H. Protter, H. W. Reddick, Mina Rees, Irving Reiner, Eric Reissner, J. F. Ritt, M. S. Robertson, S. L. Robinson, J. B. Rosser, Herman Rubin, Charles Saltzer, H. E. Salzer, Arthur Sard, I. J. Schoenberg, Lowell Schoenfeld, Pincus Schub, George Shapiro, H. N. Shapiro, R. H. Shaw, Max Shiffman, S. S. Shū, D. N. Silver, James Singer, P. A. Smith, J. J. Sopka, E. P. Starke, N. E. Steenrod, S. K. Stein, Fritz Steinhardt, Wolfgang Sternberg, F. M. Stewart, J. J. Stoker, E. G. Straus, M. M. Sullivan, Fred Supnick, Otto Szász, Olga Taussky-Todd, J. M. Thomas, D. L. Thomsen, Hing Tong, C. A. Truesdell, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, J. R. Van Andel, W. G. Warnock, Alexander Weinstein, Louis Weisner, E. T. Welmers, J. H. White, G. W. Whitehead, K. B. Whitehead, D. V. Widder, Albert Wilansky, John Williamson, Jacob Wolfowitz, L. C. Young, J. R. Ziegler, J. A. Zilber, H. J. Zimmerberg, O. J. Zobel.

On Saturday morning there were two sections for contributed papers: one in Analysis, in which Professor E. R. Lorch presided, and one in Algebra, Topology and Applied Mathematics, in which Professor P. A. Smith presided.

At 2:00 P.M. Professor Eric Reissner of Massachusetts Institute of Technology gave an address on *Boundary value problems in aerodynamics of oscillating wings*. Professor L. R. Ford presided.

The Council met at 1:00 P.M. in the Faculty Room of Main Building.

The Secretary announced the election of the following two hundred and nineteen persons to ordinary membership in the Society:

Mr. Asger Hartvig Aaboe, Washington University;
Professor Joseph Jesse Abernethy, Texas State University for Negroes, Houston, Tex.;
Mr. Richmond G. Albert, Sampson College, Sampson, N. Y.;
Mr. Arlo Dean Anderson, University of Oregon;
Miss Florence Rosamond Anderson, University of Southern California;
Mr. Rodney E. Anderson, Northern Illinois State Teachers College, DeKalb, Ill.;
Mr. Silas Stuart Anderson, Ohio University, Athens, Ohio;
Mr. Donald Norris Armstrong, Massachusetts Institute of Technology;
Natascha Artin (Mrs. Emil), Washington Square College, and Institute for Mathematics and Mechanics, New York University;
Dr. Rafael Artzy, The Galilee Secondary School, Tiberias, Palestine;
Mr. William George Bade, University of California at Los Angeles;
Professor Harry Millard Beatty, The Ohio State University;
Mr. Leonard Becker, New York University;
Professor Guy Gaillard Becknell, University of Tampa, Tampa, Fla.;
Mr. Earl Louis Bell, North Central College, Naperville, Ill.;
Mr. Robert Louis Belzer, University of Santa Clara;
Professor Robert John Bickel, Drexel Institute of Technology;
Mr. Dewey Blair, University of Cincinnati;
Dr. Albert Laurence Blakers, Princeton University;
Mr. Bartolomé C. Blanco, General Electric Company, Schenectady, N. Y.;
Mr. Daniel Block, Yeshiva University;
Mr. Francis Ronald Britton, McMaster University, Hamilton, Ontario;
Mr. John Ben Butler, 3rd, Franklin and Marshall College;
Mr. Lewis Clark Butler, Alfred University;
Miss Willie Bee Campbell, Southern University, Baton Rouge, La.;
Mr. Philip Noah Cassen, Sampson College, Sampson, N. Y.;
Mr. Arthur I. Chalfant, Pratt and Whitney Aircraft Company, East Hartford, Conn.;
Mr. Pei Ping Chen, Providence, R. I.;
Mr. Jack Chernick, Brookhaven National Laboratory, Upton, N. Y.;
Mr. Morris I. Chernofsky, The College of the City of New York;
Mr. Allen Bruce Clarke, Brown University;
Mr. Robert Francis Cotellessa, Stevens Institute of Technology;
Mr. Micha Cotlar, Facultad de Ciencias Exactas, Buenos Aires, Argentina;
Mr. Earl Hicks Crisler, West Virginia University;
Professor John Cecil Currie, Louisiana State University;
Mr. Robert Fred Deniston, Washington University;
Mr. Verne Eugene Dietrich, Purdue University;

Professor Helen Walter Dodson, Goucher College;
Mr. Joseph Vincent Downey, Regis College, Denver, Colo.;
Mr. Chester Lee Dunsmore, University of California at Los Angeles;
Dr. Agustin Duranona y Vedia, Instituto de Matematicas de la Universidad de la Plata, Buenos Aires, Argentina;
Dr. Aryeh Dvoretzky, Einstein Institute of Mathematics, Hebrew University;
Mr. Sherman Eldon Dyer, University of Texas;
Mr. Harvey Nelson Ebersole, Lafayette College, Easton, Pa.;
Miss Myrtle Edwards, Mars Hill College, Mars Hill, N. C.;
Mr. (Francis) Edward Ehlers, Brown University;
Capt. Harry Van Horn Ellis, Jr., United States Military Academy;
Mr. Daniel Jacob Ewy, Bethel College, North Newton, Kan.;
Professor Basil M. Fedorovsky, St. Joseph's College, Philadelphia, Pa.;
Mr. Arlin Martin Feyerherm, State University of Iowa;
Miss Sarah Grace Fleming, University of Alabama;
Mr. José Salvador Florio, Mendoza, Argentina;
Mr. Andrew Coulthard Free, State University of Iowa;
Mr. John Clinton Freeman, Jr., Brown University;
Professor John E. Freund, Alfred University;
Professor Arthur Earl Fulton, Georgia School of Technology;
Mr. Bernard Aaron Galler, University of California at Los Angeles;
Professor Robert Lee Garrett, Athens College, Athens, Ala.;
Professor Conklin Blain Gentry, Oakwood College, Huntsville, Ala;
Mr. Merton Taylor Goodrich, Keene Teachers College, Keene, N. H.;
Mr. George Elihu Gourrich, University of California at Los Angeles;
Mr. John Wagner Graham, Ripon College, Ripon, Wis.;
Professor Laura Zazle Greene, Washburn Municipal University, Topeka, Kan.;
Mr. Ronald LeRay Greene, Clarkson College, Potsdam, N. Y.;
Mr. Robert Todd Gregory, Iowa State College of Agriculture and Mechanic Arts;
Mr. Emil Grosswald, University of Puerto Rico;
Professor Blanche Bennett Grover, University of Houston;
Miss Elizabeth Hahnemann, University of Minnesota;
Mr. Edmond Emerson Hammond, Jr., Brown University;
Mr. Arthur Gene Hansen, Purdue University;
Professor Nathan Warren Harter, Greenville, Pa.;
Dr. Thomas Watkins Hatcher, Virginia Polytechnic Institute;
Professor David Hawkins, University of Colorado;
Mr. Robert Mayo Hayes, University of California at Los Angeles;
Mrs. Emma Marie Henderson, University of Southern California;
Professor Charles Haelig Higgins, Monmouth Junior College, Long Branch, N. J.;
Miss Winifred Adella Hill, Alabama Polytechnic Institute;
Miss Ruth Bisgrove Hofstra, Syracuse University;
Mr. Ernest Ikenberry, Louisiana State University;
Mr. Wallace Henry Ito, University of Minnesota;
Professor Paul Frederick Iverson, Potomac State School at West Virginia University;
Mr. Russell Yaroslaw Iwanchuk, St. Basil's College, Stamford, Conn.;
Mr. Roscislaw Michael Iwanowski, Sweet Briar College;
Mr. Woodrow Allan Jaffee, Roosevelt College, Chicago, Ill.;
Mr. John Xavier Jamrich, Marquette University;
Professor Vojtech Jarnik, Carol University, Praha, Czechoslovakia;

Miss Marion Bruce Jeffries, Blackstone College, Blackstone, Va.;
Miss Bessie L. Jones, University of Alabama;
Mr. Harold T. Jones, Pacific Union College, Angwin, Calif.;
Professor Margaret Eloise Jones, Ohio State University;
Miss Reba Merle Jordan, Oklahoma Baptist University, Shawnee, Okla.;
Mr. Asher Dale Kantz, Southwestern College, Winfield, Kan.;
Mr. Samuel Noah Karp, Brown University;
Mr. Gilbert Kaskey, University of Delaware;
Mr. Robert Herman Kasriel, University of Virginia;
Professor Leo Katz, Michigan State College;
Mr. Murray S. Klamkin, Carnegie Institute of Technology;
Dean Olaf C. Kjosness, Western State College, Gunnison, Colo.;
Mr. Barron George Knechtel, Aurora College, Aurora, Ill.;
Miss Louise Murrell Knifley, Marshall College, Huntington, W. Va.;
Mr. C. Frederick Koehler, St. Joseph's College, Philadelphia, Pa.;
Mr. Sidney Kravitz, Newark College of Engineering;
Rev. Benedict P. Kremer, John Carroll University, Cleveland, Ohio;
Mr. William Edward Kruse, St. Peter's College, Jersey City, N. J.;
Mr. Lamar Layton, University of Illinois;
Mr. William Thomas Lenser, University of Nebraska;
Miss Marian Amelia Lesher, Fort Scott Junior College, Fort Scott, Kan.;
Mr. Charles Hadley Lewis, Bakersfield College, Bakersfield, Calif.;
Mr. H. Melvin Lieberstein, Washington University;
Mr. Robert Bernard Lowe, Polytechnic Institute of Brooklyn;
Mr. Walter Joseph Lyche, Wartburg College, Waverly, Iowa;
Professor Myles McConnon, Norwich University, Northfield, Vt.;
Miss Betty McKnight, Centenary College of Louisiana;
Professor Brown Lee Mackin, East Central State College, Ada, Okla.;
Dr. Emilio Antonio Machado, Instituto de Matematicas, La Plata, Buenos Aires, Argentina;
Mr. Joseph Simon Mamelak, Carnegie Institute of Technology;
Mr. Joseph Anthony Marano, Manhattan College, New York, N. Y.;
Miss Margaret Evelyn Martinson, Washburn Municipal University, Topeka, Kan.;
Professor José L. Massera, F. de Ingenieria, Montevideo, Uruguay;
Miss Rose Mary Miller, Vermont Junior College, Montpelier, Vt.;
Professor Christopher Raymond Mitchell, Rhode Island College of Education, Providence, R. I.;
Mr. Alex John Mock, Porterville Junior College, Porterville, Calif.;
Professor Richard Luther Moenter, Midland College, Fremont, Neb.;
Mr. Carlos Andrews de Moraes, Rensselaer Polytechnic Institute;
Mr. Robert H. Morris, Color Control Department, Eastman Kodak Company, Rochester, N. Y.;
Professor Morris Edward Mosely, Agricultural, Mechanical and Normal College, Pine Bluff, Ark.;
Mr. Simon Mowshowitz, University of Bridgeport;
Miss Elizabeth M. Muller, University of Bridgeport;
Mr. Edgar Raymond Mullins, University of Illinois;
Mr. John Paul Murray, Fairfield University, Fairfield, Conn.;
Mr. Dennis Mercer Nead, University of Cincinnati;
Miss Florence Nusim, Oswego State Teachers College, Oswego, N. Y.;
Mr. Edward Franklin Ormsby, Union College, Schenectady, N. Y.;

Mr. Donald Bruce Owen, University of Washington;
Mr. Alvin Jewel Owens, State University of Iowa;
Mr. Carl Ralph Partington, Purdue University;
Mr. Edward Paulson, University of Washington;
Mr. Lawrence Edward Payne, Iowa State College of Agriculture and Mechanic Arts;
Miss Mae Perlstein, New York, N. Y.;
Mr. Norman Conrad Perry, University of Southern California;
Professor Hector Alberico Persico, Universidad Nacional La Plata, Buenos Aires, Argentina;
Mr. Samuel Everett Peters, Long Beach City College, Long Beach, Calif.;
Mr. Lee Willard Petersen, La Salle-Peru High School, La Salle, Ill.;
Mr. Raymond Paul Peterson, University of California at Los Angeles;
Mr. Richard J. Phelps, The Teleregister Laboratories, New York, N. Y.;
Rev. Albeni Poitras, St. Joseph's University, New Brunswick, Canada;
Professor Herman Henry Price, Pasadena College;
Mr. Carl Pride, University of Oregon;
Professor William Walker Proctor, Morgan State College, Baltimore, Md.;
Mr. Charles Newton Putt, Kansas State College;
Mr. (Thomas) Benjamin Ripton, Sampson College, Sampson, N. Y.;
Mr. Murray B. Ritterman, Long Island University;
Miss Martha Ethyl Rivers, University of Tennessee;
Mr. Nathan Henry Rochmes, University of Illinois;
Miss Marguerite Katherine Roscoe, Montana State College;
Professor Ralph M. Ross, Rose Polytechnic Institute, Terre Haute, Ind.;
Dr. William C. Roudebush, New Mexico Military Institute;
Rev. Charles Harry Rust, St. Louis University;
Mr. Donald Raleigh Ryan, Gonzaga University;
Mr. Louis Sacks, Carnegie Institute of Technology;
Mr. Charles T. Salkind, Brooklyn College;
Mr. John Wesley Sawyer, University of Missouri;
Mr. George Van Schliestett, Office of Naval Research;
Professor Lilah Godfrey Schlotthauer, Walla Walla College, College Place, Wash.;
Mr. Nicholas Christ Scholomiti, De Paul University;
Carol S. Scott (Mrs. W. C.), St. Petersburg Junior College;
Mr. Fariebee Parker Self, Centenary College of Louisiana;
Mr. Edward I. Shapiro, Brooklyn College;
Mr. Julius Canoy Shepherd, University of Maryland;
Mr. Bernard Sherak, Newark Colleges, Rutgers University;
Miss Miriam Elmore Shi, Alabama Polytechnic Institute;
Professor Edwin Theodore Sheffield, University of Alberta, Edmonton, Alberta, Canada;
Professor William Archibald Sherratt, Presbyterian College, Clinton, S. C.;
Mr. Marvin R. Sitts, Flint Junior College, Flint, Mich.;
Professor Floyd B. Sloat, Kansas State College of Agriculture and Applied Science;
Miss Georgia Caldwell Smith, Spelman College, Atlanta, Ga.;
Mr. Roland Frederick Smith, LeMoyne College, Memphis, Tenn.;
Mr. Guilford Lawson Spencer, II, Massachusetts Institute of Technology;
Professor Lois Leavitt Splinter, Nebraska Wesleyan University, Lincoln, Neb.;
Mr. Philip Charles Stanger, Oklahoma Agricultural and Mechanical College;
Mr. Marvin Leonard Stein, University of California at Los Angeles;

Mr. Harold William Stephens, University of Florida;
Mr. Joel Eric Strandberg, Riverside College, Riverside, Calif.;
Mr. George Ralph Strohl, Jr., United States Naval Academy;
Professor Edward W. Suppiger, Princeton University;
Mr. William Charles Taylor, Jr., University of Tennessee Junior College;
Professor William Benson Temple, Agricultural and Mechanical College of Tex.;
Professor Ruth Fike Terry, Florida Southern College;
Miss Florence Gertrude Tetreault, University of Detroit;
Mr. Layton Oscar Thompson, University of Detroit;
Mr. Marvin L. Tomber, University of Pennsylvania;
Mr. Charles J. Tremblay, Bard College;
Miss Evelyn Ladene Trennt, Springfield Junior College, Springfield, Ill.;
Mr. Deonisie Trifan, Brown University;
Miss Frances Belle Tunstall, Stratford College, Danville, Va.;
Mr. Ethan M. Turley, University of Illinois;
Miss Lona Lee Turner, University of Chicago;
Mr. Veras Dean Turner, University of Illinois;
Mr. Harold Robert Uhl, Champlain College, Plattsburg, N. Y.;
Professor Louisa Amelia Van Dyke, Flora Macdonald College, Red Springs, N. C.;
Mr. Robert Zeno Vause, Jr., Clemson Agricultural College, Clemson, S. C.;
Mr. Sumner Ira Vrooman, Rensselaer Polytechnic Institute;
Professor Theodore Weaver, Michigan State Normal College, Ypsilanti, Mich.;
Mr. Herschel Weil, Brown University;
Miss Frances Welsbecker, Milwaukee-Downer College;
Mr. Ralph Martin Whitmore, Southwestern University;
Professor Mary Elizabeth Wilcox, Southwestern University;
Professor Ross Brooke Wildermuth, Capital University, Columbus, Ohio;
Mr. John Fox Williams, University of Tennessee;
Professor Joseph Bassett Williams, Champlain College, Plattsburg, N. Y.;
Mr. Harold Witz, Chicago Technical College;
Mr. John George Wozencraft, University of Illinois;
Mr. Lawrence Edgar Yancey, Morehouse College, Atlanta University;
Professor John Edward Yarnelle, Hanover College, Hanover, Ind.;
Mr. Chester Grant Young, University of Santa Clara;
Professor Neil Ferguson Young, Emory and Henry College, Emory, Va.;
Mr. Ai-ting Yu, Lehigh University;
Professor Harold LeRoy Zeiders, Midland College, Fremont, Neb.;
Mr. Joshua Vung-yuan Zia, University of Maryland.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

California Institute of Technology: Messrs. Paolo Gustavo Comba, John Edward Denby-Wilkes, and Daniel Talbot Finkbeiner, Dr. Samuel Karlin, Mr. Jack Enloe McLaughlin.
University of Chicago: Messrs. Matthew Page Gaffney, Jr., Clair Eugene Miller, Hugh Brent Prendergast, and Bethuhne Vanderburg.
University of Illinois: Mr. Richard Francis Burns.
State University of Iowa: Mr. Royal Keith Zeigler.
Kenyon College: Mr. Vasile Gh. Gorciu.
Purdue University: Messrs. Dean Norman Arden and John S. Lomont.

Stanford University: Mr. Michael I. Aissen, Miss Helen Ann Jackson.
Wesleyan University: Mr. Frank Dyer Bosworth.

The Secretary reported that a reciprocity agreement had been established between the Society and the Matematisk Forening i København.

Duquesne University (Pittsburgh, Pennsylvania) was elected to institutional contributing membership in the Society.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Mr. John Todd, Kings College, University of London, and National Bureau of Standards, Washington, D. C.; Matematisk Forening i København: Professor Aksel Frederik Andersen, Danmarks Tekniske Højskole, Copenhagen; Lektor Kai Rander Buch, Danmarks Tekniske Højskole, Copenhagen; Lektor Svend Bundgaard, Matematisk Institut, Copenhagen; Mr. Thoeger Busk, University of Copenhagen; Professor Frederik Fabricius-Bjerre, Danmarks Tekniske Højskole, Copenhagen; Dr. Werner Fenchel, Danmarks Tekniske Højskole, Copenhagen; Professor David Fog, Den kgl. Veterinär- og Landbohöjskole; Mr. Anders Hjorth Hald, University of Copenhagen; Mr. Hans Henrik Hansen, Danish Meteorological Institute, Copenhagen; Dr. Svend Edvard Lauritzen, Danmarks Tekniske Højskole, Copenhagen; Professor Jakob Nielsen, The Technical University of Denmark, Copenhagen; Professor Richard Petersen, Danmarks Tekniske Højskole, Copenhagen; Dr. Georg William Rasch, State Serum Institute, Copenhagen; Société Mathématique de France: Professor Albert Chatelet, Faculté des Sciences de Paris; Mr. Lucien Droussent, Ingenieur civil au Ministere de l'Armement, Clermont-Ferrand; Professor André Lichnerowicz, University of Strasbourg; Professor Frederic Amede Emile Roger, University of Bordeaux.

The following appointments by President Einar Hille were reported: Professor B. P. Gill as a member of the Library-Housing Committee, to replace Dr. Warren Weaver; Professor T. F. Cope and Dean A. E. Meder as auditors of the Society's accounts for 1948; Professor L. M. Graves as representative of the Society on the Editorial Board of the American Year Book for the period 1948–50; Professors R. G. Helsel (Chairman), H. M. Beatty, H. M. Gehman, L. H. Miller and C. V. Newsom as a Committee on Arrangements for the 1948 Annual Meeting; Professor Guy Stevenson as representative of the Society at the Sesquicentennial and inauguration of the President of the University of Louisville on February 9–10, 1948; Professor J. M. Clarkson as representative of the Society at the inauguration of John Decatur Messick as President of East Carolina Teachers College on March 6, 1948; Dean T. M. Simpson as representative of the Society at the inauguration of Joseph Hillis Miller as President of the University of Florida on March 5, 1948; Professor J. A. Clarkson as representative of the Society at the Fifty-second Annual Meeting of The American Academy of Political and Social Science on April 2–3, 1948; Professor E. W. Chittenden as representative of the Society at the Ninetieth Anniversary of the founding of Iowa State College of Agriculture and Mechanic Arts on March 22, 1948.

The Secretary reported that Professors L. M. Graves and O. E. Neugebauer will be the voting members on the Council during 1948 for the American Journal and Mathematical Reviews, respectively.

The resignation of Associate Secretary R. H. Bruck, to be effective after the 1948 Summer Meeting, was reported.

The Secretary reported that Dean A. E. Meder has been appointed as the representative of the Association for Symbolic Logic on the Policy Committee for Mathematics, to succeed Professor Alonzo Church.

For the Committee to Select Hour Speakers for Western Sectional Meetings, it was reported that Professors P. R. Halmos and Saunders MacLane have been invited to deliver hour addresses at the November, 1948, meeting in Chicago.

The election by the Trustees of Professor P. A. Smith as a member of the Board of Trustees to fill the unexpired term of Professor G. W. Mullins (to December 31, 1948) was reported.

The Council approved December 28–30, 1948, as the dates for the Annual Meeting and authorized the President to include December 27, if such action proved necessary.

The Council accepted the following slate of officers and committees for the International Congress of Mathematicians, presented by the Committee on Nomination of Officers and Committees of the International Congress:

Secretariat: Professor J. R. Kline, Secretary; Dr. R. P. Boas, Associate Secretary.

Editorial Committee: Professors Salomon Bochner (Chairman), Einar Hille, P. A. Smith, Oscar Zariski.

Financial Committee: Professors M. H. Stone (Chairman), J. L. Coolidge, Dean M. H. Ingraham, Professors John von Neumann, W. L. G. Williams.

Organizing Committee: Professors Garrett Birkhoff (Chairman), W. T. Martin (Vice Chairman), G. C. Evans, J. R. Kline, Solomon

Lefschetz, Saunders MacLane, Dean R. G. D. Richardson, Professors J. L. Synge, Oswald Veblen, J. L. Walsh, D. V. Widder, Norbert Wiener, R. L. Wilder.

President Hille was authorized to appoint a representative of the Society on the Cooperating Committee for the International Congress of Theoretical and Applied Mechanics, to be held in London in 1948.

The Council accepted for publication in the Colloquium Series a volume entitled *Differential algebra* by Professor J. F. Ritt. The Council also voted its approval of Professor G. A. Hedlund's request that Professor W. H. Gottschalk be made a co-author of his Colloquium volume.

The Council voted to recommend to the Board of Trustees that the number of pages in the 1948 volumes of the Transactions be increased to 1600.

A report of the Committee on the Role of the Society in Mathematical Publication, dealing with the possibility of inaugurating a project for the publication of translations of important Russian mathematical articles, was presented. The Council voted as follows: (1) To authorize President Hille to appoint a committee to serve as a committee of selection for articles (in Russian and other languages to be chosen at the discretion of the committee) of sufficient importance and demand to warrant a translation into English and publication thereof and to handle such other details as are necessary to inaugurate the project. (2) To authorize President Hille to appoint a committee to prepare a Russian-English vocabulary of important mathematical terms, supplemented by a pamphlet to include the Russian alphabet, rules of syntax, and short bibliography of works available in parallel Russian and English versions. (3) To recommend to the Board of Trustees that the President and Secretary be authorized to request the Office of Naval Research for $25,000 for the purpose of carrying out the projects mentioned above.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 188 was read by Dr. Chandrasekharan and paper number 193 by Dr. Hirschman. Professor Wright was introduced by Professor Antoni Zygmund.

ALGEBRA AND THEORY OF NUMBERS

180. I. S. Cohen: *Preservation of descending chain condition.*

Let R and S be commutative integral domains with identity, $R\subseteq S$. Assume that R satisfies the ascending chain condition and that every prime ideal is maximal; this is equivalent to the assumption of the descending chain condition modulo every nonzero ideal. It is proved that if S is integrally dependent on R and if the quotient field of S is finite over that of R, then S will also satisfy the restricted descending chain condition. Special cases of this theorem have been proved by Krull, Grell, F. K. Schmidt, and Akizuki. A result recently announced without proof by Grell is shown to be equivalent to the one given here. (Received February 20, 1948.)

181*t*. H. E. Goheen: *An elementary application of the Hamilton-Cayley theorem.*

Using the fact that a square matrix satisfies its own characteristic equation, the author develops the theory of the linear system with constant coefficients, $dx/dt=Ax$, in which A is an $n\times n$ matrix of constants and x a column vector of variables. The results are believed new in case A is a singular matrix, but the point of the paper is its use of the Hamilton-Cayley theorem as a tool. Similar methods may be used to discuss the nonhomogeneous system, $dx/dt=Ax+p(t)$, in which $p(t)$ is a column vector of functions of t. (Received January 21, 1948.)

182*t*. L. K. Hua and Irving Reiner: *On the generators of the symplectic modular group.*

Let Γ denote the group of $2n$-rowed matrices $\mathfrak{M}$ with rational integral elements for which $\mathfrak{M}\mathfrak{F}\mathfrak{M}'=\mathfrak{F}$, where $\mathfrak{M}'$ denotes the transpose of $\mathfrak{M}$ and $\mathfrak{F}$ is the matrix of the coefficients of the bilinear form $\sum_{i=1}^{n}(x_iy_{i+n}-y_ix_{i+n})$. The structure of the symplectic modular group Γ_n (L. K. Hua, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 441–481) defined as the factor group of Γ over its centrum is investigated. It is shown that there are exactly four (multiplicatively) independent generators of Γ_n if $n>1$, and these generators are found explicitly. (For $n=1$, it is well known that there are two independent generators.) The same method is used to find independent generators of the generalized Picard group Γ_0' obtained by permitting the elements of $\mathfrak{M}$ to be Gaussian integers, and also to find independent generators of the group of automorphs of the quadratic form $\sum_{i=1}^{n}2x_ix_{2n-i+1}$. (Received January 23, 1948.)

183. Eri Jabotinsky: *On Newtonian sequences of polynomials.* Preliminary report.

A sequence of polynomials $P_n(x)$ is said to be Newtonian if: (1) $P_0(x)=1$ and $P_n(x+y)=\sum_{k=0}^{n}P_k(x)\cdot P_{n-k}(y)$ for $n=0,1,2,\cdots$. As particular cases of Newtonian sequences we mention: $P_n(x)=x^n/n!$; $P_n(x)=C_{x,n}$; $P_n(x)=C_{x+n-1,n}$ and Nielsen's Stirling polynomials suitably normalized. It can be proved that conditions (1) are equivalent to: (2) $P_0(x)=1$ and $\sum_{k=0}^{n}P_k(-x-k)\cdot P_{n-k}(x+k)/(x+k)=0$ for $n=1,2,\cdots$. The proof is obtained by constructing the most general Newtonian sequence and using identities of which the following is typical: $(x+a+b+c+d)^3-\sum(x+a+b+c)^3+\sum(x+a+b)^3-\sum(x+a)^3+x^3=0$. The particular case of this theorem for the Stirling polynomials is contained implicitly in Nielsen, *Recherche sur les polynômes de Stirling*, Host and Son, Kobenhavn, 1920. *Application*: Since $P_n(x)=x(x+n)^{n-1}/n!$ satisfy (2), they also satisfy (1) so that: (3) $(x+y)(x+y+n)^{n-1}=\sum_{k=0}^{n}C_{n,k}x(x+k)^{k-1}\cdot y(y+n-k)^{n-k-1}$, whence, identifying the coefficients of xy: (4) $\sum_{k=1}^{n-1}C_{n,k}k^{k-1}\cdot(n-k)^{n-k-1}=2(n-1)n^{n-2}$. Identity (3) can also be deduced from a result of Pólya. (Received January 26, 1948.)

184. R. E. Johnson: *Conjugate modules.*

The present paper is the outgrowth of an attempt to examine more closely the relationship between the right and left primitive ideal structures of a ring R. If M is a right R- and left S-module, the conjugate, $C(M)$, of M relative to the ring S is the set of all S-homomorphisms of M into S. The natural definitions of operations make $C(M)$ into a left R- and right S-module. If S is taken to be a division ring, the elements of $C(M)$ correspond to the maximal S-submodules of M. Finally, by restricting M to be a simple right R-module and S the centralizer of R relative to M, every nonzero submodule of $C(M)$ is shown to have the same R-annihilator as M, and thus, if $C(M)$ has a simple left R-submodule, the R-annihilator of M is both right and left primitive. In case elements of R have finite nonzero S-order over M, $C(M)$ is shown to be subdirectly irreducible so that a simple left R-submodule does exist. (Received January 22, 1948.)

185*t*. B. H. Neumann: *On ordered division rings.* I, II.

Using results previously presented (Bull. Amer. Math. Soc. Abstract 53-9-305) the first part shows that every ordered group can be imbedded in the multiplicative group of an ordered division ring of formal power series. The method elaborates a proof sketch of O. F. G. Schilling (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 297–304). It follows that the free group of any number of generators can be so embedded, which answers a question raised by R. Moufang (J. Reine Angew. Math. vol. 176 (1937) pp. 203–223). The group ring of the free group of two generators over any field of coefficients is, however, shown to contain two elements without nontrivial common left-multiples. In the second part it is shown that every ordered division ring can be so extended as to contain in its centre a field order-isomorphic to the field of all real numbers. It is also shown that in an ordered division ring an element which is algebraic over the centre lies in the centre. (Received January 20, 1948.)

186*t*. B. J. Pettis: *Remarks on the extension of lattice functionals.*

The chief result established is the following: a lattice L is distributive if and only if any bounded modular functional on a sublattice can be extended (mod a constant) to be bounded and modular on L. (Received February 2, 1948.)

ANALYSIS

187*t*. Richard Bellman: *On the existence and boundedness of solutions of nonlinear partial differential equations of parabolic type.*

The nonlinear partial differential equation of parabolic type is studied. The space variables range over the interior of a cube, and the time variable over $(0, \infty)$. The partial differential equation is transformed into a nonlinear integral equation, and the method of successive approximations is applied. Crucial for the method is the result that the solution of the linear problem is a continuous operation on the initial value (the temperature of the cube at $t=0$) under slight restrictions on the type of solution considered. This result is a consequence of an inequality of Eversull. Further applications of the method are indicated. (Received January 24, 1948.)

188. Salomon Bochner and K. Chandrasekharan. *On the localization property for multiple Fourier series.*

If the function $f(x_1, \cdots, x_k)$ is of period 2π in each of the k variables, belongs to L_1 in $0 \leq x_r < 2\pi$, $r=1, \cdots, k$, and belongs to L_2 in a neighborhood of the point $(x_1, \cdots, x_k)$, and if $\int_0^t \{f_p(y)\}^2 dy = o(t)$ as $t \to 0$, then $\int_0^R \{S^\delta(R)\}^2 dR = o(R)$ as $R \to \infty$, for $\delta = p + (k-1)/2$, where $f_p(t)$ is the spherical mean of order p of the function f at the point $(x_1, \cdots, x_k)$ and $S^\delta(R)$ is the Riesz mean of order δ of the multiple Fourier series of the function (when summed spherically). (Received January 14, 1948.)

189t. John DeCicco: *Functions of several complex variables and multiharmonic functions.*

This paper is a brief introduction to the theory of polygenic functions of several complex variables. The importance of the mean and phase derivatives is demonstrated in the study of the Kasner clocks. The author considers how analytic polygenic functions can be extended into spaces of double the original number of dimensions. Applications are made to the theory of monogenic functions. The author proves that if a multiharmonic function (the real or imaginary part of a monogenic function) is rational, algebraic, or entire, then the associated monogenic function is rational, algebraic, or entire. A characterization of multiharmonic functions is studied. Finally, a discussion is given of several polygenic functions of several complex variables. An expression for the Jacobian is obtained in terms of the mean and phase derivatives. Application is made to the pseudo-conformal group which may be defined as the group preserving the pseudo-angle of Kasner. (Received December 29, 1947.)

190. Aryeh Dvoretzky. *On changing the signs of the coefficients of Taylor series and of Dirichlet series.*

It is well known that if (1) $\sum_0 a_n z^n$ has a finite positive radius of convergence R, then a sequence $\{\epsilon_n\}$ $(\epsilon_n = \pm 1; n = 0, 1, 2, \cdots)$ can be found such that (2) $\sum \epsilon_n a_n z^n$ has $|z| = R$ as a natural boundary. Pólya (C. R. Acad. Sci. Paris vol. 184 (1927) pp. 1526–1528) has shown that when $R = \infty$ a sequence $\{\epsilon_n\}$ can be found so that every direction will be a Julia direction for (2), *provided that* (1) *represents a function of infinite order*. The author proved (Ph.D. thesis, Jerusalem, 1941) that the above result holds for *every transcendental integral function*. When information on the growth of the function is available much more can be said, and essentially narrower domains which exhibit the Picard phenomenon given. Also when $0 < R < \infty$ similar results may be obtained. Thus if $\limsup |a_n| R^n = \infty$ a sequence $\{\epsilon_n\}$ can be found so that (2) assumes every value with one exception at most infinitely many times in *every* sector $|z - Re^{i\phi}| < \delta$, $\psi_1 < \arg (z - Re^{i\phi}) < \psi_2$ lying inside $|z| = R$. These and more general and precise theorems are obtained through the use of the theory of normal families. The methods and results carry over mutatis mutandis to general Dirichlet series. (Received January 22, 1948.)

191. Paul Erdős: *On the strong law of large numbers.*

Kac, Salem and Zygmund proved that if $f(x+1) = f(x)$, $\int_0^1 f(x) = 0$, $\int_0^1 f(x)^2 = 1$, $n_{k+1}/n_k > c > 1$ and if, for any ϵ, $\int_0^1 (f(x) - S_n(f))^2 = O(1/(\log n)^\epsilon)$, where $S_n(f)$ denotes the nth partial sum of the Fourier series of $f(x)$, then for almost all x, $\lim (1/N) \sum_{k=1}^{N} f(n_k x) = 0$. It is shown that the conclusion remains true if $1/(\log n)^\epsilon$ is replaced by $1/(\log\log n)^{1+\epsilon}$. On the other hand some restriction on $f(x)$ is necessary since an $f(x)$ is constructed satisfying $f(x+1) = f(x)$, $\int_0^1 f(x) = 0$, $\int_0^1 (f(x))^2 = 1$ such that there exists a sequence $n_{k+1}/n_k > c > 1$ for which $\lim (1/N)(\sum f(n_k x)) = \infty$ for almost all x. (Received January 19, 1948.)

192. G. J. Haltiner: *On the theory of linear differential boundary systems when based upon a new definition of the adjoint.*

The author considers the formal aspects of the theory of an ordinary differential boundary system and of the associated theory of the expansion of an arbitrary function. Specifically, the boundary problem studied is 1. $\sum_{j=0}^{n} p_j(x)d^{n-j}u(x)/dx^{n-j}+\lambda q(x)u(x)=0$, $p_0(x)=1$, 2. $\sum_{j=1}^{n}\alpha_{ij}(\lambda)u^{(n-j)}(a,\ \lambda)+\sum_{j=n+1}^{2n}\alpha_{ij}(\lambda)u^{(2n-j)}(b,\ \lambda)=0$, $i=1, 2, \cdots, n$, in which the coefficients $p_j(x)$ and $q(x)$ are differentiable functions of x on some closed interval (a, b); λ is a complex parameter; and the $\alpha_{ij}(\lambda)$ are polynomials in λ. The treatment of the theory is based upon a new definition of the adjoint system introduced and applied to differential systems of order 2 by R. E. Langer. Some of the features in which the present paper differs from previous discussions of the nth order linear system are: (a) the new definition affords the means for an explicit and inclusive representation of the entire adjoint boundary problem; (b) specific expressions are derived for the solutions of the adjoint system in terms of the solutions and coefficients of the given problem; and (c) in the derivation of generalized orthogonality and in the determination of the residues of the Green's function for use in the expansion theory, no restriction is placed on the index of the characteristic values. (Received January 26, 1948.)

193. I. I. Hirschman and D. V. Widder: *Inversion of a general class of convolution transforms.*

Let $E(s)=e^{-cs^{2}}\prod_1^\infty[1-(s/a_k)]e^{s/a_k}$, where the roots a_k are real (not necessarily distinct) and such that $\sum_1^\infty a_k^{-2}<\infty$. It is shown that in any vertical strip of the complex s-plane that is free of zeros a_k, $1/E(s)$ is the bilateral Laplace transform of some function $G(t)$. With this kernel the integral equation $f(x)=\int_{-\infty}^{\infty}G(x-t)\phi(t)dt$ is solved; in fact $\phi(x)=E(D)f(x)$, where $Df(x)=f'(x)$ and $e^{cD}f(x)=f(x+c)$. The above convolution of G with ϕ behaves very differently in the following two cases: (a) the a_k are all >0 or all <0, (b) there are infinitely many a_k of each sign. In case (a) it has convergence properties like a Laplace transform; in case (b), like a Stieltjes transform. Hence proofs of the above inversion are different in the two cases. No conditions are imposed on $\phi(x)$ except those implicitly imposed by the convergence of the improper integral of the transform. The inversion then holds almost everywhere. In particular if $E(s)=1/\Gamma(s)$ the convolution reduces to the unilateral Laplace transform, and the inversion becomes the Post-Widder formula. If $E(s)=\sin \pi s$ we have the Stieltjes transform, and the inversion reduces to a formula due to Widder. All the iterates of the Laplace transform appear as special cases of our theory since the a_k may be multiple roots. (Received January 21, 1948.)

194*t*. J. B. Rosser: *The complete monotonicity of certain functions derived from completely monotone functions.*

By means of a generalization of Rolle's Theorem, several theorems are proved, of which the following is an instance. Let $F(x)$ be completely monotonic and have derivatives for $0\leqq x\leqq\infty$. That is, $(-1)^kF^{(k)}(x)\geqq 0$ for $0\leqq x\leqq\infty$. If λ_i and C_i are constants, $C_i\geqq 0$ so that $G(x)=F(x)-\lambda_1\exp(-C_1x)-\lambda_2\exp(-C_2x)-\cdots-\lambda_n\exp(-C_nx)$, and $G(x)$ and its first $2n-1$ derivatives all vanish at the origin, then $G(x)/x^{2n}$ is completely monotonic for $0\leqq x\leqq\infty$. (Received January 26, 1948.)

195. Arthur Sard: *Best approximate integration formulas; best approximation formulas.*

Let $\mathcal{A}_{m,n}$ be the set of approximations $A=c_0x(0)+c_1x(1)+\cdots+c_mx(m)$ of the integral $I=\int_0^m x(t)dt$ which are exact whenever $x(t)$ is a polynomial of degree n. Assume that $x(t)$ is a function with absolutely continuous nth derivative and such that

$x^{(n+1)}(t)^2$ is integrable on $[0, m]$. For each approximation in $\mathcal{A}_{m,n}$ the remainder can be expressed as $R[x]=I-A=\int_0^m x^{(n+1)}(t)k(t)dt$, where $k(t)$ is an appropriate function. The best formula in $\mathcal{A}_{m,n}$ is defined as that one for which $\int_0^m k(t)^2dt$ is minimal. The best formula leads to the least appraisal of $|R[x]|$ in terms of $x^{(n+1)}(t)$, if Schwarz's inequality is used. The best formulas are given for the cases $m=1, 2, \cdots, 6$; $n=0, 1, 2, 3$. More generally, let $\mathfrak{R}$ be a set of functionals $R[x]$, each of which is linear (additive and continuous) on the space C_q of functions $x(t)$ with continuous qth derivative on $a \leqq t \leqq b$ and each of which vanishes whenever x is a polynomial of degree $n \geqq q$. For each functional in $\mathfrak{R}$, $R[x]=\int_a^b x^{(n+1)}(t)k(t)dt$, where $k(t)$ is an appropriate function. A best functional in $\mathfrak{R}$ is defined as a functional (if any) which minimizes $|K|\int_K k(t)^2dt$, where K is the set on $[a, b]$ on which $k(t)\neq 0$ and $|K|$ is the measure of K. (Received January 14, 1948.)

196. F. M. Stewart: *Integration in noncommutative systems.*

Let $\mathfrak{D}: \lambda, \mu, \cdots$ be a topological space with a binary associative operation, $\cdot$, such that $\lambda\cdot\mu\in\mathfrak{D}$ for all $\lambda, \mu\in\mathfrak{D}$ and is continuous in λ and μ jointly. If $\mu\{t, \sigma\}\in\mathfrak{D}$ is defined for each t in an interval $[a, b]$ and each measurable subset, σ, of $[a, b]$, Riemann and Lebesgue integrability of μ over $[a, b]$ are defined. If $\mathfrak{D}$ is the additive group of real numbers with its usual topology and $\mu\{t, \sigma\}=f(t)\cdot m(\sigma)$ then the Riemann and Lebesgue integrals of μ over $[a, b]$ coincide with the ordinary Riemann and Lebesgue integrals of $f(t)$ over $[a, b]$. Other specializations of $\mathfrak{D}$ and $\mu\{t, \sigma\}$ yield the product integrals of Volterra and Birkhoff, the Riemann-Stieltjes integral, and certain integrals of functions with values in a topological linear space. Generalizations of theorems familiar in some or all of these specific cases are proved when the following conditions are imposed on $\mathfrak{D}$: (1) The topology in $\mathfrak{D}$ is given by a metric, $\rho(\lambda, \mu)$. (2) $\mathfrak{D}$ contains a two-sided identity, e. (3) Given $\epsilon>0$, B, there is a $\delta>0$ such that $\sum_{i=1}^n\rho(\lambda_i, e)<B$ and $\sum_{i=1}^n\rho(\lambda_i, \mu_i)<\delta$ imply $\rho(\prod_{i=1}^n\lambda_i, \prod_{i=1}^n\mu_i)<\epsilon$. (Received January 21, 1948.)

197*t*. E. M. Wright: *The asymptotic expansion of integral functions and of the coefficients in their Taylor series.*

Let $f(x)$ be an integral function and let $f(x+\lambda)=\sum_0^\infty c_\lambda(n)x^n$, $g(t)=t^\beta e^{\psi(t)}(e/\kappa t)^{\kappa t}$, $\mathfrak{R}(\kappa)>0$, $\psi(t)=\sum_1^J a_j t^{b_j}$, $\mathfrak{R}(b_j)<1$. If either (i) $c_0(t)\sim g(t)$ for all large, positive, integral t or (ii) $c_0(t)\sim g(t)$ in a sector of the t-plane enclosing the positive half of the real axis, then $f(x)\sim(2\pi)^{1/2}\kappa^{-1-\beta}X^{1/2+\beta}e^{P(X)}$ for large x in part (or all) of the x-plane. Here X is a suitable κth root of x and $P(X)-X=o(X)$ is a finite sum of fractional powers of X. It follows from (ii) that $c_\lambda(n)\sim c_0(n)e^{\lambda\psi(n,\lambda)}$, where $\psi(n, \lambda)$ is a calculable polynomial in λ (and is zero if $\mathfrak{R}(\kappa)>1$). The same result follows very simply from (i) if and only if $\mathfrak{R}(\kappa)\geqq 1$. In the above $h_1(y)\sim h_2(y)$ denotes that $h_1(y)/h_2(y)=1+O(y^{-K})$, with $K>0$ and y one of x, t, n. (Received December 2, 1947.)

198. L. C. Young: *Some applications of the Dirichlet integral to the theory of surfaces.*

A vector function $x(u, v)$ in the closed unit square R of u, v is termed Dirichlet representation (N) if $x(u, v)$ is absolutely continuous on almost every line parallel to the axes of u and v, and its Dirichlet integral is less than N. A Frechet surface so representable is a Dirichlet surface (N), or a Dirichlet surface when N is not specified. Dirichlet surfaces (N) actually coincide with Dirichlet surfaces of Lebesgue area less than N. Use is also made of intrinsic area. An ϵ-modification of $x(u, v)$ is an $\bar{x}(u, v)$ coincident with $x(u, v)$ outside an open set on which the total intrinsic area of $\bar{x}(u, v)$ is less than ϵ. An ϵ-modification of a surface S is defined similarly. Actually a Dirichlet representation $x(u, v)$ of Lebesgue area less than A possesses for each $\epsilon>0$ an ϵ-modification $\bar{x}(u, v)$ of intrinsic area less than A. One main theorem states: a sequence of Dirichlet representations N, equi-continuous on the boundary of R, can be ϵ-modified into Dirichlet representations equi-continuous in R. It is deduced that given a sequence of surfaces S_n with bounded Lebesgue areas, and with (say) simple boundary curves converging to a simple curve, there exists, for each $\epsilon>0$, a compact sequence of ϵ-modifications. (Received February 12, 1948.)

APPLIED MATHEMATICS

199t. Stefan Bergman: *On linear partial differential equations with singular coefficients*. I.

In the present paper solutions of the equation $L(\psi)=\psi_{ZW}+F\psi=0$, $Z=Z_1+iZ_2$ $W=Z_1-iZ_2$, $Z_k=x_k+iy_k$, $k=1, 2$, are considered. Let F be regular in B^4. Then the singularities of ψ can lie only on characteristic planes $Z=c=$const., or $W=c$. To every $Z=c$ (or $W=c$) there exist functions possessing, on $Z=c$, singularities of the same order of infinity and the same behavior as in the case of harmonic functions. Suppose F is regular in $B^4=R^4-\sum_{\mu=1}^{n}S_\mu^3$ where $R^4=[|Z|<\infty,\ |W|<\infty]$, and S_ν^3, $\nu=1, 2, \cdots, n$, are singularity surfaces of F. To every point $P_0(Z_0, W_0)\in B^4$ there exists a system of solutions $\psi_\nu=\mathrm{P}(Z-Z_0)^\nu$, $\psi_{2\nu+1}=\mathrm{P}[(W-W_0)^\nu]$, $\nu=0, 1, 2, \cdots$, $\mathrm{P}(s)=s-\int_{Z_0}^{Z}\int_{W_0}^{W}Fs\,dz_1dw_1+\int_{Z_0}^{Z}\int_{W_0}^{W}F\int_{Z_0}^{z_1}\int_{W_0}^{w_1}Fs\,dz_2dw_2dz_1dw_1$, each of which is regular in B^4 and such that every solution ψ can be developed in a sufficiently small neighborhood of P_0 in the form $\psi=\sum a_\nu\psi_\nu$. If P_0 completes a closed circuit along a curve C^1 which cannot be reduced to a point without cutting S_μ^3, then the ψ_ν undergo a transformation $\mathcal{A}=\{a_{m,n}\}$, $m=1, 2, \cdots, n=1, 2, 3, \cdots$. Suppose $\{\chi_\nu\}$ is another complete system of solutions of L and $\mathfrak{B}$ is its transformation matrix of the χ_ν corresponding to C^1. Then $\mathfrak{B}=\mathfrak{D}^{-1}\mathcal{A}\mathfrak{D}$, where $\mathfrak{D}$ is a suitably chosen matrix. (Received February 27, 1948.)

200t. Stefan Bergman: *On linear partial differential equations with singular coefficients*. II.

The author considers solutions of the equation $\psi_{ZW}+F\psi=0$, $Z=Z_1+Z_2$, $W=Z_1-iZ_2$, where F is an analytic function of two complex variables $Z_k=X_k+iY_k$, $k=1, 2$, which is regular in $N^4-S_1^3-S_2^3$ and single-valued in N^4. Here $N^4=[|Z_k|<\mathrm{P}]$, P sufficiently small, $S_k^3=[Z_k=\sum_{\nu=1}^{\infty}a_\nu^{(k)}Z_{3-k}^\nu]$. The problem of the determination of functions which are multiplied by a factor λ_k when the variable makes a closed circuit C_k^1 around S_k^3 can be reduced to the determination of functions $f(Z)$ and $g(W)$, satisfying a system of two integral equations, first of which is $(\lambda_1-1)f(\zeta)+(\lambda_1-1)g(\tau)-(\lambda_1-1)\cdot\int_{Z=z}^{Z=\zeta}v[Z,\tau;Z;W]f'(Z)dZ+(\lambda_1-1)\int_{W=\tau}^{W=w}v(\zeta,W;z,w)g'(W)dW=\int_{\phi=0}^{\phi=2\pi}[v(A,\tau;z,w)f'(A)+iv(\zeta,B;z,w)g'(B)]\rho e^{i\phi}d\phi$ and the second has a similar structure. Here $v(\zeta, \tau; z, w)$ is the "generalized Riemann's function," $A=(\rho\cos\phi-Y^{(0)})+i(\rho\sin\phi+X_2^{(0)})$, $B=(\rho\cos\phi+Y_2^{(0)})+i(\rho\sin\phi-X_2^{(0)})$; ρ, $X_2^{(0)}$, $Y_2^{(0)}$ are suitably chosen constants, and λ_k are multiplication factors to be determined from the equations. The author considers two special cases, namely $F=(WZ)^{-1}C_1+W^{-1}C_2(Z)+Z^{-1}C_3(W)+C_4(Z, W)$, and $F=Z_1^{-2}C_4+Z_2^{-2}C_5+Z^{-1}C_6(Z_2)+Z_2^{-1}C_7(Z_1)+C_8(Z, W)$, C_n being regular functions or constants. In the first case the author obtains characteristic solutions of the form $W^\rho Z^{C_1/\rho}\sum\sum B_{mn}W^mZ^n$, where ρ and B_{00} are arbitrary and in the second case $Z_1^rZ_2^s\sum\sum B_{mn}Z_1^mZ_2^n$, where r and s are determined from the "indicial equations." In the second case a denumerable subset of the B_{mn} are arbitrary. (Received February 27, 1948.)

201. Lipman Bers: *On the continuation of a potential gas flow across the sonic line.*

Let there be given a two-dimensional potential subsonic gas flow occupying some domain D. Let G be a continuously curved boundary arc of D along which the Mach number (ratio of local speed to local speed of sound) attains the value one. It is shown that if the velocity components possess continuous partial derivatives on G, and if neither the normal derivative of the speed nor the tangential component of the velocity vector vanish along this curve, then the flow may be continued across G as a potential supersonic flow without shocks or weak discontinuities. The supersonic flow is uniquely determined in some neighborhood of G. The exact domain of dependence as well as the flow itself can be effectively computed. (The domain of dependence comprises the entire supersonic region in the case of a transonic flow past an obstacle.) The proof of these statements depends upon the solution of a Cauchy problem for a linear partial differential equation of mixed type with Cauchy data on the critical line. A new treatment of this problem is given. It is believed to be simpler and more convenient than the previous ones due to Khristianowitch and Frankl (see Bull. Acad. Sci. USSR. vol. 8 (1944) pp. 195–245). (Received January 19, 1948.)

202*t*. A. D. Fialkow and Irving Gerst: *The transfer function of an L-network.*

The following theorem is proved: The necessary and sufficient conditions that $A(p)$ be the transfer function of a passive L-network containing resistance and capacitance only are: (1) $A(p)=K\prod_{i=1}^{N}(p+\delta_i)/\prod_{j=1}^{M}(p+\gamma_j)$ where the γ_j are distinct positive numbers and the δ_i are non-negative real numbers and $M=N$ or $M=N+1$. (2) The number of δ's less than a particular γ_j $(j=1, 2, \cdots, M)$ is equal to or one greater than the number of γ's less than that γ_j. (3) The number K satisfies the inequalities $0<K\leqq K_0$ where K_0 is the least of the three quantities K_d, $\prod_{j=1}^{M}\gamma_j/\prod_{i=1}^{N}\delta_i$, 1 if $M=N$ and of the first two quantities if $M=N+1$. Here K_d is the least positive root (if it exists) of the equation in K obtained by equating the discriminant of $\prod_{j=1}^{M}(x+\gamma_j)-K\prod_{i=1}^{N}(x+\delta_i)=0$ to zero. (Received February 5, 1948.)

203*t*. A. D. Fialkow and Irving Gerst: *The transfer function of a ladder network.*

The transfer function of a four terminal network is the ratio of the voltage at the output terminals to the voltage at the input terminals. The necessary and sufficient conditions that $A(p)$ be the transfer function of any ladder network containing resistance and capacitance only are determined. An algorithm for the synthesis of the network corresponding to $A(p)$ is given. Analogous results obtain for networks which contain resistance and inductance only or inductance and capacitance only. These theorems are applied to various engineering design problems. (Received February 5, 1948.)

204. F. G. Gravalos: *Two new theorems in hydrodynamics and their technical applications.*

Instead of Euler's well known formula $H_2-H_1=\omega(Cu_2r_2-Cu_1r_1)$ for the energy input (or output) in the flow about a compressor (or turbine) the first theorem gives

$H_2-H_1=-\int_{t_1}^{t_2}U\cdot\omega r dt$ where U is the whirl component of the aerodynamic reaction. The second formula describes the flow *inside* the rotor and applies to *any* two points. Consequently it is shown that $\omega(Cu_2r_2-Cu_1r_1)=-\int_{t_1}^{t_2}U\cdot\omega r dt$ and this amounts to a generalization of Euler's formula as obtained in the ordinary way. The second theorem gives a general expression, $(1/\rho)$ $\text{grad}_{u_1}p=(1/H_1)(1/2)(\partial/\partial u_1H_2^2)u_2+(1/H_1)(1/2)$ $(\partial/\partial u_1H_3^2)u_3$, for the pressure gradient along the u_1-direction in terms of intrinsic geometric elements of the coordinate surface of revolution $u_1=\text{constant}$. As a consequence of these two theorems the necessary conditions for the determination of the flow about a compressor (or turbine) of general shape are determined and a method for the calculation of the characteristics of the flow follows together with new relations between the energy changes and the variation of axial velocity. (Received February 24, 1948.)

205. C. A. Truesdell: *The energy theorem for Newtonian continua.*

A formula for the rate of change of the kinetic energy of an arbitrary finite material volume of a Newtonian continuum has been deduced. The general formula is specialized to the case of compressible viscous fluids, generalizing the classical theorem of Bobyleff for incompressible viscous fluids. It is shown that in a viscous compressible fluid the vorticity and the rate of change of the density play completely separate and exactly analogous roles in the viscous dissipation of energy. (Received February 2, 1948.)

GEOMETRY

206*t*. Edward Kasner and John DeCicco: *Physical families of curves in space.*

The authors investigate the geometric character of a system of twisted curves S_k of space connected with an arbitrary positional field of force. Such a system S_k consists of curves along which a constrained motion is possible such that the osculating plane at each point contains the force vector and the pressure is proportional to the normal component of the force vector. There are four cases of physical interest: (1) The system S_0 of trajectories. (2) The system S_{-3} of brachistochrones (in a conservative field of force). (3) The system S_1 of general catenaries. (4) The system S_∞ of velocity curves. After discussing various geometric properties of the general systems S_k, the transformation theory of such systems S_k is given. (Received December 29, 1947.)

STATISTICS AND PROBABILITY

207. E. J. Gumbel: *Normal extremes.*

The numerical values of the probability functions, averages and moments of the extreme values taken from a normal distribution have been calculated, mainly by Tippett, up to sample sizes $n=1000$. The distribution of these normal extremes converges toward the double exponential distribution $f(x)=\exp[-\exp(-y)]$ where $y=\alpha(x-u)$. Here, u is the expected largest value for a sample of n, and α the corresponding intensity function. The calculated mean (median) differs from the asymptotic mean (median) by 0.6 per cent of the calculated value for $n=1000$ $(n=10)$. The asymptotic distribution can be used for relatively small sample sizes by an adequate estimate of the parameters, either from the definitions, or from the calculated distribution of the extremes. Both procedures lead to practically identical results for u. The probability paper shows that the asymptotic distribution of the normal extremes may be applied for $n=365$ up to a 95 per cent level, a fact observed by Barricelli. The probability paper gives a criterion for the regularity of observed normal extremes. The theoretical straight line may be obtained from the means and standard deviations of the observations and of the reduced values y, the latter depending exclusively upon n. The modal mth extremes are obtained from the mode of the extreme through a recurrent procedure. The distribution of the intervals between extremes converges toward an exponential function. (Received February 2, 1948.)

TOPOLOGY

208*t*. E. E. Floyd: *On the extension of homeomorphisms on regions.*

Let P be a region in a compact space A and let Q be a subset of a compact space B. Let f be a continuous map from $\overline{P}$ to $\overline{Q}$ such that $f|P$ is a homeomorphism from P onto Q. A study is made of the action of the map $f|F(P)$, $F(P)$ the boundary of P, when various restrictions are placed on Q. In the first place, if Q is a region of an n-sphere S_n, then the map $f|F(P)$ is shown to be non-alternating. Secondly, attention is given to the case in which Q is *ul* $i-c$, $0\leq i\leq k$. If P is also *ul* $i-c$, $0\leq i\leq k-1$, then $f|F(P)$ is n-monotone. If P is *ul* $i-c$, $0\leq i\leq n$, then a necessary and sufficient condition that Q be *ul* $i-c$, $0\leq i\leq n$, is that $f|F(P)$ be n-monotone. (Received December 4, 1947.)

209. R. H. Fox: *On the imbedding of polyhedra in 3-space.*

By a *tubular figure* is meant a regular neighborhood of a finite (but not necessarily connected) linear graph. Consider a system of m non-intersecting surfaces of genus $k_1, \cdots, k_m$ polyhedrally imbedded in spherical 3-space S. It was shown by Alexander (Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 6–8) that in the case $m=1$, $k=0$ each of the two (closed) complementary domains is a tubular figure and in the case $m=1$, $k=1$ at least one of them is. In the general case, although there need be no tubular figures among the complementary domains, each complementary domain is homeomorphic to the complement of some tubular figure. By means of this partial extension of Alexander's theorem necessary conditions for a given polyhedron of dimension not greater than 3 to be a polyhedral subset of 3-space may be derived. For example, from a theorem about 3-dimensional manifolds due to Reidemeister (Monatshefte für Mathematik und Physik vol. 43 (1936) pp. 20–28) one obtains the following theorem: If the first Betti number of a connected polyhedral subset of S is greater than 2 then its fundamental group is non-abelian. (Received January 21, 1948.)

210*t*. Deane Montgomery: *Dimension of factor spaces.*

Let G be a locally compact topological group of finite dimension and let H be a closed abelian subgroup, then the dimension of G/H is finite. (Received December 24, 1947.)

211*t*. Deane Montgomery: *Subgroups of locally compact groups.*

Let G be a locally compact connected group of dimension n. If H is an n-dimensional closed subgroup of G, then H is all of G. (Received December 24, 1947.)

212*t*. Hing Tong (National Research Fellow): *Note on minimal bicompact spaces.* Preliminary report.

In his *Treatise on set topology*, Part I (Madras, India, 1947) R. Vaidyanathaswamy raised the question whether there exist minimal bicompact non-Hausdorff spaces (loc. cit. Preface). It is not difficult to see that a minimal bicompact space is necessarily a T_1. On the other hand, it can be shown readily that the space R constructed below is minimal bicompact non-Hausdorff: The points of R are the symbols α_1, α_2 and all ordered pairs (m, n) of natural numbers. A neighborhood system of R consists of all sets of single point (m, n), all sets of the form $\{\alpha_1$ and all points of the form $(m_{2i+1}, 2i+1)$ for $m_{2i+1} > M_{2i+1}$, $i=0, 1, 2, \cdots \}$, and all sets of the form $\{\alpha_2$, all points of the form $(m_{2i}, 2i)$ for $m_{2i} > M_{2i}$, $i=1, 2, 3, \cdots$, and all points of the form (m, n) for $n > N\}$. Other properties of minimal bicompact spaces are also discussed. (Received January 24, 1948.)

213. G. W. Whitehead: *On spaces aspherical in dimensions less than n.*

Let X be a pathwise connected topological space whose ith homotopy group $\pi_i(X)$ vanishes for $i<n$. If $n>2$, superposition on the essential element of $\pi_{n+1}(S^n)$ defines a homomorphism $\eta:\pi_n(X)\rightarrow\pi_{n+1}(X)$. Let $H_k(X)$ be the integral singular homology group of X, $S_k(X)$ the group of spherical homology classes. Then if $n>2$, $H_{n+1}(X) \approx \pi_{n+1}(X)/\text{Image } \eta$, and $H_{n+2}(X)/S_{n+2}(X) \approx \text{Kernel } \eta/2\pi_n(X)$ if $n>3$. If moreover $\pi_{n+2}(X)=0$, then $H_{n+3}(X)/S_{n+3}(X) \approx H_{n+1}(X)/2H_{n+1}(X) + {}_2(\pi_n(X))$. By means of these results the groups $H_{n+k}(\pi, n)$ defined by Eilenberg and MacLane (Ann. of Math. vol. 46 (1945) pp. 480–509) are calculated for $k \leq 3$ and $n>3$. (Received January 5, 1948.)

J. R. KLINE,
Secretary

THE FEBRUARY MEETING IN CHICAGO

The four hundred thirty-third meeting of the American Mathematical Society was held at the University of Chicago, Chicago, Illinois, on Saturday, February 28, 1948, with a total attendance of one hundred seventeen, including the following one hundred ten members.

E. J. Akutowicz, A. A. Albert, W. R. Allen, Reinhold Baer, J. H. Bell, Herman Betz, S. F. Bibb, R. H. Bing, D. G. Bourgin, H. R. Brahana, G. W. Brown, R. H. Bruck, P. B. Burcham, Lee Byrne, K. H. Carlson, R. E. Carr, E. D. Cashwell, T. E. Caywood, E. W. Chittenden, A. M. Christiansen, H. J. Cohen, J. J. Corliss, J. B. Crabtree, H. J. Curtis, W. M. Davis, D. M. DeWitt, Flora Dinkines, John Dyer-Bennet, J. H. Engel, G. M. Ewing, J. V. Finch, Harley Flanders, R. S. Fouch, J. S. Frame, Evelyn Frank, C. G. Fry, R. E. Fullerton, Albert Furman, H. H. Goldstine, S. H. Gould, Charles Hatfield, I. R. Hershner, Fritz Herzog, Edwin Hewitt, E. H. C. Hildebrandt, J. D. Hill, D. L. Holl, C. C. Hsiung, H. K. Hughes, T. J. Jaramillo, Meyer Jerison, Madeline Johnsen, W. C. Kalinowski, Irving Kaplansky, William Karush, Leo Katz, L. M. Kelly, M. Z. Krzywoblocki, E. P. Lane, W. G. Leavitt, Walter Leighton, M. M. Lemme, Benjamin Lepson, C. T. Loo, W. J. Lyche, A. W. McGaughey, Morris Marden, M. S. Matchett, H. F. Mathis, C. W. Moran, E. J. Moulton, S. B. Myers, E. P. Northrop, F. S. Nowlan, E. N. Oberg, E. H. Ostrow, Gordon Pall, P. M. Pepper, Everett Pitcher, J. C. Polley, D. H. Potts, A. L. Putnam, O. W. Rechard, W. T. Reid, Haïm Reingold, L. D. Rodabaugh, G. F. Rose, Arthur Rosenthal, C. H. Rust, H. J. Ryser, J. M. Sachs, R. G. Sanger, L. J. Savage, A. C. Schaeffer, W. T. Scott, I. E. Segal, Seymour Sherman, M. F. Smiley, A. H. Smith, R. H. Stark, M. P. Steele, H. E. Stelson, B. M. Stewart, M. H. Stone, D. G. Velesz, C. P. Wells, L. R. Wilcox, C. O. Williamson. J. L. Zemmer, Antoni Zygmund.

At 9:00 A.M., with Professor A. A. Albert presiding, Professor André Weil delivered an address entitled, *Abelian varieties*. A session for short research papers was held at 10:15 A.M. with Professor Gordon Pall presiding. The meeting was closed at 12:30 P.M.

In the interval between the two sessions the Associate Secretary read a letter from Professor Inzinger, President of the Austrian Mathematical Society, asking for names of mathematicians who would be interested in receiving free a copy of the Nachrichten der Mathematischen Gesellschaft in Wein.

Local arrangements for this meeting were in charge of Dr. J. B. Crabtree.

There follow the complete abstracts of the papers presented at this meeting. The papers presented by title are designated by the letter "*t*." Paper no. 216 was presented by Professor Hatfield.

ALGEBRA AND THEORY OF NUMBERS

214*t*. Marshall Hall: *Subgroups of finite index in free groups.*

A standard representation for subgroups U of free groups is given in a paper (M. Hall and T. Radó, *On Schreier systems in free groups*, Bull. Amer. Math. Soc. Abstract 53-11-357) to appear in the Transactions of the American Mathematical Society. Using this representation, some new relationships are established for subgroups of finite index. In particular, the total length of a system of generators for U is given in terms of the total length of coset representatives. Also a formula is found giving the number of distinct subgroups of index n in the free group with r generators. (Received January 22, 1948.)

215*t*. Sam Perlis and Gordon Walker: *A generalization and new proof of a theorem on scalar extensions of a field.*

The following theorem is established. Let $K(\phi)$ be a separable finite extension of the field K, and F be any scalar extension of K. Let the minimum function $f(x)$ of ϕ over K have the irreducible factorization $f(x)=\prod_{i=1}^{r} f_i(x)$ in the polynomial domain $F[x]$; then regarding $K(\phi)$ as an algebra over K, its scalar extension $K(\phi)_F$ has the structure $K(\phi)_F=F(\phi_1)\oplus\cdots\oplus F(\phi_r)$ where $f_i(\phi_i)=0$ and $F(\phi_i)$ is the stem field of $f_i(x)$. When F is (or contains) the scalar root field over K of $K(\phi)$, each $f_i(x)$ has degree one and a well known theorem results (A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, Theorem 26). The proof, which consists of showing that the minimum function of $\phi_1+\cdots+\phi_r$ is $f(x)$, is simpler than known proofs of the corollary. (Received January 20, 1948.)

ANALYSIS

216. R. H. Cameron and Charles Hatfield: *On the summability of certain orthogonal developments of nonlinear functionals.*

The authors show that the Fourier-Hermite expansion (Ann. of Math. vol. 48 (1947) pp. 385–392) of a bounded Wiener measurable (nonlinear or linear) functional is Abel summable (in an appropriate infinite-dimensional sense) to the value of the functional at each point where the functional is continuous in the Hilbert topology. The Fourier-Hermite set of orthogonal functionals is a set of finite products of Hermite polynomials of Stieltjes integrals $\int_0^1\alpha_j(t)dx(t)$ in which x is the variable function and the $\alpha_j(t)$ form a C. O. N. on $0\leq t\leq 1$. In the present case, the authors have taken $\alpha_j(t)$ to be $2^{1/2}\cos(j-1/2)\pi t$. (Received January 19, 1948.)

217. Evelyn Frank: *Orthogonality properties of C-fractions.* II.

This paper constitutes an extension to the general case of special orthogonality relations for α-regular and regular C-fractions previously considered by the author (Bull. Amer. Math. Soc. Abstract 54-3-132). Consider the polynomials $B_p^*(z)=z^{n_p}B_p(1/z)$, $p=0, 1, \cdots$. Here $n_p=s_p$ if $s_p\geq t_p$, or $n_p=t_p$ if $s_p<t_p$; s_p is the degree of the numerator and t_p the degree of the denominator $B_p(z)$ of the pth approximant of *any* C-fraction $1+a_1z^{\alpha_1}/1+a_2z^{\alpha_2}/1+\cdots$ (cf. Frank, Amer. J. Math. vol. 68 (1946)). Let the sequence $\{c_p\}$ be the coefficients of the power series corresponding to the C-fraction, and let S be defined as the operator which replaces every z^p by c_{p+1} in any polynomial upon which it operates. Then the orthogonality relations $S(B_p^*(z)B_q^*(z))=0$ if $n_q=0, 1, \cdots, \sum_{i=1}^{p+1}\alpha_i-n_p-2$, $S(B_p^*(z)B_q^*(z))=(-1)^p a_1a_2\cdots a_{p+1}$ if $n_q=\sum_{i=1}^{p+1}\alpha_i-n_p-1$ hold relative to S. A necessary condition for the existence of the polynomials $B_p^*(z)$ is the vanishing of certain determinants Δ with elements c_p (op. cit. p. 93), and the nonvanishing of certain others. Conversely, from

the c_p and these determinant conditions, there may be determined uniquely the polynomials $B_p^*(z)$ such that the above orthogonality relations hold. (Received January 21, 1948.)

218. Morris Marden: *On the zeros of certain real rational and meromorphic functions.*

The Jensen-Walsh theorems on the zeros of the derivative of a real polynomial are generalized to real rational functions of the form $F(z)=\sum_{j=1}^{n}\rho_j e^{i\omega_j}/(z-c_j)$ with $|\omega_j|<\pi/2$ in this paper. By K_j is denoted the circle passing through the points c_j and $\bar{c}_j$ and having its center at a point k_j such that angle $\bar{c}_j, c_j, k_j$ measured counterclockwise is ω_j. It is then proved that all the real zeros of $F(z)$ lie in the interval containing both all the real c_j and all the k_j, that between any two real c_j lie an odd number of zeros of $F(z)$ and that on any interval of the real axis containing no c_j there lies at most one real zero of $F(z)$. It is also proved that no non-real zero of $F(z)$ may lie outside of all the circles K_j and that, if $n_1\leq n$ points c_j lie in a region R consisting of the closed interiors of some K_j but not overlapping the closed interiors of the remaining K_j, then at least n_1-1 and at most n_1+1 zeros of $F(z)$ lie in R. By iteration these theorems applied to the sequence of real rational functions $f_{p+1}(z)/f_p(z)=F_p(z)=\sum_{j=1}^{n-p}\rho_{jp}e^{i\omega_{jp}}/(z-c_{jp})$, where $|\omega_{jp}|<\sigma_p<\pi/2$ and $p=0, 1, \cdots, n-1$, show that no real zero of $F_p(z)$ lies outside of each circle with center $(c_{j0}+\bar{c}_{j0})/2$ and diameter $|c_{j0}-\bar{c}_{j0}|(\cos\sigma_0)\prod_{j=0}^{p}\csc\sigma_j$. The theorems are extended to real meromorphic functions of the form $F(z)=A-B^2z+(r_0/z)+\sum_1^\infty\{[r_j/(z-c_j)]+m(r_j/c_j)\}$ with $m=0$ or 1, and A and B real. (Received January 23, 1948.)

219. O. W. Rechard: *A note on the extension of rectangle functions.*

Let $\phi(R)$ be a real, finite, single-valued, non-negative function defined on the class K of oriented closed rectangles R contained in a fixed oriented closed rectangle R_0. A horizontal or vertical line segment s with end points on the sides of R_0 is said to be regular in case for every rectangle R in K which is divided by s into two rectangles R_1 and R_2 in K it is true that $\phi(R)=\phi(R_1)+\phi(R_2)$. From an earlier result obtained by Rechard and Reichelderfer (Bull. Amer. Math. Soc. Abstract 53-5-224 accepted for publication in the Duke Math. J. vol. 15 (1948)) it follows that ϕ admits a completely additive extension to an additive class of subsets of R_0, including all Borel sets in R_0, if and only if it satisfies the following conditions: (i) the class of regular lines in R_0 is everywhere dense both vertically and horizontally; (ii) if a descending sequence $R_1, R_2, \cdots$, of rectangles in K has as a product the rectangle R in K, then $\lim\phi(R_n)=\phi(R)$. A similar result is established for functions of oriented open rectangles. In this case, the sequence of rectangles in condition (ii) above is ascending with its sum equal to the rectangle R. (Received January 23, 1948.)

APPLIED MATHEMATICS

220. M. Z. Krzywoblocki: *On two-dimensional steady laminar jets in compressible viscous fluids.*

Bickley and Schlichting solved the equations of the velocity distribution in a steady two-dimensional laminar jet in incompressible fluid. In the present paper the problem of jets in compressible viscous fluids is solved. The following equations were taken into account: two equations of motion, continuity, state and energy. The two velocity components, the density and the temperature are given in the forms of the sequences of functions. (Received January 14, 1948.)

221*t*. M. J. Norris: *Coordinate transformations from the operational viewpoint.*

It is shown that space-time coordinate transformation equations similar to the Lorentz equations can be obtained following the operational viewpoint of Bridgman. The problem treated is that in which the motions are restricted to one dimension. A local time for each observer is defined operationally, and then the time and distance of an observed event for a particular observer are defined operationally in terms of the observer's local time. The transformation equations are then derived. The methods are entirely elementary and are adequate to handle the case of non-uniform motion. (Received January 26, 1948.)

GEOMETRY

222. P. M. Pepper: *Local symmetry in metric spaces.*

Let m be a point of a metric space S, δ a real number not less than 0, and $S_\delta(m)$ the set of all points x in S for which $xm<\delta$. If Δ is the least upper bound of numbers $\delta\geqq 0$ with the property that if x is in $S_\delta(m)$ then there exists a point y_x in S such that $xy_x=2xm=2my_x$, then S is said to have local symmetry about the center m with the modulus Δ. The modulus may be infinite in which case the word local is dropped. If S is locally symmetric about two of its points m_0, m_1 ($m_0\neq m_1$) with respective moduli Δ_0, Δ_1, and if $m_0m_1<\max(\Delta_0, \Delta_1)$, then the diameter of S is at least $\max[2\min(\Delta_0, \Delta_1)-2m_0m_1, 2m_0m_1]$. In case both moduli are infinite, the diameter of S is infinite. A corollary of this theorem is: If S is a complete, convex, metric space (of at least two points) which is locally symmetric about each of its points with moduli bounded away from 0 by a number $\Delta>0$, then the diameter of S is not less than 2Δ. The metric space S is said to have local open-symmetry about its point m with modulus Δ if and only if Δ is the least upper bound of numbers δ such that for each $\epsilon>0$ there exists a function f_ϵ which maps the points x of $S_\delta(m)$ into S in such a way that $xf_\epsilon(x)\geqq 2xm-\epsilon$ and $|f_\epsilon(x)m-xm|\leqq\epsilon$. The first theorem stated above for local symmetry holds also for local open-symmetry. (Received January 21, 1948.)

STATISTICS AND PROBABILITY

223*t*. Francis Regan and W. C. Kalinowski: *A postulational treatment of the probability for certain types of emissions.*

The transmutation of radioactive substances is accompanied with the emission of certain types of rays. Since these emissions occur spontaneously and at random, the problem of radioactive transmutation can be treated statistically, that is, according to the laws of probability. This physical phenomenon is given a mathematical interpretation, in that a system of postulates is set up whereby the probability of the occurrence of n points of a set E (half open interval closed on the left) beginning at the instant t may be obtained. Probabilities are also found for a finite number of nonabutting intervals. These results are then used to include a set of points E whose frontier points are of measure zero. (Received January 21, 1948.)

TOPOLOGY

224. R. H. Bing: *A convex metric for a locally connected continuum.*

If $D(x, y)$ is a distance function, it is said to be a convex metric if for each pair of points x, y, there is a point z such that $D(x, z)=D(z, y)=(1/2)D(x, y)$. If M_i ($i=1, 2$)

is a compact continuum with a convex metric D_2 and M_1 intersects M_2, then there is a convex metric D_3 on M_1+M_2 that preserves D_1 on M_1. A necessary and sufficient condition that a compact n-dimensional metric space have a convex metric is that it be locally connected. (Received January 24, 1948.)

225. S. B. Myers: *Spaces of continuous functions.*

Let $B(X)$ be the Banach space of all real bounded continuous functions on a completely regular space X. A subspace M of $B(X)$ is called completely regular over X if given any closed set $K\subset X$ and any $x_0\in X-K$, there is a $b\in M$ such that $b(x_0)=\|b\|$ and, for $x\in K$, $\sup |b(x)| < \|b\|$. The author has previously shown (Ann. of Math. vol. 49 (1948)) that if B is a Banach space there is at most one compact X such that B is equivalent to a linear subspace of $B(X)$ completely regular over X. A further study is made here of the space $B(X)$ and its subspaces completely regular over X. For example, although $B(X)$ cannot be reflexive unless X is finite, and $B(X)$ cannot be separable unless X is compact metric, nevertheless (1) if X is separable metric, there is a closed separable linear subspace of $B(X)$ which is completely regular over X, (2) if X is homeomorphically imbeddable in euclidean n-space, there is an $n+2$-dimensional linear subspace of $B(X)$ completely regular over X. (Received January 24, 1948.)

R. H. Bruck,
Associate Secretary

BOOK REVIEWS

Eigenfunction expansions associated with second order differential equations. By E. C. Titchmarsh. Oxford, Clarendon Press, 1946. 8+174 pp. $7.00.

The subject of this book has its origin in the Sturm-Liouville expansions. The author deals with the problem of expanding an arbitrary function in terms of the e.f's (eigenfunctions) of a second order ordinary differential equation, with emphasis on the singular theory. He puts aside H. Weyl's method of handling of the singular theory on the basis of integral equations and also bypasses the use of the general theory of linear operators in Hilbert spaces (M. H. Stone); instead, use is made of contour integrations and the Cauchy calculus of residues. The material in Chapters 1, 2, 4, 5, 7 is in some essential parts due to the author; Chapters 9, 10 are entirely due to him; the highly important Chapter 3 involves some very heavy technical equipment and is due essentially to the author and to H. Weyl; the material in Chapter 6 is due to M. H. Stone. The book is written with complete rigor in a very readable style. Specifically is studied the operator $L=q(x)-d^2/dx^2$, where $q(x)$ is a given function, defined on some finite or infinite interval (a, b). The values of λ and the corresponding solutions (subject to suitable boundary conditions) of

$$Ly = \lambda y \tag{1}$$

are termed the e.v's (eigenvalues) and e.f's, respectively.

In the regular case (chap. 1) the following is proved. If real q is continuous on a finite interval (a, b), then (1) has a solution ϕ so that $\phi(a)=\sin\alpha$, $\phi'(a)=-\cos\alpha$ (α assigned); if, in addition, f is integrable over (a, b), then the Sturm-Liouville expansion $s(x)$ behaves with regard to convergence as an ordinary Fourier series, while $2^{-1}(f(x+0)+f(x-0))=s(x)$ when f is of bounded variation near x. In chap. 2 the author treats singular cases when the expansion is still a series and the interval is $(0, \infty)$. It is assumed that q is continuous on every finite subinterval of $(0, \infty)$. The general solution of (1) is of the form $\theta+l\phi$, where θ, ϕ are solutions of (1) such that $\theta(0)=-\phi'(0)=\cos\alpha$, $\theta'(0)=\phi(0)=\sin\alpha$. If solutions are considered for which $\{\theta(b)+l\phi(b)\}\cos\beta+\{\theta'(b)+l\phi'(b)\}\sin\beta=0$, one obtains $l=l(\lambda)=-[\theta(b)\operatorname{ctg}\beta+\theta'(b)][\phi(b)\operatorname{ctg}\beta+\phi'(b)]^{-1}$. As β varies, l describes a circle C_b; $C_b\to$ a "limit-circle" or a "limit-point" as $b\to\infty$. For every non-real λ, (1) has a solution $\psi(x, \lambda)=\theta(x, \lambda)+m(\lambda)\phi(x, \lambda)\subset L^2(0, \infty)$, where $m(\lambda)$ is the limit-point or is any point on the limit-circle. It is assumed that the only singularities of $m(\lambda)$ are poles $\lambda_0, \lambda_1, \cdots$, with corresponding residues $r_0, r_1, \cdots$. If f and Lf are $L^2(0, \infty)$, $f(0) \cos \alpha + f'(0) \sin \alpha = 0$ and the $\lim_{x\to\infty} W(\psi(x, \lambda), f(x)) = 0$ (W is the wronskian) for all non-real λ, then

$$f(x) = \sum c_n \psi_n(x)). \tag{2}$$

Moreover $\int_0^\infty f^2 dx = \sum c_n^2$. Form $\Phi(x, \lambda) = \psi(x, \lambda)\int_0^x \phi(y, \lambda)f(y)dy + \phi(x, \lambda)\int_x^\infty \psi(y, \lambda)f(y)dy$; if f is $L^2(0, \infty)$ and λ is distinct from the λ_n, then $\Phi(x, \lambda) = \sum c_n(\lambda - \lambda_n)^{-1}\psi_n(x)$. Indications are given for the case $(-\infty, \infty)$.

In the general singular case (chap. 3) $m(\lambda)$ is not restricted. Form $k(\lambda) = \lim_{\delta\to 0} \int_0^\lambda [-\operatorname{Im}(u+i\delta)]du$ $(\delta > 0)$. If f is $L^2(0, \infty)$, then the $g_n(\lambda) = \int_0^n \phi(y, \lambda)f(y)dy$ converge in the mean square with respect to $k(\lambda)$ over $(-\infty, \infty)$ to a limit $g(\lambda)$, while $\pi\int_0^\infty f^2 dx = \int_{-\infty}^\infty g^2(\lambda)dk(\lambda)$. The author gives some indications for the case $(-\infty, \infty)$. The concept of s. (spectrum) enters as follows. When $(a, b) = (0, \infty)$ the s. is the λ-set which is the complement of the set of points in the neighborhood of which $k(\lambda)$ is constant. It is only the s. that contributes in the representation formula $\pi f(x) = \int_{-\infty}^\infty \phi(x, u)d\int_0^\infty Q(y, u)f(y)dy$, where $Q(y, u) = \int_0^u \phi(y, \lambda)dk(\lambda)$ (valid when f is subject to the conditions imposed in (2)). If $m(x)$ is meromorphic, there is a point-s. (consisting of the set of poles). The continuous s. is the set of intervals in each of which $k(\lambda)$ is steadily increasing.

Chap. 4 contains numerous applications to Hermite, Legendre, Bessel, and so on, expansions. In chap. 5 the following is proved. If q is $L(0, \infty)$, then the s. is continuous on $(0, \infty)$ and there is a point-s. on $(-\infty, 0)$; when $q \to \infty$ (steadily), $q' \geqq 0$, $q' = O(|q|^c)$ $(0 < c < 3/2)$ and q'' is eventually of constant sign, then the s. is discrete. Let $q \leqq 0$, $q' < 0$, $q \to -\infty$, $q' = O\{|q|^c\}$ $(0 < c < 3/2)$ and q'' be of constant sign; then, if $\sigma = \int^\infty |q|^{-1/2}dx = \infty$, there is a continuous s. over $(-\infty, \infty)$ and, if $\sigma < \infty$, a point s. on $(0, \infty)$ and a continuous s. on $(-\infty, 0)$. When $q \to \infty$ as $x \to \infty$, there are discrete e.v's, while the e.f associated with λ_n has n zeros.

In chap. 6 it is shown that the expansion of f, given in chap 3, will hold when the conditions on f are relaxed, while additional ones are imposed on q. The expansion of f is given when q is $L(0, \infty)$ and f is $L^2(0, \infty)$ and also $L(0, \infty)$. In chap. 7 a study is presented of the discrete e.v's $\lambda_0, \lambda_1, \cdots$ when $q \to \infty$. The following is one of the results. Let $N(\lambda) = n$ $(\lambda_{n-1} < \lambda \leqq \lambda_n)$; if $q' \to \infty$, $q'' \geqq 0$, $q'' \leqq (q')^\gamma (1 < \gamma < 4/3;\ x > x_0)$, then $\pi N(\lambda) = \int_0^p (\lambda - q(x))^{1/2}dx + O(1)$ (here $p = p(\lambda)$ arises from the equation $q(p) = \lambda$). In chap. 8 a further approximation to $N(\lambda)$ is given on the basis of a rigorous method, suggested by a heuristic argument used by physicists in connection with the quantum mechanics equation $\psi''+8\pi^2mh^{-2}(E-V(x)\psi=0$ (Brillouin, Wentzel, Kramer). In chap. 9 it is proved that if $q'>0$, $q''\geqq 0$, $q''\leqq (q')^\gamma(1<\gamma<4/3;\ x\geqq x_0)$ and f is $L^2(0,\ \infty)$, then $f(x+0)+f(x-0)=2\sum c_n\psi_n(x)$, provided f is of bounded variation near x. In chap. 10 the author proves the following summability theorem. If continuous $q\rightarrow\infty$ monotonically and f is $L^2(0,\ \infty)$, then $f(x)=\lim_{v\rightarrow\infty}\sum_n(v+\lambda_n)^{-1}vc_n\psi_n(x)$ for every x for which $\int_0^\eta|f(x+y)-f(x)|dy=o(\eta)$, as $\eta\rightarrow 0$ (that is, almost everywhere). With the aid of the above result the expansion theorem of chap. 9 is then proved anew, which presents an analogy with the situation in the ordinary Fourier theory.

Some developments analogous to those in the book under review have been carried out in the field of partial differential equations by T. Carleman [Arkiv för Mathematik, Astronomi och Fysik vol. 24 B (1934) pp. 1–7] and by the present reviewer [Ann. of Math. vol. 43 (1942) pp. 1–55; also, Rec. Math. (Mat. Sbornik) N.S. 20 (1947) pp. 365–430]. The field of partial differential equations being so much more difficult than that of ordinary differential equations, much more remains to be done. The book of Titchmarsh may serve as a useful guide in the line of investigation just mentioned.

W. J. TRJITZINSKY

The theory of functions of real variables. By L. M. Graves. New York and London, McGraw-Hill, 1946. 10+300 pp. $4.00.

The theory of functions of real variables occupies a central position in present day analysis, and it is typical for graduate schools in the United States to offer students in mathematics a one year introduction to the subject. It has always been a problem to find a suitable text for such a course, needing, as it does, something shorter and crisper than one of the standard treatises. The volume under review is written with this end in view.

The first chapter is a short exposition of the ideas and methods of deductive logic. The notions of negation, conjunction and alternation of propositions are given symbolic notations, and regarded as undefined (but "generally understood") operations: a list of laws by which their use is to be governed (for example, double negation, excluded middle) is discussed. The calculus of classes is briefly described, and the student is introduced to a technique of translating English sentences involving logical and class relationships into briefer symbolic formulae. We are warned about possible paradoxes which may arise through unguarded use of these notions, and the writer describes "common sense" safeguards which he feels are generally adequate to avoid trouble.

Chap. II gives the traditional construction of the reals, starting with Peano's axioms and proceeding via equivalence classes and Dedekind cuts to the real numbers, which are shown to form a complete ordered field. In chap. III point sets in n dimensions are treated, and a chain of theorems giving their fundamental topological properties culminates in the Heine-Borel theorem. The concepts of continuity and limit for functions of one or several real variables are defined in chap. IV; superior and inferior limits, semi-continuity and uniform continuity are discussed, and their relations and properties exhibited by a series of standard theorems, including max and min theorems for semi-continuous functions. The definition of the limit of $f(x)$ at $x=a$ is such that if it exists it must equal $f(a)$ (when the latter is defined), contrary to most texts. A point with which the reviewer believes some readers might take issue is the author's addition of ideal points to Euclidean spaces of all dimensions, leading, for example, to the statement that if a function is continuous on a closed set S, it is uniformly continuous on S.

Chap. V furnishes a detailed exposition of the fundamental definitions and theorems on differentiation of functions of one variable, including mean value theorems and a discussion of derivates. Total differentiability for functions of several variables, and theorems on partials of various orders, receive due attention. The Riemann integral for functions of one variable is treated rapidly and succinctly in chap. VI, and chap. VII is devoted to various aspects of uniform convergence, including a brief introduction to multiple series, the function space C, and non-differentiable functions.

The next two chapters, while well within the spirit of the book's title, have an element of novelty. Chap. VIII deals with functions defined implicitly by a relation $f(x, y)=0$. In connection with investigating the solvability for one variable in terms of the other, some fixed point theorems are proved, involving a short excursion into the theory of simplicial partitions. Chap. IX is on existence theorems for ordinary differential equations.

The last three chapters are on integration theory. Chaps. X and XI deal with Lebesgue integration, using the Riesz device of approach by step functions for the definition. The detailed discussion is carried out for functions of one variable, but a good deal of side comment is furnished concerning generalizations. We are led to an introductory exposition of the spaces L_p, and thence to a discussion of orthonormal sets for the case $p=2$. (It is rather surprising not to see Hilbert's name mentioned in this connection.) The last chapter is on several types of Stieltjes' integration.

Very few misprints or misstatements have come to the reviewer's attention. Especial mention should be made of the valuable and well chosen lists of references to the literature which conclude each chapter.

It seems to the reviewer that in scope and choice of subject matter this text is nicely calculated to suit the needs of introductory classes in real variable theory. On the basis of having used the text for such a class for one term, he would suggest only one respect in which it proved to be somewhat troublesome: namely, in the free use from the start of the logical symbols introduced in chap. I. It is suggested that students embarking on this subject have a good many new ideas, and an essentially novel ideal of precision, to struggle with, both of which are inherent in the subject itself. It seems open to question whether we really help them by replacing a possibly tedious, but clear, English sentence by such a formula as

$$m \neq p : \supset : \exists q \ni \cdot m + q = p \cdot V \cdot \exists n \ni \cdot m = p + n$$

for them to cope with in addition, almost at the outset of their voyage.

JAMES A. CLARKSON

Topological methods in the theory of functions of a complex variable. By Marston Morse. (Annals of Mathematics Studies, no. 15.) Princeton University Press, 1947. 2+145 pp. $2.50.

The theory of analytic functions is related to topology in two ways. On one hand it can serve as a powerful tool in the study of topological questions. On the other hand many of the basic theorems in the theory of functions are essentially topological in character and can be proved by such methods. It is the latter observation that forms the starting point for Morse's booklet. If the purely topological properties of analytic functions are to be isolated, it is natural to study the class of functions which, in the small, share the topological properties of analytic functions. One of the main problems will then be to find out to what extent this new class retains the topological properties in the large.

The author actually considers two classes of topologically defined functions, *interior transformations* and *pseudo-harmonic functions.* They arise, respectively, from analytic and harmonic functions by sense-preserving local homeomorphisms. Interior transformations have been studied previously by Stoilow and Whyburn, from different points of view, while the concept of pseudo-harmonic functions is new except for pioneer work by Morse himself and in collaboration with M. H. Heins. It is clear from the definition that the local behavior of these functions at an interior point is perfectly known, but on the boundary of the region of definition a careful and detailed analysis is required, even in the case of relatively strong conditions of regularity.

The main theorems are enumerative in character, dealing with the number of singularities and similar characteristics. As a typical case, take the formula

$$M + m - S - s = 2 - v,$$

where M is the number of logarithmic poles of a pseudo-harmonic function U in a region G, m the number of relative minima of U on the boundary B, S and s the number of saddle points (critical points) on G and B, v the number of contours. For harmonic functions, which remain such on the boundary, formulae of this type are of course familiar as more or less immediate consequences of Cauchy's integral theorem, except for the fact that the number m of relative minima is usually not displayed. One recognizes, however, that m serves to measure the turning of the tangent to the image of B under the analytic transformation $U+iV$ whose real part is U. In proving this theorem under more general assumptions a delicate point is the counting of singularities on the boundary; instead of counting half-multiplicities, as one would ordinarily do in the theory of analytic functions, Morse shows good cause for adopting a different procedure. In fact, the main point in the proof, which of course proceeds along purely topological lines, consists in a careful scrutiny of the saddle-points on the boundary.

The first four chapters deal with this and similar questions. Various types of boundary conditions are discussed in great detail, and certain enumerative theorems which have received special attention in the theory of functions are singled out. Throughout, important details are investigated with all the skill expected from the author. If the reader has any doubts, they will not be concerned with the validity of the arguments, but rather with the degree of importance that can be attached to these questions. Since the appreciation of Morse's work must necessarily hinge on such considerations, it is a pity that the limited size of this tract has prevented its author from presenting his material against a broader background. Personally, the reviewer believes that a good case could be made for Morse's point of view.

In the fifth and last chapter deformations of locally simple curves are studied. The main theorem that two locally simple curves can be deformed into each other (by an admissible transformation) if and only if they have the same angular order is proved in its full generality. An admissible transformation is defined as one under which the intermediate curves are uniformly locally simple. The same question is treated for the case of O-deformations under which the curves are not allowed to pass through a point O. Here the characteristics are the angular order and the order with respect to O. These are of course highly important questions whose systematic treatment in the non-regular case has been long overdue.

There is finally a brief introduction to the deformation for interior transformations which has been more extensively treated in the author's joint paper with M. H. Heins (Acta Math. vol. 80 (1947)). This interesting theory aims at a complete characterization of the homotopy properties of meromorphic functions as opposed to those of more general interior transformations. As an introduction to this theory the book serves a very laudable purpose.

L. AHLFORS

Vector and tensor analysis. By Louis Brand. New York, Wiley, 1947. 9+439 pp. $5.50.

This volume bears a relation to the average elementary book on vector analysis similar to the relation of the author's well known *Vectorial mechanics* to the average elementary book on mechanics. In each case all of the expected topics are included and treated thoroughly, carefully, and with rigor and sophistication. But a number of related subjects are brought into the picture.

Thus in the volume under review one finds chapters on motor algebra, quaternions, dyadics, the application of vectors to perfect fluids and to elementary differential geometry. A long chapter on tensor analysis carries the subject beyond covariant differentiation and includes surface geometry in tensor notation. The author expects to include further applications of tensor analysis such as those made by Einstein and Kron in a later volume.

Many exercises are worked out in the text. These often include interesting results not found in conventional texts. These, together with some of the exercises to be worked out, should do a great deal for the reader interested in increasing the breadth of his mathematical education.

The book can be used as the basis of an elementary course in vector

NOTES

The Society's Third Symposium on Applied Mathematics will be held in the summer of 1949. Invitations from institutions wishing to act as hosts should be sent to the Chairman of the Committee on Applied Mathematics, Professor J. L. Synge, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania. All invitations received will be considered simultaneously, the closing date for receipt of invitations being June 30, 1948. It is requested that each institution indicate a proposed subject for the Symposium, and state whether adequate facilities for meetings and dormitory accommodation would be available.

The Institute of Mathematical Statistics has announced the election of the following officers: Professor Abraham Wald of Columbia University as president; Dr. Churchill Eisenhart, Chief of the Statistical Engineering Laboratory of the National Bureau of Standards, and Associate Professor Henry Scheffé of the University of California at Los Angeles as vice presidents.

A new journal, *Communications on Applied Mathematics*, has been announced by the Institute for Mathematics and Mechanics at New York University. It will be issued quarterly and the first number appeared in January 1948.

Professor Y. C. Wong of National Sun Yat-Sen University has been awarded an honorary doctorate of science by the University of London.

Dr. Mina Rees of the office of Naval Research and Dr. Warren Weaver of the Rockefeller Foundation have received the British award of the King's Medal for Service in the Cause of Freedom in recognition of valuable services rendered to the allied war effort in various fields of scientific research and development.

Professor Selig Brodetsky of the University of Leeds has retired.

Professor T. G. Cowling of the University College of North Wales has been appointed to a professorship at the University of Leeds.

Dr. C. Y. Pauc of the University of Marseille has been appointed a lecturer at the University of Cape Town.

Dr. H. W. E. Schwerdtfeger of the University of Adelaide has been appointed to the position of senior lecturer at the University of Melbourne.

Dr. Valentine Bargmann of Princeton University has been appointed to an associate professorship at the University of Pittsburgh.

Assistant Professor Richard Bellman of Princeton University has been appointed to an associate professorship at Stanford University.

Professor Richard Brauer of the University of Toronto has been appointed to a professorship at the University of Michigan.

Mr. A. G. T. Carlton has been appointed to an assistant professorship at the University of Illinois.

Assistant Professor F. O. Duncan of the University of Missouri has retired.

Dr. H. H. Germond has accepted a position as research scientist at the Douglas Aircraft Company, Santa Monica, California.

Professor L. K. Hua of Tsing Hua University has been appointed a lecturer at Princeton University.

Associate Professor Witold Hurewicz of the Massachusetts Institute of Technology is at Princeton University as a visiting professor.

Mr. Sidney Kravitz of Newark College of Engineering has accepted a position as mathematician with the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Dr. G. R. MacLane of Harvard University has been appointed to an assistant professorship at Rice Institute.

Mr. Richard Otter of Princeton University has been appointed to an assistant professorship at the University of Notre Dame.

Professor Anna Pell-Wheeler of Bryn Mawr College will retire in June 1948.

Professor Joseph Pierce of Atlanta University has been appointed to an associate professorship at Wayne University.

Professor J. B. Reynolds of Lehigh University will retire at the end of this academic year.

Mr. E. D. Schell has accepted an appointment as chief of the Mathematics and Electronic Computor Branch, Office of the Comptroller, United States Air Forces.

Associate Professor Max Shiffman of New York University has been appointed to a professorship at Stanford University.

Professor J. L. Synge of Carnegie Institute of Technology has been appointed to a professorship in the School of Theoretical Physics of the Dublin Institute for Advanced Studies.

Professor A. H. Taub of the University of Washington has been appointed to a research professorship of applied mathematics at the University of Illinois.

Assistant Professor P. M. Whitman of Tufts College has accepted a position at the Applied Physics Laboratory, Silver Spring, Maryland.

Professor C. R. Wylie of the Army Air Forces Institute of Technology, Wright Field, has been appointed to a professorship at the University of Utah.

The following promotions are announced:

R. E. Basye, Agricultural and Mechanical College of Texas, to an assistant professorship.

W. M. Borgman, Wayne University, to the position of assistant dean of administration.

Alonzo Church, Princeton University, to a professorship.

T. F. Cope, Queens College, to a professorship.

E. R. Heineman, Texas Technological College, to a professorship.

P. W. Ketchum, University of Illinois, to a professorship.

Harry Levy, University of Illinois, to an associate professorship.

E. R. Lorch, Columbia University, to a professorship.

C. W. Mendel, University of Illinois, to an associate professorship.

H. J. Miles, University of Illinois, to a professorship.

Vivian R. Nuess, University of Illinois, to an assistant professorship.

Bob Parker, Texas Technological College, to an assistant professorship.

J. W. Peters, University of Illinois, to an associate professorship.

H. V. Price, State University of Iowa, to an associate professorship.

L. W. Swanson, Coe College, to an associate professorship.

H. P. Wirth, College of the City of New York, to an associate professorship.

Leo Zippin, Queens College, to an associate professorship.

The following appointments to instructorships are announced: University of Illinois: Mr. Furio Alberti, Dr. M. D. Springer; University of Michigan: Mr. S. W. Hahn; New York State College for Teachers: Mr. Rudolph Cherkauer; Pennsylvania State College: Mr. M. M. Resnikoff; Princeton University: Dr. H. N. Shapiro; Texas Technological College: Mrs. Susie Kammerdiener, Miss Lillian McGlothlin, Mr. E. H. Thomas, Mrs. Naomi Thompson; Wayne University: Mr. J. M. Patterson.

Professor Emeritus M. W. Haskell of the University of California died on January 15, 1948 at the age of eighty-four years. He was a member of the Society from 1899 to 1929.

Mr. A. L. McCarty of Merced, California, died on June 20, 1947. He had been a member of the Society since 1923.

Professor Emeritus A. B. Turner of the College of the City of New York died on February 5, 1948 at the age of seventy-five years.

Professor G. E. Wahlin of the University of Missouri died on February 11, 1948 at the age of sixty-eight years. He had been a member of the Society for thirty-nine years.

The following one hundred two doctorates, with mathematics, mathematical physics, or statistics as a major subject, were conferred during 1947 in universities in the United States and Canada. The university, month in which degree was conferred, minor subject (other than mathematics) and the title of the dissertation are given in each case if available.

D. M. Adelman, California, June, *Some arithmetic properties of sequences of integers satisfying linear recursion relations.*

E. J. Akutowicz, Harvard, September, *The third iterate of the Laplace transform.*

H. L. Alder, California, June, *The existence and nonexistence of certain identities in the theory of partitions.*

H. P. Atkins, Rochester, June, *On fractional derivatives of univalent functions and bounded functions.*

Jacob Bearman, Minnesota, August, *Rotations in the product of two Wiener spaces.*

A. L. Blakers, Princeton, February, *Some relations between homology and homotopy sequences.*

J. V. Breakwell, Harvard, June, *An approximate method of calculating the energy levels of configurations* $l^n l'$ *in complex spectra.*

F. L. Brown, Notre Dame, June, *Remarks concerning tri-operational algebra.*

R. C. Buck, Harvard, September, *Uniqueness, interpolation, and characterization theorems for functions of exponential type.*

W. C. Carter, Harvard, September, *On the cohomology theory of fields.*

Abraham Charnes, Illinois, June, minor in physics, *Wing-body interaction in linear supersonic flow.*

Y. S. Chin, Harvard. September, *Curve systems covering a surface.*

K. L. Chung, Princeton, June, *On the maximum partial sums of sequences of independent random variables.*

E. L. Cohen, Duke, June, minor in philosophy, *Sums of an even number of squares in* $GF[p^n, x]$.

Richard Cohn, Columbia, June, *Properties of the limit of the ratio of arc to chord.*

S. L. Crump, Iowa State, June, minor in genetics, *The estimation of components of variance in multiple classifications.*

D. A. Darling, California Institute of Technology, June, minor in physics, *Continuous stochastic processes.*

H. F. DeBaggis, Notre Dame, June, *Hyperbolic geometry.*

S. P. Diliberto, Princeton, June, *Reduction theorems for systems of linear differential equations.*

W. W. Dolan, Oklahoma, June, minor in physics, *Metric differential geometry of reciprocal rectilinear congruences.*

Mary P. Dolciani, Cornell, June, *On the representation of integers by quadratic forms.*

D. B. Duncan, Iowa State, June, minor in genetics, *Significance tests for differences between ranked variates drawn from normal populations.*

H. W. Ellis, Toronto, June, *Mean continuous integrals.*

Bernard Epstein, Brown, June, *A method for the solution of the Dirichlet problem for certain types of domains.*

C. D. Firestone, Cornell, September, *Sufficient conditions for the modelling of axiomatic set theory.*

M. P. Fobes, Harvard, September, *On a conjectured inequality related to the product of Lipschitz skeleton cochains.*

Marianne R. Freundlich, Illinois, October, minor in physics, *On normed rings.*

Irving Gerst, Columbia, February, *On the integration of a certain class of differential equations.*

A. W. Goodman, Columbia, June, *On some determinants related to p-valent functions.*

W. S. Gustin, California (Los Angeles), June, I. *A bilinear integral identity for harmonic functions.* II. *Sets of finite planar order.*

Franklin Haimo, Harvard, June, *Division by the integers in Abelian groups.*

Edwin Halfar, Iowa, August, *Reduction of topological operators and their canonical representation.*

G. J. Haltiner, Wisconsin, September, *On the theory of linear differential systems when based upon a new definition of the adjoint.*

T. E. Harris, Princeton, June, *Some theorems on the Bernoullian multiplicative process.*

A. R. Harvey, Harvard, June, *Some studies on functions of exponential type.*

Sister M. Agnes Hatke, Purdue, June, minor in physics, *A certain cumulative probability function.*

I. R. Hershner, Harvard, June, *The moduli of univalence and of p-valence of functions analytic in the unit circle.*

I. I. Hirschman, Harvard, June, *Some representation and inversion problems for the Laplace transform.*

Marshall Holt, Pittsburgh, June, minor in engineering, *A study of the beam column problem.*

W. N. Huff, Pennsylvania, February, *On the type of polynomials generated by* $f(xt)\phi(t)$.

P. B. Johnson, California Institute of Technology, June, minor in physics, *Some applications of functional analysis to parallel displacement in Riemannian geometry.*

W. C. Kalinowski, St. Louis, August, minor in physics, *A postulational treatment of the probability for certain types of emissions.*

L. H. Kanter, Stanford, October, *On the zeros of orthogonal polynomials and the related Christoffel numbers.*

Samuel Karlin, Princeton, June, *Independent functions.*

J. R. F. Kent, Illinois, October, minor in physics, *Separation theorems for differential equations of the third and fourth order.*

J. H. Laning, Massachusetts Institute of Technology, September, minor in physics, *Mathematical theory of lubrication type-flow.*

E. H. Larguier, Michigan, June, *Homology bases with applications to local connectedness.*

W. G. Leavitt, Wisconsin, May, *A normal form for matrices whose elements are holomorphic functions.*

J. S. Leech, Yale, June, *On convex sets in complex Banach spaces.*

K. B. Leisenring, Michigan, June, *Hyperbolic geometry and trigonometry on the completed Lambert sphere.*

M. M. Lemme, Purdue, June, *On a new cumulative frequency function.*

W. J. LeVeque, Cornell, September, *On the distribution of values of number-theoretic functions.*

Sister M. Teresine Lewis, Catholic University, June, minor in physics and education, *The construction and application of magic rectangles modulo p for small values of p.*

P. J. McCarthy, Princeton, February, *Approximate solutions for means and variances in a certain class of box problems.*

W. R. McEwen, Minnesota, March, minor in physics, *An upper bound for the number of orthogonal polynomials of any specified degree associated with an algebraic curve in space.*

A. M. Mark, Cornell, September, *Limit theorems in the theory of probability.*

F. J. Massey, California, June, *Estimation of a distribution function by confidence limits.*

C. W. Mathews, Illinois, June, minor in physics, *Cauchy type integral representation for functions of a complex variable.*

B. E. Meserve, Duke, June, minor in physics, *Inequalities of higher degree.*

W. H. L. Meyer, Chicago, December, *Multiple integral problems involving n dependent functions in the calculus of variations.*

E. E. Moise, Texas, June, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua.*

B. N. Moyls, Harvard, September, *Extensions of valuations with prescribed value groups and residue class fields.*

Sigurd Mundhjeld, Nebraska, August, minor in physics, *On certain doubly periodic functions of the third kind.*

W. D. Munro, Minnesota, March, minor in physics, *Orthogonal trigonometric sums with auxiliary conditions.*

M. L. Nelson, California, February, *An Abel integral equation with constant limits of integration.*

Paul Olum, Harvard, June, *Homology with operators and mapping theory.*

T. G. Ostrom, Minnesota, June, minor in physics, *The solution of linear integral equations by means of Wiener integrals.*

L. J. Paige, Wisconsin, May, *Neofields.*

N. H. Painter, Massachusetts Institute of Technology, February, minor in physics, *Application of Lagrangian procedures to a wave guide attenuator.*

S. T. Parker, Cincinnati, June, *Convergence factor and regularity theorems for convergent integrals.*

Harry Polachek, Columbia, February, *Stirling's series.*

D. H. Potts, California Institute of Technology, June, minor in physics, *Mean value derivatives.*

F. M. Pulliam, Illinois, October, minor in astronomy, *Existence of a two-dimensional potential flow with wake past a symmetric convex profile.*

Irving Reiner, Cornell, June, *A generalization of Meyer's theorem.*

J. O. Reynolds, North Carolina, August, *On the irreducibility of certain polynomials.*

R. A. Rosenbaum, Yale, June, *Sub-additive functions.*

Pierre Samuel, Princeton, June, *Ultrafilters and compactification of uniform spaces.*

W. R. Scott, Ohio State, June, *On essentially absolutely continuous transformations.*

Mary A. Seybold, Illinois, October, minor in education, *Isomorphism groups of metabelian groups generated by four independent operators of order p.*

E. B. Shanks, Illinois, October, minor in education, *Homothetic correspondences between Riemannian spaces.*

H. N. Shapiro, Princeton, June, *On the iterates of arithmetic functions.*

W. H. Simons, California, September, *Modular functions of stufe* 2.

S. S. Shū, Brown, June, *On the method of successive approximations applied to compressible flow problems.*

E. H. Spanier, Michigan, June, *Cohomology theory for general spaces.*

M. D. Springer, Illinois, June, minor in economics, *Joint sampling distribution of mean and standard deviation for a chi-square universe.*

F. M. Stewart, Harvard, September, *Non-commutative integration and abstract differential equations.*

W. M. Stone, Iowa State, June, minor in physics and theoretical mechanics, *The generalized Laplace transformation with applications to problems involving finite differences.*

L. W. Swanson, Minnesota, August, minor in physics, *Solutions of certain differential equations associated with the theory of orthogonal polynomials.*

J. D. Swift, California Institute of Technology, June, minor in physics, *Periodic functions over fields of characteristic p.*

D. L. Thomsen, Massachusetts Institute of Technology, September, minor in aeronautical engineering, *On the separation of variables in the two-dimensional scalar Helmholtz equation.*

Hing Tong, Columbia, June, *The elements of the theory of transfinite numbers.*

E. A. Trabant, California Institute of Technology, June, minor in physics, *Application of the tensor calculus to problems arising in dynamics.*

L. B. Tuckerman, Princeton, June, *The embedding of products and joins of complexes in Euclidean space.*

D. F. Votaw, Princeton, June, *Testing compound symmetry in a normal multivariate distribution.*

N. A. Wiegmann, Wisconsin, May, *The theory of normal matrices with some analogs of the generalized principal axis transformation.*

Albert Wilansky, Brown, June, *An application of Banach linear functionals to the theory of summability.*

Margaret Willerding, St. Louis, June, minor in physics, *The determination of all classes of positive quaternary quadratic forms which represent all (positive) integers.*

Christine S. Williams, Yale, June, *A theory of normal chains.*

A. G. Wilson, California Institute of Technology, June, minor in physics, *Axially symmetric thermal stresses in a semi-infinite solid.*

R. L. Wilson, Wisconsin, May, *A finite method for the determination of the Galois group of an equation with an application to the problem of reducibility.*

Miriam L. Yevick, Massachusetts Institute of Technology, September, minor in physics, *Lebesgue density theorem in abstract measure spaces.*

Bertram Yood, Yale, June, *Linear topological algebras.*

The following doctorate was conferred in 1946, but was not included in the list in the preceding volume of this Bulletin (vol. 53, pp. 471–474).

Irma R. Moses, Cornell, September, *On the representation in the ring of p-adic integers of a quadratic form in n variables by one in m variables.*

NEW PUBLICATIONS

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FABER POLYNOMIALS IN THE THEORY OF UNIVALENT FUNCTIONS

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Introduction. The Faber polynomials play an important role in the theory of univalent functions. Grunsky [1][1] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalency of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [2] gave a differential equation for univalent functions solving certain extremum problems with respect to the coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (cf. Schiffer [3]; see also Schaeffer-Spencer [4]).

It seems, therefore, of interest to study these Faber polynomials more closely, in particular their dependence on the given function with respect to which they are defined, their variation with the latter and certain characteristic inequalities for them and their coefficients. This investigation is carried out in the present paper. In §1 we establish a generating function for all Faber polynomials with respect to a given function. In §2 we establish variation formulas for the Faber polynomials and their coefficients. In §3 we solve certain extremum problems with respect to the coefficients of these polynomials and find again all the inequalities which have been established by Grunsky. In §4 we use our method in order to generalize our results and to find inequalities for the Faber polynomials themselves.

1. Identities for Faber polynomials.

Consider a function $f(z)$ which has in the neighborhood of $z=\infty$ a development of the form:

$$f(z) = z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots . \tag{1}$$

$F_m(t)$ is called the mth Faber polynomial with respect to $f(z)$ if it is a polynomial of degree m in t and if we have at $z=\infty$ a development:

$$F_m[f(z)] = z^m + \sum_{n=1}^{\infty} c_{mn} z^{-n}. \tag{2}$$

The existence and uniqueness of all Faber polynomials with respect to a given $f(z)$ are easily shown by recursion.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

We shall now construct a generating function for all $F_m(t)$. For this purpose, we consider the function

$$U(z, w) = \log \frac{f(z) - f(w)}{z - w} \tag{3}$$

which is regular for z and w in the neighborhood of infinity. We choose the principal branch of the logarithm, such that if either z or w converge to infinity, U converges to zero, as follows immediately from (1). Hence, we have the following series development for the function $U(z, w)$ of two complex variables:

$$U(z, w) = \log [f(z) - f(w)] - \log (z - w) = \sum_{m,n=1}^{\infty} d_{mn} z^{-m} w^{-n}. \tag{4}$$

If we construct next

$$\log \frac{f(z) - t}{z} = - \sum_{m=1}^{\infty} \frac{1}{m} F_m(t) z^{-m}, \tag{5}$$

where the development is valid again in a neighborhood of $z = \infty$ and the $F_m(t)$ are polynomials of degree m, we find from (5) and (4):

$$\begin{aligned} - \sum_{m=1}^{\infty} \frac{1}{m} F_m[f(w)] z^{-m} &= \log \frac{f(z) - f(w)}{z - w} + \log \left(1 - \frac{w}{z}\right) \\ &= - \sum_{m=1}^{\infty} \frac{1}{m} \left(w^m - \sum_{n=1}^{\infty} m d_{mn} w^{-n} \right) z^{-m}. \end{aligned} \tag{6}$$

A comparison of equal powers of z on both sides gives finally

$$F_m[f(w)] = w^m - \sum_{n=1}^{\infty} m d_{mn} w^{-n}. \tag{7}$$

This shows that the coefficients $F_m(t)$ are in fact just the Faber polynomials defined above and that we possess in the left-hand side of (5) a generating function for them.

Comparing (2) and (7), we conclude

$$c_{mn} = - m d_{mn}. \tag{8}$$

Now we have, because of the symmetry of $U(z, w)$ in z and w,

$$d_{mn} = d_{nm} \tag{9}$$

whence

$$n c_{mn} = m c_{nm}, \tag{10}$$

an identity previously proved by Grunsky [1] and Schur [5].

Differentiating (5) with respect to t, we obtain:

$$\frac{1}{f(z)-t} = \sum_{m=1}^{\infty} \frac{1}{m} F_m'(t) z^{-m}, \tag{11}$$

a generating function for the derivatives of Faber's polynomials which will be of use later on.

Numerous further relations might be obtained, as for example the following: Differentiating (11) with respect to t we find:

$$\frac{1}{(f(z)-t)^2} = \sum_{m=1}^{\infty} \frac{1}{m} F_m''(t) z^{-m} = \sum_{m,n=1}^{\infty} \frac{1}{mn} F_m'(t) F_n'(t) z^{-(m+n)}, \tag{12}$$

whence

$$\frac{1}{k} F_k''(t) = \sum_{m+n=k} \frac{1}{mn} F_m'(t) F_n'(t). \tag{13}$$

For our further developments, however, formulas (5), (10) and (11) will be sufficient.

2. A variation formula for the Faber polynomials.

Let us suppose now that the function $f(z)$ is regular and univalent in a domain D of the z-plane which contains the point at infinity and is bounded by a finite number of proper continua. It will, therefore, map D conformally upon a domain Δ of the ζ-plane. If ζ_0 is an arbitrary point in the ζ-plane which does not belong to Δ and ρ any positive constant, there exist infinitely many functions which are univalent in Δ and have in the domain $|\zeta-\zeta_0| > 4\rho$ a development of the form:

$$v(\zeta) = \zeta + \frac{a\rho^2}{\zeta-\zeta_0} + \frac{b\rho^3}{(\zeta-\zeta_0)^2} + \frac{c\rho^4}{(\zeta-\zeta_0)^3} + \cdots, \tag{14}$$

where $|a| \leq 4^2$, $|b| \leq 4^3$, $|c| \leq 4^4$, $\cdots$ (see Schiffer [2]).

The function

$$f^*(z) = v[f(z)] = f(z) + \frac{a\rho^2}{f(z)-\zeta_0} + \cdots \tag{15}$$

is again regular and univalent in the initial domain D and has still a development of the form (1) at infinity. It maps the domain D upon a new domain Δ^* in the ζ-plane. If ρ is small enough, Δ^* will be very near to Δ and we may conceive the mapping (14) as a small variation of this domain. Variations of the type (15) which transform a univalent function $f(z)$ into a function $f^*(z)$ of the same type are of great value in solving extremum problems within the class of normalized univalent functions.

Suppose now that to a given univalent function $f(z)$ the nth Faber polynomial $F_n(t)$ has been determined. If we subject $f(z)$ to a variation (15), how does the corresponding Faber polynomial change? In order to settle this question, we make use of the generating function (5). We have in view of (15)

$$(16)\qquad \begin{aligned} -\sum_{m=1}^{\infty}\frac{1}{m}F_m^*(t)z^{-m} &= \log\frac{f^*(z)-t}{z} \\ &= \log\frac{f(z)-t}{z}+\frac{a\rho^2}{[f(z)-t][f(z)-\zeta_0]}+o(\rho^2), \end{aligned}$$

where $o(\rho^2)$ shall denote henceforth a term containing at least ρ^2 as a factor. We transform (16) by making again use of identity (5) and applying (11):

$$(17)\qquad \begin{aligned} -\sum_{m=1}^{\infty}\frac{1}{m}F_m^*(t)z^{-m} &= -\sum_{m=1}^{\infty}\frac{1}{m}F_m(t)z^{-m} \\ &\quad + a\rho^2\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{F_m'(t)-F_m'(\zeta_0)}{t-\zeta_0}z^{-m}+o(\rho^2). \end{aligned}$$

Comparison of equal powers of z on both sides gives, therefore, the following variation formula for $F_m(t)$:

$$(18)\qquad F_m^*(t)=F_m(t)-a\rho^2\frac{F_m'(t)-F_m'(\zeta_0)}{t-\zeta_0}+o(\rho^2).$$

Let us now consider the value $F_m[f(w)]$ as a functional of $f(z)$ for a fixed point $w\in D$. Its variation is given by:

$$(19)\qquad F_m^*[f^*(w)]=F_m^*[f(w)]+a\rho^2\frac{F_m'[f(w)]}{f(w)-\zeta_0}+o(\rho^2),$$

whence, in view of (18),

$$(20)\qquad F_m^*[f^*(w)]=F_m[f(w)]+a\rho^2\frac{F_m'(\zeta_0)}{f(w)-\zeta_0}+o(\rho^2).$$

Using the series development (2) for $F_m[f(w)]$, an analogous representation for $F_m^*[f^*(w)]$, and the identity (11) for the second right-hand term, we obtain by comparison of the coefficients of z^{-n} on both sides:

$$c^*_{mn} = c_{mn} + a\rho^2 \cdot \frac{1}{n} F'_m(\zeta_0) F'_n(\zeta_0) + o(\rho^2). \tag{21}$$

This variation formula for the c_{mn} exhibits, of course, again the symmetry of the matric nc_{mn}.

Finally, let us differentiate (15) with respect to z; we obtain:

$$f^{*\prime}(z) = f'(z)\left[1 - \frac{a\rho^2}{(f(z) - \zeta_0)^2}\right] + o(\rho^2), \tag{22}$$

which may be written as a variation formula for $\log f'(z)$ as follows:

$$\log f^{*\prime}(z) = \log f'(z) - \frac{a\rho^2}{(f(z) - \zeta_0)^2} + o(\rho^2), \tag{23}$$

and which will be used in this form later on. By differentiating (20) with respect to w and applying (22) we get after elementary transformations:

$$F'^{*}_m[f^*(w)] = F'_m[f(w)] + a\rho^2 \frac{F'_m[f(w)] - F'_m(\zeta_0)}{(f(w) - \zeta_0)^2} + o(\rho^2). \tag{24}$$

This is a variation formula for the functional $F'_m[f(w)]$ and is interesting in that it contains only the functional itself in addition to $f(w)$ and ζ_0.

3. Application of the variation formulas in extremum problems. We use now the formula (21) in order to solve certain extremum problems with respect to the family Φ of all functions (1) which are univalent in D. Let $x = (x_1, x_2, \cdots, x_N)$ denote a vector of N complex numbers, not all zero, and consider the quadratic form

$$Q(x, x) = \sum_{m,n=1}^{N} nc_{mn}x_m x_n. \tag{25}$$

For each given function $f(z)$ of the family Φ this form has a well defined complex value. We ask now the following question:

What is the maximum modulus of $Q(x, x)$, if $f(z)$ is a function of the family Φ, and for which functions $f(z) \in \Phi$ is this maximum attained?

It follows easily from the theory of normal families that this question is significant and that there exists at least one $f(z) \in \Phi$ for which the maximum value is obtained. We may, therefore, restrict ourselves to the task of characterizing these extremal functions and computing the corresponding value of $|Q(x, x)|$.

Suppose that we know already an extremal function $f(z)$ and the corresponding value of $Q(x, x)\cdot f(z)$ will map D conformally upon a domain Δ in the ζ-plane. Let ζ_0 be a point in this plane which does not belong to Δ and subject Δ to a variation (14) as described in §2. Under this variation Δ is transformed into a domain Δ^* which is obtained from D by means of a mapping with a function $f^*(z)$, given by (15). Since $f^*(z)$ also belongs to the family Φ, its corresponding value $|Q^*(x, x)|$ cannot be larger than the maximal $|Q(x, x)|$. On the other hand, we may easily compute $|Q^*(x, x)|$ by means of formula (21). We have, in view of this formula,

$$Q^*(x, x) = Q(x, x) + a\rho^2\left(\sum_{n=1}^{N} x_n F_n'(\zeta_0)\right)^2 + o(\rho^2). \tag{26}$$

The extremal property of $|Q(x, x)|$ implies

$$\begin{aligned} |Q^*(x, x)| = \left|Q(x, x) + a\rho^2\left(\sum_{n=1}^{N} x_n F_n'(\zeta_0)\right)^2 + o(\rho^2)\right| \\ \leq |Q(x, x)|, \end{aligned} \tag{27}$$

which is equivalent to

$$\operatorname{Re}\left\{a\rho^2 Q^{-1}(x, x)\left(\sum_{n=1}^{N} x_n F_n'(\zeta_0)\right)^2 + o(\rho^2)\right\} \leq 0. \tag{28}$$

This inequality has to be fulfilled for every function $v(\zeta)$ as defined in (14). We now make use of the following lemma (Schiffer [2]):

Let Δ be a domain in the ζ-plane whose complement consists of continua Γ_ν $(\nu = 1, \cdots, c)$ and $\sigma(\zeta) \neq 0$ a function analytic in each of the Γ_ν. Let, further, the inequality

$$\operatorname{Re}\{a\rho^2\sigma(\zeta_0)\} + o(\rho^2) \leq 0 \tag{29}$$

hold for every function (14), univalent in Δ. Then each Γ_ν is an analytic curve with the parametric representation $\zeta(s)$ and satisfying for properly chosen (real) parameter s the differential equation

$$\zeta'(s)^2\sigma\{\zeta(s)\} = 1. \tag{30}$$

Application of this lemma to the particular inequality (28) leads to the result: The extremal function $f(z)$ maps the domain D upon a domain Δ in the ζ-plane which is bounded by analytic slits $\zeta(s)$ each of which satisfies the differential equation

$$\zeta'(s)^2 Q(x, x)^{-1}\left(\sum_{n=1}^{N} x_n F_n'[\zeta(s)]\right)^2 = 1. \tag{31}$$

We see now that our particular extremum problem leads to a differential equation (30) with a term $\sigma(\zeta)$ which is a complete square. Thus, this differential equation becomes immediately integrable in closed form. Among all extremum problems arising in the theory of univalent functions, the particular class of those problems with the above property plays an important role, since these problems permit a particularly easy solution.

We conclude from (31) that the boundary curves Γ_ν of Δ satisfy the requirement

$$\operatorname{Im}\left\{Q(x, x)^{-1/2}\sum_{m=1}^{N} x_m F_m(\zeta)\right\} = k_\nu, \tag{32}$$

$$\zeta \in \Gamma_\nu;\ k_\nu = \text{constant};\ \nu = 1, 2, \cdots, c.$$

The function

$$S(z) = Q(x, x)^{-1/2}\sum_{m=1}^{N} x_m F_m[f(z)] \tag{32'}$$

is regular in D, except for an Nth order pole at infinity and possesses constant imaginary parts on each boundary continuum of D.

In order to exploit these conditions, we introduce the following class of functions with respect to D. We define associate pairs of functions $A_m(z)$ and $B_m(z)$, having at $z = \infty$ the respective developments

$$A_m(z) = z^m + \sum_{n=1}^{\infty} a_{mn} z^{-n}, \tag{33}$$

$$B_m(z) = \sum_{n=1}^{\infty} b_{mn} z^{-n} \tag{33'}$$

which are regular in D except for the pole of $A_m(z)$ at infinity. On each boundary continuum of D we require:

$$A_m(z) = \overline{B_m(z)} + \text{const.} \tag{34}$$

where the complex constant depends on m and the particular boundary continuum. The existence and uniqueness of these function-pairs is easily proved (cf. Grunsky [1]).

We construct now the function

$$T(z) = Q(x, x)^{-1/2}\sum_{m=1}^{N} x_m A_m(z) + \overline{Q(x, x)^{-1/2}}\sum_{m=1}^{N} \bar{x}_m B_m(z). \tag{35}$$

This function has at $z=\infty$ the same principal part as $S(z)$ and, in view of the characteristic property (34) of $A_m(z)$ and $B_m(z)$, it has on each boundary continuum of D a constant imaginary part. The function $S(z)-T(z)$ is, therefore, regular throughout D and has on each boundary continuum a constant imaginary part. Hence, it follows by the usual considerations that this function is a constant, that is,

$$\begin{aligned} Q(x, x)^{-1/2}\sum_{m=1}^{N} x_m F_m[f(z)] = {} & Q(x, x)^{-1/2}\sum_{m=1}^{N} x_m A_m(z) \\ & + \overline{Q(x, x)^{-1/2}}\sum_{m=1}^{N} \bar{x}_m B_m(z) + C. \end{aligned} \tag{36}$$

Comparing in this equation the terms in z^0, we find from (2), (33), and (33′) at once that $C=0$. Let us introduce the real parameter δ by the definition

$$e^{i\delta} = \operatorname{sgn} Q(x; x); \tag{37}$$

then (36) may be written in the form

$$\sum_{m=1}^{N} x_m F_m[f(z)] = \sum_{m=1}^{N} x_m A_m(z) + e^{i\delta}\sum_{m=1}^{N} \bar{x}_m B_m(z). \tag{36′}$$

Let us now apply the formulas (2), (33), and (33′); comparing the coefficients of z^{-n} on both sides yields:

$$\sum_{m=1}^{N} x_m c_{mn} = \sum_{m=1}^{N} (x_m a_{mn} + e^{i\delta}\bar{x}_m b_{mn}). \tag{38}$$

Multiply the nth equation with nx_n and add all resulting equations for $n=1$ till $n=N$. We obtain:

$$\sum_{m,n=1}^{N} nc_{mn}x_m x_n = \sum_{m,n=1}^{N} na_{mn}x_m x_n + e^{i\delta}\sum_{m,n=1}^{N} nb_{mn}\bar{x}_m x_n, \tag{39}$$

whence:

$$\left|\sum_{m,n=1}^{N} nc_{mn}x_m x_n\right| \leq \left|\sum_{m,n=1}^{N} na_{mn}x_m x_n\right| + \sum_{m,n=1}^{N} nb_{mn}\bar{x}_m x_n. \tag{39′}$$

The last right-hand side term is non-negative, since it can be shown that (nb_{mn}) is a positive-definite Hermitian matric; we mention here that (na_{mn}) is a symmetric matric (cf. Grunsky [1]).

The significant feature of (39′) is that the right-hand terms depend only upon D and not upon the mapping function $f(z)$. The inequality (39′), proved for the extremal function, holds, therefore, a fortiori for every other admissible function also.

The form of (39) leads us to the following extremum problem: What is the maximum modulus of $P(x,x) = Q(x,x) - \sum_{m,n=1}^{N} n a_{mn} x_m x_n$, if $f(z)$ is a function of the family Φ?

One sees easily that the above considerations have to be repeated and that the same sequence of formulas appears, except that instead of $Q(x, x)$ we have always to introduce $P(x, x)$. We obtain finally, by putting

$$e^{i\gamma} = \operatorname{sgn} P(x, x), \tag{40}$$

the equation

$$\sum_{m,n=1}^{N} n(c_{mn} - a_{mn}) x_m x_n = e^{i\gamma} \sum_{m,n=1}^{N} n b_{mn} \bar{x}_m \bar{x}_n \tag{41}$$

for the coefficients c_{mn} belonging to the extremal function $f(z)$. Hence, we have in general:

$$\left| \sum_{m,n=1}^{N} n(c_{mn} - a_{mn}) x_m x_n \right| \leq \sum_{m,n=1}^{N} n b_{mn} \bar{x}_m x_n. \tag{42}$$

These are just Grunsky's inequalities, derived here by variational methods. We have obtained, moreover, the following equation for extremal functions:

$$\sum_{m=1}^{N} x_m F_m[f(z)] = \sum_{m=1}^{N} x_m A_m(z) + e^{i\gamma} \sum_{m=1}^{N} \bar{x}_m B_m(z), \tag{42′}$$

which shows that $f(z)$ is an algebraic function of the A_m and B_m $(m = 1, \cdots, N)$.

4. Inequalities for the Faber polynomials. The extremum problems of the last paragraph were so easily solved since they led to a variational differential equation with a complete square expression. It will, therefore, be useful to discuss other extremum problems of the same class.

Consider, for example, the functional

$$R(w, x) = \sum_{m,n=1}^{N} n c_{mn} x_m x_n + 2 \sum_{m=1}^{N} x_m F_m[f(w)] - \log f'(w), \tag{43}$$

which depends upon the function $f(z)$, assumed to be of the class Φ, and upon a fixed point $w \in D$. In view of (20), (21) and (23), we have for $R(w, x)$ the following variation formula:

$$(44)\quad R^*(w, x) = R(w, x) + a\rho^2\left(\sum_{m=1}^{N} x_m F_m'(\zeta_0) + \frac{1}{f(w) - \zeta_0}\right)^2 + o(\rho^2).$$

We see that $R(w, x)$ has a variation formula containing again a perfect square term and we expect, therefore, that the connected extremum problem will permit an easy solution. Therefore, we propose the following problem:

If $f(z)$ is an arbitrary function of the class Φ, what is the maximum modulus of $R(w, x)$ and for which function $f(z)$ is this extremum attained?

Assuming that $f(z)$ is the desired extremum function and that it maps the original domain D upon a domain Δ, we find, by reiterating literally the conclusions of the last paragraph, that Δ is bounded by analytic slits $\zeta(s)$, each of which satisfies the differential equation

$$(45)\qquad R(w, x)^{-1}\cdot\zeta'(s)^2\left[\sum_{m=1}^{N} x_m F_m'[\zeta(s)] + \frac{1}{f(w) - \zeta(s)}\right]^2 = 1.$$

This equation may be integrated and yields

$$(46)\qquad \operatorname{Im}\left\{R(w, x)^{-1/2}\left[\sum_{m=1}^{N} x_m F_m(\zeta) - \log(\zeta - f(w))\right]\right\} = \text{const.}$$

along each boundary continuum of Δ. The function

$$(47)\qquad U(z) = R(w, x)^{-1/2}\left[\sum_{m=1}^{N} x_m F_m[f(z)] - \log(f(z) - f(w))\right]$$

is regular throughout D, except for an Nth order pole at infinity and a logarithmic pole at $z = w$. It has a constant imaginary part along each boundary continuum of D.

In order to make use of this fact as in the last paragraph, we have to define another pair of functions. Let $A(z, w)$ and $B(z, w)$ be univalent functions of z in D which map it upon the whole ζ-plane, slit along circular arcs with center at the origin, and along straight segments pointing to the origin respectively, and such that $w \in D$ corresponds to the origin in the ζ-plane. We suppose at infinity the developments:

$$(48)\qquad \log A(z, w) = \log(z - w) + \sum_{n=1}^{\infty} \alpha_n(w) z^{-n},$$

$$(48')\qquad \log B(z, w) = \log(z - w) + \sum_{n=1}^{\infty} \beta_n(w) z^{-n}.$$

On each boundary continuum of D, $\log A$ will have constant real parts while $\log B$ will possess constant imaginary parts. The functions

$$(49)\quad \phi(z, w) = \frac{1}{2}\log [A(z, w)\cdot B(z, w)] = \log (z - w) + \sum_{n=1}^{\infty} a_n(w)z^{-n}$$

and

$$(49')\qquad \psi(z, w) = \frac{1}{2}\log\frac{B(z, w)}{A(z, w)} = \sum_{n=1}^{\infty} b_n(w)z^{-n}$$

are, therefore, regular in D with the exception of the logarithmic poles at w and ∞ for $\phi(z, w)$. On each boundary continuum of D, we have obviously:

$$(50)\qquad \phi(z, w) = \overline{\psi(z, w)} + \text{const.}$$

where the constant depends on the particular boundary continuum and on w.

There exist, of course, close relations between the functions ϕ, ψ on the one hand, and the functions $A_m(z)$, $B_m(z)$ on the other. In fact, let us consider the integral around the whole boundary C of D:

$$(51)\quad \frac{1}{2\pi i}\oint_C A_m(z)\phi'(z, w)dz = A_m(w) + ma_m(w) - w^m; \quad \phi' = \frac{d}{dz}\phi.$$

The above equation is an elementary consequence of Cauchy's residue theorem. On the other hand, we have in view of (34), (50) and the single-valuedness in D of all functions concerned:

$$(52)\quad \begin{aligned}\frac{1}{2\pi i}\oint_C A_m(z)\phi'(z, w)dz &= \frac{1}{2\pi i}\oint_C \overline{B_m(z)}\,\overline{d\psi(z, w)}\\ &= -\overline{\frac{1}{2\pi i}\oint_C B_m d\psi}.\end{aligned}$$

Again, we apply the residue theorem which shows that the above integral is zero, B_m and ψ being regular throughout D. Hence, we have proved:

$$(52')\qquad A_m(w) = -ma_m(w) + w^m.$$

Using (33) and (49), we find, for fixed w and z near infinity, the development

$$(53)\qquad \phi(z, w) = \log (z - w) - \sum_{m,n=1}^{\infty} m^{-1}a_{mn}w^{-n}z^{-m}.$$

Since (na_{mn}) is a symmetric matric, we see that $\phi(z, w) - \log (z-w)$ is a symmetric analytic function of z and w.

Let us consider next the following consequence of the residue theorem:

$$\frac{1}{2\pi i}\oint_C A_m(z)\psi'(z, w)dz = mb_m(w). \tag{54}$$

In view of (34) and (50), we may write this integral also in the form:

$$\begin{aligned}\frac{1}{2\pi i}\oint_C A_n(z)d\psi &= \overline{\frac{1}{2\pi i}\oint_C \overline{B_m(z)}\, d\phi} \\ &= -\overline{\frac{1}{2\pi i}\oint_C B_m(z)\phi'(z, w)dz},\end{aligned} \tag{55}$$

which gives, again because of the residue theorem,

$$mb_m(w) = -\overline{B_m(w)}. \tag{56}$$

Therefore, in view of (33′) and (49′):

$$\psi(z, w) = -\sum_{m,n=1}^{\infty} \frac{1}{m}\bar{b}_{mn}\bar{w}^{-n}z^{-m} = -\sum_{m,n=1}^{\infty} \frac{1}{mn} n\bar{b}_{mn}\bar{w}^{-n}z^{-m}. \tag{57}$$

Using the Hermitian character of mb_{mn}, we find finally:

$$\psi(z, w) = -\sum_{m,n=1}^{\infty} \frac{1}{n} b_{nm}\bar{w}^{-n}z^{-m}; \tag{57′}$$

(57) and (57′) show that ψ is analytic in z and $\bar{w}$.

Let us return now to the function $U(z)$, defined in (47). We prove easily, as in the last paragraph, the identity:

$$\begin{aligned}R(w, x)^{-1/2}&\left[\sum_{m=1}^{N} x_m F_m[f(z)] - \log (f(z) - f(w))\right] \\ &= R(w, x)^{-1/2}\left[\sum_{m=1}^{N} x_m A_m(z) - \phi(z, w)\right] \\ &\quad + \overline{R(w, x)}^{-1/2}\left[\sum_{m=1}^{N} \bar{x}_m B_m(z) - \psi(z, w)\right].\end{aligned} \tag{58}$$

Putting again

$$e^{i\tau} = \operatorname{sgn} R(w, x), \tag{58′}$$

we may write (58) in the form:

$$(59)\qquad \begin{aligned} &\sum_{m=1}^{N} x_m F_m[f(z)] - \log (f(z) - f(w)) \\ &\qquad = \sum_{m=1}^{N} x_m A_m(z) - \phi(z, w) + e^{i\tau}\left[\sum_{m=1}^{N} \bar{x}_m B_m(z) - \psi(z, w)\right]. \end{aligned}$$

Comparing the coefficients of z^{-n} on both sides, we find in view of (2), (5), (33), (33′), (49), (49′), (52′) and (56):

$$(60)\qquad \begin{aligned} &\sum_{m=1}^{N} x_m c_{mn} + \frac{1}{n} F_n[f(w)] \\ &\qquad = \sum_{m=1}^{N} x_m a_{mn} + \frac{1}{n} A_n(w) + e^{i\tau}\left[\sum_{m=1}^{N} \bar{x}_m b_{mn} + \frac{1}{n}\overline{B_n(w)}\right]. \end{aligned}$$

Multiplying with nx_n and summing up for $1 \leq n \leq N$, we get:

$$(61)\qquad \begin{aligned} &\sum_{m,n=1}^{N} n c_{mn} x_m x_n + \sum_{m=1}^{N} x_m F_m[f(w)] = \sum_{m,n=1}^{N} n a_{mn} x_m x_n \\ &\qquad + \sum_{m=1}^{N} A_m(w) x_m + e^{i\tau}\left[\sum_{m,n=1}^{N} n b_{mn} \bar{x}_m x_n + \sum_{m=1}^{N} \overline{B_m(w)}\, x_m\right]. \end{aligned}$$

On the other hand, we obtain from (59) in the limit $z \to w$:

$$(62)\qquad \begin{aligned} &\sum_{m=1}^{N} x_m F_m[f(w)] - \log f'(w) = \sum_{m=1}^{N} x_m A_m(w) - \chi(w) \\ &\qquad\qquad + e^{i\tau}\left[\sum_{m=1}^{N} \bar{x}_m B_m(w) - \psi(w, w)\right], \end{aligned}$$

where $\chi(w) = \lim_{z\to w} [\phi(z, w) - \log (z - w)]$. Adding together (61) and (62) yields:

$$(63)\qquad \begin{aligned} &\sum_{m,n=1}^{N} n c_{mn} x_m x_n + 2\sum_{m=1}^{N} x_m F_m[f(w)] - \log f'(w) \\ &= \sum_{m,n=1}^{N} n a_{mn} x_m x_n + 2\sum_{m=1}^{N} x_m A_m(w) - \chi(w) \\ &\qquad + e^{i\tau}\left[\sum_{m,n=1}^{N} n b_{mn} \bar{x}_m x_n + 2 \operatorname{Re}\left\{\sum_{m=1}^{N} \bar{x}_m B_m(w)\right\} - \psi(w, w)\right]. \end{aligned}$$

From (63) we derive:

$$(64)\quad \begin{aligned} |R(w, x)| \leq \Big| \sum_{m,n=1}^{N} n a_{mn} x_m x_n + 2 \sum_{m=1}^{N} x_m A_m(w) - \chi(w) \Big| \\ + \Big| \sum_{m,n=1}^{N} n b_{mn} \bar{x}_m x_n \\ + 2 \operatorname{Re} \left\{ \sum_{m=1}^{N} \bar{x}_m B_m(w) \right\} - \psi(w, w) \Big|. \end{aligned}$$

Since the right-hand side depends only on D and not on the particular function $f(z)$, this inequality, proved for the maximum of $|R(w, x)|$ only, holds a fortiori for the rest of the family Φ.

As before, we may improve our result and prove:

$$(65)\quad \begin{aligned} \Big| R(w, x) - \sum_{m,n=1}^{N} n a_{mn} x_m x_n - 2 \sum_{m=1}^{N} x_m A_m(w) + \chi(w) \Big| \\ \leq \Big| \sum_{m,n=1}^{N} n b_{mn} \bar{x}_m x_n + 2 \operatorname{Re} \left\{ \sum_{m=1}^{N} \bar{x}_m B_m(w) \right\} - \psi(w, w) \Big|. \end{aligned}$$

This result contains of course Grunsky's inequalities as particular cases. If, on the other hand, we put $x_1 = x_2 = \cdots = x_N = 0$, we get from (65)

$$(66)\quad |\log f'(w) + \chi(w)| \leq |\psi(w, w)|$$

or, in view of (53) and (57'), the following inequality which is valid as long as all series concerned are convergent:

$$(66')\quad \Big| \log f'(w) + \sum_{m,n=1}^{\infty} \frac{1}{m} a_{mn} w^{-(m+n)} \Big| \leq \sum_{m,n=1}^{\infty} \frac{1}{n} b_{mn} \bar{w}^{-n} w^{-m}.$$

In the particular case that D is the domain $|z| > 1$, we have $a_{mn} = 0$, $b_{mn} = \delta_{mn}$, whence:

$$(66'')\quad |\log f'(w)| \leq \log \frac{|w|^2}{|w|^2 - 1},$$

a well known inequality.

It would be of interest to derive estimates for $F_n[f(w)]$ without combining this functional with other terms. But since the variational differential equation in this problem is much more complicated, no easy solution is to be expected. To indicate the difficulty of these questions, let us remark that the determination of exact bounds for $|F_n'[f(w)]|$, even only in the case that D is the exterior of the unit circle, would solve simultaneously the Bieberbach problem for functions which are univalent inside the unit circle. On the other hand, the determination of exact bounds for the coefficients c_{mn} in the Faber development would lead to such bounds for the c_{1n}, that is, the coefficients of univalent functions of the type (1). Since this problem is yet unsolved, we see that little progress is at the moment to be expected in the general question of the c_{mn}. It appears, therefore, particularly interesting that for the functionals discussed above such an easy and complete answer is possible. Numerous other functionals of such variational behavior might be constructed and new inequalities be established.

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HEBREW UNIVERSITY AND
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THE MULTIPLICATIVE COMPLETION OF SETS OF FUNCTIONS

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1. **Introduction.** A set $\{f_n(x)\}_1^\infty$ of functions of $L^2(a, b)$, where (a, b) is finite or infinite, is called complete if $g(x) \in L^2$ and $\int_a^b f_n(x)g(x)dx = 0$, $n = 1, 2, \cdots$, imply that $g(x) = 0$ almost everywhere on (a, b); a well known equivalent property ("closure") is that every element of L^2 can be approximated in the L^2 metric by finite linear combinations of the $f_n(x)$.

Suppose that $\{f_n(x)\}$ is not complete. It will sometimes be possible to find a function $m(x)$ such that the set $\{m(x)f_n(x)\}$ is complete. This can also be considered as completeness after a change of weight function or a change of measure; but we shall not attempt to consider the most general change of measure here. We give some results on when a set can or cannot be completed by multiplication; the problem of finding necessary and sufficient conditions is left open.

We first state our results.

THEOREM 1. *If $\{f_n(x)\}_1^\infty$ is an orthonormal set which is not complete, but can be completed by the addition of a finite number of functions to the set, then there is a bounded measurable function $m(x)$ such that $\{m(x)f_n(x)\}_1^\infty$ is complete.*

The condition of Theorem 1, while necessary, is not sufficient, as Theorem 2 shows.

THEOREM 2. *The orthogonal set $\{e^{-x/2}L_{2n}(x)\}_0^\infty$, where $L_{2n}(x)$ is the 2nth Laguerre polynomial, cannot be completed on $(0, \infty)$ by the addition of a finite number of functions, but is completed on multiplication by $m(x) = e^{-x/2}$.*

Our next three theorems give examples of sets which cannot be completed by multiplication.

THEOREM 3. *A set of even functions cannot be completed by multiplication by an integrable function in any interval containing* 0.

THEOREM 4. *The set $\{e^{2inx}\}_{-\infty}^{\infty}$ cannot be completed in $(-\pi, \pi)$ by multiplication by an integrable function.*

THEOREM 5. *The set $\{x^{\lambda_n}\}$, where $\lambda_n > 0$, $\sum 1/\lambda_n < \infty$, cannot be completed in any interval by multiplication by a continuous function.*

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2. Proof of Theorem 1. For simplicity we suppose that the functions are real. We first consider the case where the set has deficiency 1, that is, can be completed by the addition of one function f_0. We may suppose that f_0 is orthogonal to f_n, $n=1, 2, \cdots$; for otherwise we could replace f_0 by $f_0 - \sum_1^\infty f_n(x)\int_a^b f_0(t)f_n(t)dt$, where $\sum$ implies convergence in L^2, that is, convergence in mean square. Suppose that $m(x)$ is measurable, bounded, never 0, but such that $f_0(x)/m(x)$ does not belong to L^2 on (a, b); then if

$$\int_a^b m(x)g(x)f_n(x)dx = 0, \qquad n = 1, 2, \cdots, \tag{1}$$

with $g(x) \in L^2$, we have, since $\{f_n(x)\}_0^\infty$ is orthogonal and complete, $m(x)g(x) = cf_0(x)$ almost everywhere for some constant c. Then c must be 0, since otherwise $g(x) = cf_0(x)/m(x)$ would not belong to L^2 on (a, b); that is, since $m(x)$ is never 0, $g(x) = 0$ almost everywhere. In other words, the set $\{m(x)f_n(x)\}_1^\infty$ is complete.

It remains to construct $m(x)$. Let E be a bounded set of positive measure on which $|f_0(x)| \geqq \epsilon > 0$; choose $m(x)$ on E so that $m(x)$ is bounded and measurable and $1/m(x)$ is never ∞ but does not belong to L^2 on E; let $m(x) = 1$ elsewhere. This function has the desired properties.

We now consider the general case. Here there are k functions $f_n(x)$, $n=0, -1, -2, \cdots, -k+1$, such that $\{f_n(x)\}_{-k+1}^\infty$ is complete. We may again suppose that $\{f_n(x)\}_{-k+1}^\infty$ is an orthogonal set. It is enough to construct a bounded measurable $m(x)$, never 0, such that $\{m(x)\}^{-1}\sum_{-k+1}^0 a_jf_j(x)$ belongs to L^2 only if all the a_j are zero. For, if (1) is true,

$$m(x)g(x) = \sum_{j=-k+1}^{0} a_jf_j(x), \tag{2}$$

and unless all the a_j are zero, (2) contradicts the fact that $g(x)$ belongs to L^2.

We now construct $m(x)$. Let E_0 be a bounded set of positive measure on which $f_0(x) \neq 0$. Construct a bounded measurable $m_0(x)$, never 0, such that $f_0(x)/m_0(x)$ does not belong to L^2 on E_0 and $m_0(x) = 1$ outside. We now proceed by induction. Suppose that $0 \leqq n < k-1$ and that we have determined a bounded measurable $m_n(x)$ and a bounded set E_n of positive measure such that $\{m_n(x)\}^{-1}\sum_{-n}^0 a_jf_j(x)$ belongs to $L^2(a, b)$ only if all a_j are zero. Consider all linear combinations $F = a_0f_0 + a_1f_{-1} + \cdots + a_{-n}f_{-n} + f_{-n-1}$. At most one function $F(x)/m_n(x)$ can belong to $L^2(a, b)$, since, if F_1/m_n and F_2/m_n both did, their difference would not involve f_{-n-1} and so could not belong to L^2 by the induction hypothesis. If no $F(x)/m_n(x)$ belongs to L^2, take $E_{n+1}=E_n$, $m_{n+1}(x)=m_n(x)$. If some $F(x)/m_n(x)$ belongs to L^2, $|F(x)|>0$ on some bounded measurable set E^* of positive measure; by making E^* smaller if necessary, we may suppose that either (a) E^* is disjoint from E_n or (b) E^* is a subset of E_n. In case (a), let $E_{n+1}=E_n+E^*$; in case (b) let $E_{n+1}=E^*$. Define a bounded measurable $m_{n+1}(x)$ so that $m_{n+1}(x)\leqq m_n(x)$ and so that $F(x)/m_{n+1}(x)$ does not belong to L^2 on E_{n+1}. Then $m_{n+1}(x)$ has the properties of $m_n(x)$, since $m_n(x)\leqq m_{n+1}(x)$, and, in addition, $\{m_{n+1}(x)\}^{-1}\sum_{-n-1}^{0} a_jf_j(x)$ does not belong to L^2 unless all the a_j are zero. This completes the induction; take $m(x)=m_{k-1}(x)$.

3. **Proof of Theorem 2.** (a) The functions $e^{-t/2}L_{2n+1}(t)$ are orthogonal to all $e^{-t/2}L_{2n}(t)$; hence no finite set of additional functions will complete the latter set.

(b) Suppose that $g(t)\in L^2(0,\ \infty)$ and $\int_0^\infty e^{-t}g(t)L_{2n}(t)dt=0$, $n=0, 1, 2, \cdots$. That is,

$$\sum_{k=0}^{2n} C_{2n,k}\frac{(-1)^k}{k!}\int_0^\infty e^{-t}g(t)t^k dt = 0,$$

or $\Delta^{2n}\mu_0=0$, where

$$\mu_k = (1/k!)\int_0^\infty t^k e^{-t}g(t)dt.$$

Also,

$$|\mu_k| \leqq \frac{\{(2k)!\}^{1/2}}{2^{k+1/2}k!}\left\{\int_0^\infty g^2(t)dt\right\}^{1/2} \leqq \text{constant};$$

that is, $\{\mu_k\}$ is bounded. By a theorem of Agnew [1][1] and Fuchs [2], [3], $\mu_k=0$, $k=0, 1, 2, \cdots$. Hence [4, p. 20] $g(t)=0$ almost everywhere.

4. **Proof of Theorem 3.** Let the functions $f_n(x)$ be even and belong to $L^2(-a, a)$. We shall show that for every integrable $m(x)$ there is a bounded $g(x)$, not almost everywhere zero, such that

$$\int_{-a}^{a} g(x)m(x)f_n(x)dx = 0, \qquad n = 1, 2, \cdots. \tag{3}$$

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[1] Numbers in brackets refer to the references at the end of the paper.

This is trivial if $m(x)=0$ on some set of positive measure. Otherwise, since $m(x)$ is integrable, there is a set E of positive measure in $(0, a)$ on which $m(x)$ is bounded and not zero. Let E_1 be the symmetric set in $(-a, 0)$. E_1 contains a subset E_2 of positive measure on which $m(x)$ is bounded and not zero. Let E_3 be the symmetric set in $(0, a)$. Let $g(x)=m(-x)$ in E_2, $g(x)=-m(-x)$ in E_3, $g(x)=0$ elsewhere. Then $m(x)g(x)$ is odd and $f_n(x)$ is even, so (3) follows.

5. **Proof of Theorem 4.** As in §4, if $m(x)$ is integrable we have to find $g(x)$, not almost everywhere zero, such that

$$\int_{-\pi}^{\pi} g(x)m(x)e^{2inx}dx = 0, \qquad n = 0, \pm 1, \pm 2, \cdots .$$

This is trivial if $m(x)=0$ on some set of positive measure. Otherwise, we can find E, of positive measure, in $(-\pi, 0)$, with $m(x)$ bounded and bounded from zero. The set E_1 obtained by adding π to every point of E is a subset of $(0, \pi)$; it contains a subset E_2 of positive measure on which $m(x)$ is bounded and bounded from zero. Let E_3 be the set obtained by subtracting π from every point of E_2 and let $E_4=E_2+E_3$. Let $f(x)=1/m(x)$ in E_4, $f(x)=0$ elsewhere. Then $f(x)m(x)$ has period π and so is orthogonal to $e^{(2n+1)ix}$, $n=0, \pm 1, \pm 2, \cdots$, and is different from 0 on a set of positive measure.

The same argument shows that $\{e^{2inx}\}$ cannot be completed by multiplication on any interval of length exceeding π.

6. **Proof of Theorem 5.** It is known that the L^2 span of $\{x^{\lambda_n}\}$, $\sum 1/\lambda_n < \infty$, on any interval not containing 0, contains only functions analytic in that interval [5]. Suppose that $\{m(x)x^{\lambda_n}\}$ were complete, hence closed, with $m(x)$ continuous. Let I be an interval in which $|m(x)| > \epsilon > 0$. Then to every $f(x)$ of $L^2(I)$ and every positive δ there would exist constants a_k such that

$$\int_I |m(x) \sum a_k x^{\lambda_k} - f(x)|^2 dx < \delta\epsilon^2,$$

and hence

$$\int_I | \sum a_k x^{\lambda_k} - f(x)/m(x)|^2 dx < \delta.$$

Thus $f(x)/m(x)$ would be in the span of $\{x^{\lambda_n}\}$, but we can certainly find $f(x)$ of L^2 with $f(x)/m(x)$ not analytic in I.

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BROWN UNIVERSITY AND
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DERIVATIVES OF INFINITE ORDER

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Let $f(x)$ have derivatives of all orders in (a, b). If, as $n \to \infty$, $f^{(n)}(x) \to g(x)$ uniformly, or even boundedly, dominatedly or in the mean, then $g(x)$ is necessarily of the form ke^x, where k is a constant; in fact, if $c \in (a, b)$,

$$f^{(n-1)}(x) - f^{(n-1)}(c) \to \int_c^x g(t)dt$$

and so

$$g(x) - g(c) = \int_c^x g(t)dt.$$

It then follows first that $g(x)$ is continuous, then that $g(x)$ is differentiable in (a, b), finally that $g'(x) = g(x)$ and so $g(x) = ae^x$.

If $f^{(n)}(x)$ approaches a limit only for one value of x, however, it does not necessarily do so for other values of x. On the other hand, G. Vitali [10][1] and V. Ganapathy Iyer [6] showed that if $f(x)$ is analytic in (a, b) and $f^{(n)}(x)$ approaches a limit for one $x_0 \in (a, b)$, then $f^{(n)}(x)$ converges uniformly in each closed subinterval of (a, b). Ganapathy Iyer asked two questions in this connection:

(I) If $f^{(n)}(x) \to g(x)$ for each x in (a, b), where $g(x)$ is finite, does $g(x) = ke^x$?

(II) If $f(x)$ belongs to a quasianalytic class in (a, b) and $\lim_{n\to\infty} f^{(n)}(x_0)$ exists for a single x_0, does $\lim_{n\to\infty} f^{(n)}(x)$ exist for every x in (a, b)?

We shall show that the answer to both questions is yes. We also indicate some possible generalizations.

We first answer (I).

THEOREM 1. *If $f^{(n)}(x) \to g(x)$ for each x in (a, b), where $g(x)$ is finite, then $f(x)$ is analytic in (a, b).*

It follows from Ganapathy Iyer's result that then $g(x) = ke^x$.

PROOF. At each point x of (a, b) form the Taylor series of $f(x)$. The radius of convergence of this series, as a function of x, has a positive

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[1] Numbers in brackets refer to the references cited at the end of the paper.

lower bound; in fact, it is infinite for each x. By a known theorem [2, 5, 11] (stated with an incomplete proof by Pringsheim [9, p. 180]), $f(x)$ is analytic in (a, b).

Next we answer (II).

THEOREM 2. *If* $f(x)$ *belongs to a Denjoy-Carleman quasianalytic class in the (open) interval* (a, b), *and if* $f^{(n)}(x_0) \to L$ *for one* x_0 *in* (a, b), *then* $f(x)$ *is analytic in* (a, b).

Again, by the result of Vitali and Ganapathy Iyer it follows that $f^{(n)}(x) \to Le^{x-x_0}$ in (a, b).

PROOF. We say that $f(x) \in C\{M_n\}$ if $|f^{(n)}(x)| \leq k^n M_n$, $x \in I$, for each closed subinterval I of (a, b), where k depends on $f(x)$ and on I. The class $C\{M_n\}$ is quasianalytic if $\sum M_n^{-1/n} < \infty$; in this case any two functions of the class which coincide, together with all their derivatives, at $x_0 \in (a, b)$, are identical. It is known [3, 8] that $C\{M_n\}$ is identical with the class $C\{M_n^*\}$ obtained by a certain regularizing process; the only property of M_n^* which we need here is that M_{n+1}^*/M_n^* is nondecreasing. It follows that every class $C\{M_n\}$, except the trival class $C\{0\}$, contains $C\{1\}$. This seems to have been first pointed out explicitly by T. Bang [1, p. 16]; we reproduce the simple proof.

We have to show that $k_1^n \leq k_2^n M_n$, or that $M_n^* \geq k_3^n$ for some k_3. Now we have $M_n^*/M_{n-1}^* \geq M_1^*/M_0^* = \alpha$, say. Hence $M_n^* \geq M_{n-1}^* \alpha \geq M_{n-2}^* \alpha^2 \geq \cdots \geq \alpha^n M_0^*$.

Now suppose that $f^{(n)}(x_0) \to L$ and let

$$g(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0)(x - x_0)^k/k!.$$

For some number Q, $|f^{(n)}(x_0)| \leq Q$. Hence

$$| g^{(n)}(x) | = \left| \sum_{k=0}^{\infty} f^{(n+k)}(x_0)(x - x_0)^k/k! \right| \leq Qe^{x-x_0}$$

and so $g(x) \in C\{1\}$; hence $g(x) \in C\{M_n\}$. But $g^{(n)}(x_0) = f^{(n)}(x_0)$ for every n and so $f(x) \equiv g(x)$, an analytic function.

A natural generalization of the problem is to interpret the relation $f^{(n)}(x) \to g(x)$ in a generalized sense. For example, if $f^{(n)}(x) \to g(x)$ $(C, 1)$, dominatedly, the proof given in §1 shows that $g(x) = ke^x$; this proof, in fact, applies to any generalized limit such that $s_{n-1}(x)$ converges to the same limit as $s_n(x)$ (see [4, p. 418], [7] for discussions of such generalized limits, which include, in particular, (C, k), $k > -1$).

We can also replace $f^{(n)}(x) \to g(x)$ by $f^{(n)}(x)/\lambda_n \to g(x)$, $\{\lambda_n\}$ a given sequence of constants. We give two simple theorems in this direction.

THEOREM 3. *Let*

$$\lim_{n\to\infty} f^{(n)}(x)/\lambda_n = g(x), \qquad a \leqq x \leqq b. \tag{1}$$

(i) *If* lim inf $|\lambda_{n-1}/\lambda_n| = 0$ *and* (1) *holds uniformly*, $g(x) \equiv 0$ *in* $a \leqq x \leqq b$.
(ii) *If* lim inf $|\lambda_{n-1}/\lambda_n| > 0$ *and* (1) *holds dominatedly*, $g(x) = ke^{bx}$.

The example $f(x) = 1/x$, $\lambda_n = (-1)^n n!$, $a = 1$, $b = 2$ shows that uniformity is essential in (i). It would be interesting to know whether (without uniformity) there can be an exceptional point in the interior of (a, b); if $f(x)$ is analytic, there cannot, as the next theorem shows.

THEOREM 4. *If*

$$\limsup_{n\to\infty} n^{-1} |\lambda_n|^{1/n} < \infty \tag{2}$$

and (1) *is true for each* x *in* $a < x < b$, *then* $g(x) = ke^{bx}$ *in* $a < x < b$.

If $f(x)$ is analytic, lim sup $n^{-1} |f^{(n)}(x)|^{1/n} < \infty$ for each x and hence either (2) is true or (1) implies $g(x) \equiv 0$.

PROOF OF THEOREM 3. We observe that if $a < c < b$

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n}\left\{\frac{f^{(n-1)}(x)}{\lambda_{n-1}} - \frac{f^{(n-1)}(c)}{\lambda_{n-1}}\right\} = \int_c^x g(t)dt. \tag{3}$$

If (i) of Theorem 3 is true, the left side of (3) approaches zero as $n \to \infty$ through a suitable sequence; hence $g(x) = 0$ almost everywhere; but $g(x)$ is continuous because (1) holds uniformly, and so $g(x) \equiv 0$.

If (ii) is true and lim $|\lambda_{n-1}/\lambda_n| = \infty$, $\lambda_n \to 0$ and so $f^{(n)}(x) \to 0$; otherwise, for some sequence of n's, $\lambda_{n-1}/\lambda_n \to L$, where $L \neq 0$, $L \neq \infty$. Then (3) gives

$$L\{g(x) - g(c)\} = \int_c^x g(t)dt$$

and hence $g(x) = ke^{x/L}$.

PROOF OF THEOREM 4. We have from (1), for each x and for $n > n_x$, $|f^{(n)}(x)| \leqq (1 + g(x))|\lambda_n|$, and so

$$\limsup_{n\to\infty} |f^{(n)}(x)/n!|^{1/n} \leqq \limsup_{n\to\infty} |\lambda_n|^{1/n}/(n/e) < \infty.$$

The reasoning given for Theorem 1 now applies.

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ON A NOTE OF GALBRAITH AND GREEN

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In a recent note [1][1] Galbraith and Green evaluate the integral

$$I_n = \frac{1}{\pi}\int_0^\pi \left[\frac{1+r^2}{1-r^2} - \frac{2r}{1-r^2}\cos\theta\right]^{-n-1} d\theta.$$

This may be identified with Laplace's second integral for the Legendre function. Thus if $\arg\,[(1+r^2)/(1-r^2)] < \pi/2$, by [2, §15.23],

$$I_n = P_n([1+r^2]/[1-r^2]). \tag{1}$$

By[2] [2, §§14.51, 15.22] this may be written

$$\begin{aligned} I_n = {} & [\Gamma(-2n-1)/\Gamma^2(-n)](1/r^2-1)^{n+1}F(n+1,\ n+1;\ 2n+2;\ 1-1/r^2) \\ & + [\Gamma(2n+1)/\Gamma^2(n+1)](1/r^2-1)^{-n}F(-n,\ -n;\ -2n;\ 1-1/r^2). \end{aligned}$$

From this it is readily seen that

$$\lim_{r\to 1^-} (1-r)^n I_n = 2^{-n}\Gamma(2n+1)/\Gamma^2(n+1) \quad \text{when } R(n) > -1/2, \tag{2}$$

$$\lim_{r\to 1^-} (1-r)^{-n-1} I_n = 2^{n+1}\Gamma(-2n-1)/\Gamma^2(-n) \quad \text{when } R(n) < -1/2. \tag{3}$$

The slight difference between (2) and Theorem 2 of Galbraith and Green's paper is due to an error in the latter. (3) does not appear in that paper.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] An error in §14.51 of the reference may be corrected by changing the signs of the arguments of the gamma functions of binomial argument in the formula at the top of p. 289.

ON SOME CRITERIA OF CARLEMAN FOR THE COMPLETE CONVERGENCE OF A *J*-FRACTION

H. S. WALL

1. Introduction. Carleman [1, pp. 214–215][1] derived, from his theory of integral equations, a number of criteria for the complete convergence of a real *J*-fraction

$$(1.1) \qquad -\overset{\infty}{\underset{p=1}{K}} \frac{-a_{p-1}^2}{b_p+z} \qquad (a_0=1,\ a_p\neq 0).$$

The most important one of these criteria states that the *J*-fraction is completely convergent if the series $\sum(1/c_{2p})^{1/2p}$ diverges, where $c_0, c_1, c_2, \cdots$ are the coefficients in the power series $\sum(c_p/z^{p+1})$ associated with the *J*-fraction. In [2] Carleman gave an algebraic proof of this theorem for the case where $b_p=0$, $p=1, 2, 3, \cdots$. The present note contains an algebraic proof for the general case, and some remarks concerning the other criteria of Carleman, especially with reference to their application to *J*-fractions with arbitrary complex coefficients.

2. The determinate and indeterminate cases. Let $a_0=1$, $a_p\neq 0$, b_p, $p=1, 2, 3, \cdots$, be complex constants, and consider the system of linear equations

$$(2.1) \qquad -a_{p-1}x_{p-1}+(b_p+z)x_p-a_px_{p+1}=0, \qquad p=1, 2, 3, \cdots.$$

Since the a_p are not zero, these equations determine $x_2, x_3, x_4, \cdots$ uniquely in terms of arbitrarily chosen initial values x_0, x_1. If $x_0=-1$, $x_1=0$, let $x_p=X_p(z)$, and if $x_0=0$, $x_1=1$, let $x_p=Y_p(z)$. Then, $X_{p+1}(z)/Y_{p+1}(z)$ is the pth approximant of the *J*-fraction (1.1). If the infinite series $\sum|X_p(z)|^2$ and $\sum|Y_p(z)|^2$ both converge for one value of z, then they converge for every value of z [7, p. 120]. We may accordingly distinguish two cases for a *J*-fraction (1.1) with complex coefficients. In the *indeterminate case*, both the infinite series

$$(2.2) \qquad \sum|X_p(0)|^2, \qquad \sum|Y_p(0)|^2$$

are convergent. In the *determinate case*, at least one of these infinite series is divergent. A real *J*-fraction is completely convergent if and only if the determinate case holds. This is also the case of a determinate moment problem [4].

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

On applying Schwarz's inequality to the determinant formulas

$$X_{p+1}(z)Y_p(z) - X_p(z)Y_{p+1}(z) = \frac{1}{a_p} \tag{2.3}$$

and

$$X_{p+2}(z)Y_p(z) - X_p(z)Y_{p+2}(z) = \frac{b_{p+1} + z}{a_p a_{p+1}}, \tag{2.4}$$

after setting $z = 0$, we find immediately that when the indeterminate case holds then the series $\sum(1/|a_p|)$ and $\sum|b_{p+1}/a_p a_{p+1}|$ are convergent. These and other criteria obtained by applying Schwarz's inequality to other determinant formulas were found by Dennis and Wall [3]. They were not aware of the fact that Carleman [1, p. 215] had given the first of these criteria for the case of real J-fractions. Hellinger [6] showed that the determinate case holds for a real J-fraction if lim inf $|a_p|$ is finite. This criterion, which is contained in the first criterion mentioned above, was given by Carleman [1, p. 215].

Let $\delta_1, \delta_2, \delta_3, \cdots$ be complex numbers such that $\delta \leqq |\delta_p| \leqq \Delta$, $p = 1, 2, 3, \cdots$, where δ and Δ are positive constants independent of p. *If the determinate case holds for the J-fraction* (1.1), *then the determinate case holds for the J-fraction*

$$-\mathop{K}_{p=1}^{\infty} \frac{-a_{p-1}^2\delta_{p-1}\delta_p}{b_p\delta_p + z} \qquad (\delta_0 = 1). \tag{2.5}$$

The introduction of the factors δ_k into (1.1) effects the replacement of the series (2.2) by the series

$$\sum\left|\frac{\delta_1}{\delta_p}\right| |X_p(0)|^2, \qquad \sum\left|\frac{\delta_1}{\delta_{p+1}}\right| |Y_p(0)|^2,$$

so that the theorem is obviously true. Carleman [1, p. 215] obtained this for the case of real a_p, b_p, δ_p as a corollary to his general theory.

Let α_p and β_p be complex numbers such that

$$|\alpha_p| < M, \quad |\beta_p| < M, \qquad a_p + \alpha_p \neq 0,\ p = 1, 2, 3, \cdots.$$

If the determinate case holds for the J-fraction (1.1), *then the determinate case holds for the J-fraction*

$$-\mathop{K}_{p=1}^{\infty} \frac{-(a_{p-1} + \alpha_{p-1})^2}{b_p + \beta_p + z} \qquad (\alpha_0 = 0). \tag{2.6}$$

Carleman [1, p. 214] gave this theorem for the case of real a_k, b_k,

α_k, β_k. If these numbers are complex and $\alpha_k = 0$, $k = 1, 2, 3, \cdots$, then the theorem can be proved by means of an easy modification of the proof given in [7, pp. 120–121]. (Cf. also [8, p. 557].) The method used in [7] can be modified to cover the case $\beta_k = 0$, $k = 1, 2, 3, \cdots$, and hence to establish the theorem in the general case. We shall indicate briefly those modifications. Let

$$L_p(x) = -a_{p-1}x_{p-1} + b_p x_p - a_p x_{p+1}, \qquad p = 1, 2, 3, \cdots,$$

and let $L_p'(x)$ denote the expression $L_p(x)$ in which a_p is replaced by $a_p + \alpha_p$, where the α_p are any complex constants such that $|\alpha_p| < M$, $a_p + \alpha_p \neq 0$, $p = 1, 2, 3, \cdots$. The solution of the system $L_p(x) = 0$ under the initial conditions $x_0 = -1$, $x_1 = 0$ is $x_p = X_p(0)$, and under the initial conditions $x_0 = 0$, $x_1 = 1$ the solution is $x_p = Y_p(0)$. Let x_p' be the solution of the system $L_p'(x) = 0$ under the initial conditions $x_0' = a$, $x_1' = b$, where a and b are arbitrarily chosen constants. The theorem will be established if we show that the convergence of the series $\sum |X_p(0)|^2$ and $\sum |Y_p(0)|^2$ implies the convergence of the series $\sum |x_p'|^2$.

If ξ_p and ξ_p' are arbitrary solutions of the systems $L_p = 0$ and $L_p' = 0$, respectively, then we have the Green's formula

$$\tag{2.7} \begin{aligned} \sum_{p=1}^{n} [\xi_p L_p'(\xi') - \xi_p' L_p(\xi)] = [(\xi_p \xi_{p+1}' - \xi_p' \xi_{p+1}) a_p]_0^n \\ - \sum_{p=1}^{n} (\alpha_{p-1}\xi_{p-1}' + \alpha_p \xi_{p+1}')\xi_p = 0. \end{aligned}$$

If, in particular, $\xi_p = X_p(0)$, $\xi_p' = x_p'$, this gives

$$-b + (x_n' X_{n+1} - x_{n+1}' X_n) a_n - \sum_{p=1}^{n} (\alpha_{p-1} x_{p-1}' + \alpha_p x_{p+1}') X_p = 0,$$

where we have written X_k for $X_k(0)$; and if $\xi_p = Y_p(0) = Y_p$, $\xi_p' = x_p'$, we obtain

$$-a + (x_n' Y_{n+1} - x_{n+1}' Y_n) a_n - \sum_{p=1}^{n} (\alpha_{p-1} x_{p-1}' + \alpha_p x_{p+1}') Y_p = 0.$$

On multiplying the first of these equations by Y_n, the second by $-X_n$, and adding, we find that

$$x_n' - \sum_{p=1}^{n} k_{n,p} x_p' = k_n,$$

where

$$k_{n,p} = \begin{cases} \dfrac{\alpha_p(X_{p+1}Y_n - X_nY_{p+1}) + \alpha_{p-1}(X_{p-1}Y_n - X_nY_{p-1})}{1 + (\alpha_{n-1}/a_{n-1})}, & p = 1, 2, \cdots, n-1, \\ 0, \quad p = n, & \end{cases}$$

$$h_n = \frac{bY_n - aX_n}{1 + (\alpha_{n-1}/a_{n-1})}.$$

Under the hypothesis that the series $\sum|X_p|^2$ and $\sum|Y_p|^2$ converge, it follows, as indicated before, that the series $\sum|1/a_p|$ converges, so that $\lim_{n\to\infty} |a_{n-1}| = \infty$, Since, by hypothesis, $|\alpha_p| < M$, $p = 1, 2, 3, \cdots$, we then find at once by Schwarz's inequality that the double series $\sum|k_{pq}|^2$ and the series $\sum|h_p|^2$ are convergent. It then follows exactly as in [7, p. 121] that the series $\sum|x_p'|^2$ is convergent, and the theorem is thereby established.

3. **Criterion involving the moments c_p.** Let $\sum(c_p/z^{p+1})$ be the power series expansion in descending powers of z of a real J-fraction (1.1). *If the series*

$$\sum_{p=1}^{\infty}\left(\frac{1}{c_{2p}}\right)^{1/2p} \tag{3.1}$$

diverges, then the determinate case holds for the J-fraction. We shall give a simple algebraic proof of this well known theorem of Carleman [1, p. 215].

The quadratic forms

$$F_n = \sum_{p,q=1}^{n} c_{p+q-2}x_px_q, \qquad n = 1, 2, 3, \cdots,$$

are all positive definite [4], and we may therefore write

$$F_n = (b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n)^2 + (b_{22}x_2 + b_{23}x_3 + \cdots + b_{2n}x_n)^2 + \cdots + (b_{nn}x_n)^2,$$

where the b_{pq} are real constants independent of n for $p \leqq n$, $q \leqq n$, and where $b_{pp} \neq 0$, $p = 1, 2, 3, \cdots, n$. The b_{pp} are connected with the partial numerators of the J-fraction by the formulas [9]

$$(a_0a_1 \cdots a_p)^2 = b_{p+1,p+1}^2 \qquad (a_0 = b_{11} = 1)$$

and, obviously,

$$c_{2p} = \sum_{q=1}^{p+1} b_{q,p+1}^2 \qquad (p = 0, 1, 2, \cdots)$$

so that $c_{2p} \geq b_{p+1,p+1}^2$. Consequently,

$$(a_0 a_1 \cdots a_p)^2 \leq c_{2p}.$$

Hence, by Carleman's inequality[3] [2],

$$e\sum_{p=1}^{n} \frac{1}{|a_p|} > \sum_{p=1}^{n}\left(\frac{1}{a_1^2 a_2^2 \cdots a_p^2}\right)^{1/2p} \geq \sum_{p=1}^{n}\left(\frac{1}{c_{2p}}\right)^{1/2p}.$$

Therefore, if the series (3.1) is divergent, then the series $\sum |1/a_p|$ is divergent, so that, as indicated in §2, the determinate case holds for the J-fraction.

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[3] For an algebraic proof of Carleman's inequality see, for instance [5].

A NOTE ON CONVERGENCE IN LENGTH

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1. **Introduction.** Let I be a closed linear interval $a_0 \leqq t \leqq b_0$. Let $\mathfrak{x}(t) = (x(t), y(t), z(t))$, $t \in I$, represent a vector function whose three components $x(t)$, $y(t)$, $z(t)$ are of bounded variation and continuous on I. This vector function determines in Euclidean 3-space a curve $x = x(t)$, $y = y(t)$, $z = z(t)$ whose length we denote by $L(\mathfrak{x})$. By convergence in length of a sequence of such vector functions $\mathfrak{x}_n(t) = (x_n(t), y_n(t), z_n(t))$, $n = 0, 1, 2, \cdots$, is meant that $x_n(t)$, $y_n(t)$, $z_n(t)$ converge uniformly on I to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively and that $L(\mathfrak{x}_n)$ converges to $L(\mathfrak{x}_0)$. We denote by $V(f)$ the total variation on I of a scalar function $f(t)$ which is continuous and of bounded variation on I. By convergence in variation of a sequence $f_n(t)$, $n = 0, 1, \cdots$, is meant that $f_n(t)$ is continuous and of bounded variation on I for $n = 0, 1, \cdots$, that $f_n(t)$ converges uniformly on I to $f_0(t)$, and that $V(f_n) \rightarrow V(f_0)$. These concepts are due to Adams, Clarkson, and Lewy [1, 2].[1]

We are concerned here with the problem of determining conditions under which convergence in length holds. Uniform convergence on I of the components $x_n(t)$, $y_n(t)$, $z_n(t)$ to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively implies only that $\liminf L(\mathfrak{x}_n) \geqq L(\mathfrak{x}_0)$. It is also well known (see [2, 4, 5]) that convergence in length of such a sequence $\mathfrak{x}_n$ implies convergence in variation of each of the three sequences of components—and, indeed, convergence in variation of any sequence of scalar functions obtained by projecting the curves $\mathfrak{x} = \mathfrak{x}_n(t)$, $t \in I$, $n = 0, 1, \cdots$, on any line whatever. As a consequence of this we see that convergence in length of the sequence $\mathfrak{x}_n(t)$ implies convergence in variation of the sequence $c_1x_n(t) + c_2y_n(t) + c_3z_n(t)$ for arbitrary choice of the constants c_1, c_2, c_3. Convergence in variation of each of the three sequences of components is not sufficient to ensure convergence in length of the sequence of vectors (see [2]). In connection with the work of A. P. Morse [4] there arose the question as to whether convergence in length is implied by convergence in variation of every linear combination of the components. This has already been proved by Morse [4] for the case where $\mathfrak{x}_n(t)$ is of the special form $(t, y_n(t), 0)$, $n = 0, 1, \cdots$. In this note we generalize Morse's result to the parametric case. The proof is based on a generalization,

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

due essentially to Steinhaus [6], of a remarkable formula for arc length devised by Cauchy [3].

2. **Preliminaries.** If each component of a vector function $\mathfrak{x}(t)=(x(t), y(t), z(t))$, $t\in I$, is BV (of bounded variation), then $\mathfrak{x}(t)$ is said to be BV; if each component is AC (absolutely continuous), then $\mathfrak{x}(t)$ is said to be AC; if each component is continuous, then $\mathfrak{x}(t)$ is said to be continuous. We denote by $\mathfrak{x}'(t)$ the vector $(x'(t), y'(t), z'(t))$ wherever the derivatives $x'(t)$, $y'(t)$, $z'(t)$ all exist. If the components $x_n(t)$, $y_n(t)$, $z_n(t)$ of a sequence of vectors $\mathfrak{x}_n(t)$, $t\in I$, $n=0, 1, \cdots$, converge uniformly on I to $x_0(t)$, $y_0(t)$, $z_0(t)$ respectively, we say that $\mathfrak{x}_n(t)$ converges uniformly on I to $\mathfrak{x}_0(t)$.

Let Δ denote any subinterval $t'\leqq t\leqq t''$ contained in I and let $D(I)$ denote any subdivision of I into a finite number of nonoverlapping intervals Δ. The length $L(\mathfrak{x})$ of a vector function $\mathfrak{x}(t)$, $t\in I$, is defined as

$$L(\mathfrak{x}) = \text{l.u.b.} \sum |\mathfrak{x}(t'') - \mathfrak{x}(t')|, \qquad \Delta \in D(I),$$

where the least upper bound is taken over all subdivisions $D(I)$.

We note that this definition of length of vector functions is the exact analog of total variation of scalar functions and that it agrees with the usual definition of length of a curve if $\mathfrak{x}(t)$ is BV and continuous on I and we think of $\mathfrak{x}(t)$ as determining a curve $x=x(t)$, $y=y(t)$, $z=z(t)$, $a_0\leqq t\leqq b_0$.

We mention now the following well known facts which will be used in this note.

(a) If $\mathfrak{x}(t)=(x(t), y(t), z(t))$ is BV and continuous on I, then

$$V(x) \leqq L(\mathfrak{x}) \leqq V(x) + V(y) + V(z).$$

(b) If $f_n\to f_0(V)$, then $kf_n\to kf_0(V)$ for arbitrary choice of the constant k.

(c) If $f_n(t)$ converges uniformly on I to $f_0(t)$ and if $f_n(t)$ is BV and continuous on I for all n, then $\liminf V(f_n)\geqq V(f_0)$.

(d) If $f(t)$, $t\in I$, is BV and continuous, then $f'(t)$ is summable on I and $V(f)\geqq\int_I |f'|\,dt$, the sign of equality holding if and only if $f(t)$ is AC.

(e) If $\mathfrak{x}(t)$ is BV and continuous on I and if $\mathfrak{x}=\mathfrak{x}_n(t)$, $t\in I$, $n=1, 2, \cdots$, is a sequence of polygons inscribed in the curve $\mathfrak{x}=\mathfrak{x}(t)$, $t\in I$, and converging uniformly on I to $\mathfrak{x}=\mathfrak{x}(t)$, $t\in I$, then $\mathfrak{x}_n\to\mathfrak{x}(L)$.

(f) Let $\mathfrak{x}_n(t)=(x_n(t), y_n(t), z_n(t))$, $t\in I$, $n=0, 1, \cdots$. If $\mathfrak{x}_n\to\mathfrak{x}_0(L)$, then $x_n\to x_0(V)$.

(g) If $\mathfrak{x}(t)$, $t\in I$, is BV and continuous, then $|\mathfrak{x}'(t)|$ is summable on I and $L(\mathfrak{x}) \geqq \int_I |\mathfrak{x}'|\,dt$, the sign of equality holding if and only if $\mathfrak{x}(t)$ is AC.

LEMMA. *Suppose* $\mathfrak{x}_n \to \mathfrak{x}_0(L)$. *Let* $\mathfrak{u}$ *be any fixed unit vector and* $f_n(t)$ *the scalar product of the vectors* $\mathfrak{x}_n(t)$ *and* $\mathfrak{u}$; *that is,*

$$f_n(t) = \mathfrak{x}_n(t)\cdot\mathfrak{u}, \qquad t \in I,\ n = 0, 1, \cdots .$$

Then $f_n \to f_0(V)$ *and* $V(f_n) \leqq L(\mathfrak{x}_n)$ *for* $n = 0, 1, \cdots$.

The function $f_n(t)$ defined here is the projection of the curve $\mathfrak{x} = \mathfrak{x}_n(t)$ on a line parallel to the given vector $\mathfrak{u}$. This fact is well known, as was mentioned in §1, but a proof will be included for the convenience of the reader. Let $\mathfrak{u}$ be the vector (a, b, c), $a^2+b^2+c^2 = 1$. Then $f_n(t) = ax_n(t) + by_n(t) + cz_n(t)$, $n = 0, 1, \cdots$. Let us set up a new system of rectangular coordinates x^*, y^*, z^*, such that the x^*-axis coincides with the line through the origin with direction cosines a, b, c. Then x^* is expressed in terms of the old coordinates as $ax+by+cz$. Let $\mathfrak{x}_n^*(t) = (x_n^*(t), y_n^*(t), z_n^*(t))$ denote the vector $\mathfrak{x}_n(t)$ referred to the new coordinates. Since arc length is independent of the particular coordinate system used, we have $\mathfrak{x}_n^* \to \mathfrak{x}_0^*(L)$ and hence (see (f), (a)) $x_n^* \to x_0^*(V)$ and $V(x_n^*) \leqq L(\mathfrak{x}_n^*)$. But $x_n^*(t) = ax_n(t) + by_n(t) + cz_n(t) = f_n(t)$, $t \in I$, $n = 0, 1, \cdots$. That is,

$$f_n \to f_0(V) \text{ and } V(f_n) \leqq L(\mathfrak{x}_n)$$

for all n.

3. **Cauchy's formula.** The formula for arc length which is stated in Lemma 4 applies to any continuous, rectifiable curve in Euclidean 3-space. It is a direct generalization of a formula of Cauchy [3], the method of proof given here being due to Steinhaus [6]. A proof of the formula is included in this note because the reasoning involved in it is used in §4, as well as the formula itself.

LEMMA 1. *Let* $\mathfrak{a}$ *be a fixed vector and* $\mathfrak{u}$ *a variable unit vector from the center of a unit sphere to any point* p *on the surface* S *of the sphere. Then*

$$\iint_S |\mathfrak{a}\cdot\mathfrak{u}|\,d\sigma = 2\pi|\mathfrak{a}|,$$

where $d\sigma$ *is the area-element on* S.

PROOF. Choose rectangular coordinates so that the z-axis coincides in direction with $\mathfrak{a}$. In terms of the spherical coordinates $1, \theta, \phi$ of the point p on S, we have $\mathfrak{a}\cdot\mathfrak{u} = |\mathfrak{a}|\cdot\cos\phi$ and hence

$$\begin{aligned}\iint_S |\mathfrak{a}\cdot\mathfrak{u}|\,d\sigma &= \int_0^{\pi}\int_0^{2\pi} |\mathfrak{a}|\cdot|\cos\phi|\sin\phi\,d\theta d\phi\\ &= 2\pi|\mathfrak{a}|\int_0^{\pi}|\cos\phi|\sin\phi\,d\phi\\ &= 2\pi|\mathfrak{a}|\int_0^{\pi/2}\sin 2\phi d\phi = 2\pi|\mathfrak{a}|.\end{aligned}$$

LEMMA 2. *Let* $\mathfrak{x}(t)$ *be BV and continuous on* I. *Let* S *be the surface of a unit sphere,* p *any point on* S, *and* $\mathfrak{u}$ *the vector from the center of the sphere to* p. *For each point* p *on* S *let* $V(p)$ *denote the total variation of the function* $f(t, p)=\mathfrak{x}(t)\cdot\mathfrak{u}$, $t\in I$. *Then* $V(p)$ *is summable on the surface* S.

PROOF. Since $f(t, p)$ is clearly BV and continuous on I for each point p on S, $V(p)$ is defined for every p. Consider now a sequence of points p_n on S such that $p_n\to p_0$. It is easily verified that the sequence $f(t, p_n)$ converges uniformly on I to $f(t, p_0)$, from which it follows that lim inf $V(p_n)\geqq V(p_0)$ (see (c) of §2). This means that $V(p)$ is lower semi-continuous on S and hence measureable on S. From the fact that $V(p)$ is bounded on S by $L(\mathfrak{x})$ (see Lemma of §2), we conclude that $V(p)$ is summable on S.

LEMMA 3. *Given a sequence of BV, continuous vector functions* $\mathfrak{x}_n(t)$, $t\in I$, $n=0, 1, \cdots$. *Let* S, p, $\mathfrak{u}$ *be defined as in Lemma* 2. *For each* p *on* S *let* $V_n(p)$ *denote the total variation of the function* $f_n(t, p)=\mathfrak{x}_n(t)\cdot\mathfrak{u}$, $t\in I$, $n=0, 1, \cdots$. *If the sequence* $f_n(t, p)$ *converges in variation for every point* p *on* S, *then*

$$\iint_S V_n(p)d\sigma\to\iint_S V_0(p)d\sigma,$$

where $d\sigma$ *is the area-element on* S.

PROOF. By hypothesis the sequence $f_n(t, p)$ converges in variation for every point p on S. Hence

$$V_n(p)\to V_0(p),\qquad p\in S. \tag{1}$$

Let a, b, c be any three (real) constants such that $a^2+b^2+c^2=1$. Since the point (a, b, c) lies on S, we then have by (1) convergence in variation of the sequence $ax_n(t)+by_n(t)+cz_n(t)$ and therefore

$$V(ax_n+by_n+cz_n)\to V(ax_0+by_0+cz_0).$$

In particular,

$$(2)\qquad V(x_n) \to V(x_0), \qquad V(y_n) \to V(y_0), \qquad V(z_n) \to V(z_0).$$

Summability of each function $V_n(p)$ follows from Lemma 2; $V_n(p)$ converges on S to $V_0(p)$ by (1). In order to prove convergence of their integrals it will therefore be sufficient to show that the sequence $V_n(p)$ is bounded on S. For each point p on S we have by the Lemma 1 and (a) of §2

$$(3)\qquad V_n(p) \leqq L(\mathfrak{x}_n) \leqq V(x_n) + V(y_n) + V(z_n), \qquad n = 0, 1, \cdots .$$

But (2) implies the existence of a constant M such that

$$(4)\qquad V(x_n) \leqq M, \quad V(y_n) \leqq M, \quad V(z_n) \leqq M, \qquad n = 0, 1, \cdots .$$

Inequalities (3) and (4) establish the fact that the sequence $V_n(p)$ is bounded on S and hence, as remarked above,

$$\iint_S V_n(p)d\sigma \to \iint_S V_0(p)d\sigma.$$

LEMMA 4. *Under the hypotheses of Lemma* 2,

$$L(\mathfrak{x}) = (2\pi)^{-1}\iint_S V(p)d\sigma,$$

where $d\sigma$ is the area-element on S (this is the generalized Cauchy formula, see §3).

PROOF. Let us suppose first that $\mathfrak{x}(t)$ is AC. It is clear that for fixed p the function $f(t, p)$ is AC on I and that its derivative is equal to $\mathfrak{x}'(t)\cdot\mathfrak{u}$ except on a subset of I of measure zero. Hence (by (d) of §2)

$$(1)\qquad V(p) = \int_I |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, dt.$$

But absolute continuity of $\mathfrak{x}(t)$ also implies (see (g) of §2)

$$(2)\qquad L(\mathfrak{x}) = \int_I |\mathfrak{x}'(t)|\, dt.$$

Since $V(p)$ is summable on S (see Lemma 2), we obtain by use of (1)

$$(3)\qquad \begin{aligned}\iint_S V(p)d\sigma &= \iint_S\left[\int_I |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, dt\right]d\sigma\\ &= \int_I\left[\iint_S |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, d\sigma\right]dt,\end{aligned}$$

where the theorem of Tonelli justifies the changes in the order of integration. From (3), Lemma 1, and (2) we conclude that

$$(2\pi)^{-1}\int\!\!\int_S V(p)d\sigma = (2\pi)^{-1}\int_I\left[\int\!\!\int_S |\mathfrak{x}'(t)\cdot\mathfrak{u}|\, d\sigma\right]dt$$

$$= (2\pi)^{-1}\int_I (2\pi)\,|\mathfrak{x}'(t)|\,dt = L(\mathfrak{x}).$$

Let us suppose next that $\mathfrak{x}(t)$ is merely BV and continuous on I. Define $\mathfrak{x}=\mathfrak{x}_n(t)$, $t\in I$, $n=1, 2, \cdots$, to be a sequence of polygons inscribed in the curve $\mathfrak{x}=\mathfrak{x}(t)$, $t\in I$, $\mathfrak{x}_n(t)$ converging uniformly on I to $\mathfrak{x}(t)$. By (e) of §2 we then have $\mathfrak{x}_n\to\mathfrak{x}_0(L)$ and hence

$$L(\mathfrak{x}_n)\to L(\mathfrak{x}). \tag{4}$$

By the lemma of §2 we also have convergence in variation of the sequence $f_n(t, p)=\mathfrak{x}_n(t)\cdot\mathfrak{u}$, $t\in I$, for every point p on S. Application of Lemma 3 then yields the result that

$$\int\!\!\int_S V_n(p)d\sigma\to\int\!\!\int_S V(p)d\sigma, \tag{5}$$

where $V_n(p)$ is defined as in Lemma 3 for $n=1, 2, \cdots$. But since each approximating function $\mathfrak{x}_n(t)$ is AC, we can express its length in the form

$$L(\mathfrak{x}_n) = (2\pi)^{-1}\int\!\!\int_S V_n(p)d\sigma, \qquad n = 1, 2, \cdots.$$

In conjunction with (4) and (5) this implies

$$L(\mathfrak{x}) = \lim L(\mathfrak{x}_n) = \lim\,(2\pi)^{-1}\int\!\!\int_S V_n(p)d\sigma = (2\pi)^{-1}\int\!\!\int_S V(p)d\sigma.$$

4. **The theorem.** Let $\mathfrak{x}_n(t)=(x_n(t), y_n(t), z_n(t))$, $n=0, 1, \cdots$, be a sequence of vectors which are BV and continuous on I. Then $\mathfrak{x}_n\to\mathfrak{x}_0(L)$ if and only if the sequence $c_1x_n(t)+c_2y_n(t)+c_3z_n(t)$ converges in variation for every choice of the (real) constants c_1, c_2, c_3.

PROOF. *Sufficiency.* By hypothesis the sequence $c_1x_n(t)+c_2y_n(t)+c_3z_n(t)$ converges in variation for every choice of the constants c_1, c_2, c_3. This implies uniform convergence of $\mathfrak{x}_n(t)$ to $\mathfrak{x}_0(t)$ and also convergence in variation of $f_n(t, p)$ for every point p on S, where $f_n(t, p)$ is defined as in Lemma 3. From Lemmas 3 and 4 we then conclude that

$$\lim L(\mathfrak{x}_n) = \lim (2\pi)^{-1} \int\int_S V_n(p)d\sigma = (2\pi)^{-1} \int\int_S V_0(p)d\sigma = L(\mathfrak{x}_0)$$

and hence $\mathfrak{x}_n \to \mathfrak{x}_0(L)$.

Necessity. If $\mathfrak{x}_n \to \mathfrak{x}_0(L)$, the sequence $f_n(t, p)$ converges in variation for every point p on S (see Lemma of §2). Let c_1, c_2, c_3 be any three (real) constants. If $c_1 = c_2 = c_3 = 0$, then the statement is trivial. Otherwise let $\mathfrak{u}$ be a unit vector with direction cosines proportional to c_1, c_2, c_3. The desired relation now follows readily from the lemma and (b) of §2.

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ON UNIQUE INVARIANT MEASURES

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1. Statement of the problems. Let S be a σ-field[1] of subsets (measurable sets) of an abstract group G. What can be said about the structure of S if there is a unique measure defined on S and invariant under the group operation? What are the conditions for the uniqueness of an invariant measure? These are the problems studied in this note by means of a simple lemma.

2. Definitions and results.

DEFINITION 1. "*Measure*" means in this paper a non-negative, countably additive function of the set $X \in S$ such that G is not of measure 0 and is the union of a sequence of measurable sets of finite measure. For any two measures m and n we denote by $S_{m,n}$ the σ-field of subsets of $G \times G$, defined so as to allow the application of the generalized theorem of Fubini [1, p. 87].[2]

DEFINITION 2. A measure m is called *invariant* if $A \in S$, $g \in G$ implies $gA \in S$ and $m(gA) = m(A)$. An invariant measure is called *unique* if it differs from any other invariant measure only by a multiplicative constant.

FUNDAMENTAL ASSUMPTION. It is assumed that $g \in G$, $A \in S$ implies $Ag \in S$ and that any two invariant measures m, n satisfy the following *condition* M$_1$: The transformation $[(x, y) \rightarrow (y^{-1}x, y)]$ sends every set $A \times G$ with $A \in S$ into a set of $S_{m,n}$.

DEFINITION 3. A measurable set A is called *almost congruent by finite* (*resp. denumerable*) *partition* with the measurable set A' if there is a finite (resp. infinite) sequence of disjoint measurable subsets A_k of A with $m(A - \bigcup_k A_k) = 0$ and a corresponding sequence of elements g_k of G such that the sets $g_k^{-1}A_k$ are disjoint subsets of A' and $m(A' - \bigcup_k g_k^{-1}A_k) = 0$.

The answer to our first problem is given by the following theorem.

THEOREM 1. *If the measure is unique invariant then any measurable set A, whose measure is not greater than that of a measurable set B or equal to it, is almost congruent by finite or denumerable partition with some measurable subset of B or with B, respectively.*

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[1] That is, a class of sets containing G and closed under complementation and the formation of countable unions.

[2] Numbers in brackets refer to the bibliography at the end.

COROLLARY. *A unique invariant measure m accepts any of its values that are less than $m(B)$ on some subset of B.*[2]

The answer to our second problem is contained in the following theorem.

THEOREM 2. *In order that an invariant measure m be unique, it is necessary and sufficient that $m(Xg)$ be an absolutely continuous function of the measurable set X for every g in G.*

COROLLARY. *A bi-invariant measure is unique.*

To formulate two easy consequences of Theorem 2 we introduce the following definition.

DEFINITION 4. The measure m satisfies the *condition* M_2 (resp. M_2^*) if the transformation $[(x, y)\rightarrow(yx, y)]$ (resp. $[(x, y)\rightarrow(xy, y)]$) sends every set $A\times G$ with $A\in S$ into a set of $S_{m,m}$.

THEOREM 3. *Every invariant measure which satisfies the condition* M^2 *is unique.*

THEOREM 3*. *Every invariant measure which satisfies the condition* M_2^* *is unique.*

Theorem 3 has been found before by A. Weil [**2**, pp. 140–149]. Its new proof given here seems to be simpler and more natural than that of Weil. Theorem 3* is new and completes Weil's result.

3. **Two lemmas.** The proof of the above theorems relies upon 2 lemmas.

LEMMA 1. *Let n and m be measures satisfying the condition* M_1, *and A and B measurable sets. Then we have*

$$\int_B m(Ax^{-1})dn(x) = \int_G n(y^{-1}A \cap B)dm(y), \tag{3.1}$$

$$\int_B m(x^{-1}A)dn(x) = \int_G n(Ay^{-1} \cap B)dm(y). \tag{3.2}$$

Indeed, let H be the image of the set $A\times G$ by the transformation $[(x, y)\rightarrow(y^{-1}x, y)]$ and $f_1(x, y)$ resp. $f_2(x, y)$ the characteristic functions of $H\cap B\times G$ resp. $H\cap G\times B$.

Then the above relations result from the evaluation of the integrals of $f_1(x, y)$ resp. $f_2(x, y)$ over $G \times G$ by the generalized theorem of Fubini [2, p. 87].

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[2] It follows from [**4**] that these values either are multiples of a positive number or fill a closed interval (which can be infinite).

Remark. The relations (3.1), (3.2) remain valid for the group operation $x * y = y \cdot x$.[4] So do the theorems 1 and 2, as they will be deduced from Lemma 1.

LEMMA 2. *Let m be a measure such that $m(Xg)$ is an absolutely continuous function of the measurable set X for every g in G. Then:*

(a) *For any two measurable sets A, B of positive measures there is an element g such that $g^{-1}A$ meets B in a set of positive measure, provided that the measures m, m satisfy the condition* M_1;

(b) *$m(Z) = 0$ implies $n(g^{-1}Z) = 0$ for almost all elements g of G, provided that n is a measure such that n and m satisfy the condition* M_1.

PROOF. If m is substituted for n in (3.1), then the left side of (3.1) is positive. Therefore the integrand of the right side can not vanish identically, which is the assertion (a). If Z is substituted for A and G for B in (3.1), then both sides of (3.1) are zero, which implies the assertion (b).

4. Auxiliary theorem and proof of Theorem 1.

AUXILIARY THEOREM. *Let m be a measure such that $m(Xg)$ is an absolutely continuous function of the measurable set X for every g in G. Let A and B be measurable sets and let B be of finite measure. Then one of these two sets is almost congruent by finite or denumerable partition with some subset of the other.*

PROOF. Assume that no subset of one of these sets is almost congruent by finite partition with the other and that disjoint subsets A_k of A and elements g_k of G were chosen so that the sets $g_k^{-1}A_k = B_k$ are disjoint subsets of B for $k = 1, 2, \cdots, n$.

Then both sets $A - \bigcup_{k=1}^{n} A_k$, $B - \bigcup_{k=1}^{n} B_k$ are of positive measure. Therefore the function

$$F_{n+1}(g) = m\left[g^{-1}\left(A - \bigcup_{k=1}^{n} A_k\right) \cap \left(B - \bigcup_{k=1}^{n} B_k\right)\right]$$

has, by Lemma 2(a), a positive upper bound M_{n+1}. We define g_{n+1} to be any element g in G with $F_{n+1}(g) > M_{n+1}/2$ and put

$$B_{n+1} = g_{n+1}^{-1}\left(A - \bigcup_{k=1}^{n} A_k\right) \cap \left(B - \bigcup_{k=1}^{n} B_k\right), \qquad A_{n+1} = g_{n+1}B_{n+1}.$$

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[4] Indeed, the relations (3.1) and (3.2) for the new group operation $*$ are identical with the relations (3.2) and (3.1), respectively, for the old group operation.

Thus an infinite sequence of disjoint measurable subsets A_k of A and a corresponding sequence of elements g_k of G has been defined by induction so that the sets $g_k^{-1}A_k = B_k$ are disjoint subsets of B.

If

$$m\left(A - \bigcup_{k=1}^{\infty} A_k\right)\cdot m\left(B - \bigcup_{k=1}^{\infty} B_k\right)$$

were positive, the function

$$F(g) = m\left[g^{-1}\left(A - \bigcup_{k=1}^{\infty} A_k\right)\cap\left(B - \bigcup_{k=1}^{\infty} B_k\right)\right]$$

would have a positive upper bound M and as $F_k(g) \geq F(g)$ and $M_k \geq M > 0$ for every g in G, $k = 1, 2, \cdots$, the series $\sum_{k=1}^{\infty} M_k$ would be divergent. This is in contradiction to the inequalities $M_k/2 \leq F_k(g_k) = m(B_k)$, $\sum_{k=1}^{\infty} m(B_k) \leq m(B) < \infty$.

Hence either $A - \bigcup_{k=1}^{\infty} A_k$ or $B - \bigcup_{k=1}^{\infty} B_k$ is of measure 0, which proves the assertion.

4.1. PROOF OF THEOREM 1. As $m(Xg)$ is an invariant measure for every fixed g in G, it can differ from $m(X)$ only by a multiplicative constant and is, therefore, an absolutely continuous function of the measurable set X.

First case: $m(A)$ is finite. The inequality $m(A) \leq m(B)$ and the invariance of m exclude the existence of a subset A' of A almost congruent with B by either finite or denumerable partition (symbolically: $A' \approx B$) and such that $m(A') < m(A)$. Therefore there is, by the auxiliary theorem, a subset B' of B with $A \approx B'$. If $m(B) = m(A)$, then we have $m(B - B') = 0$, which implies $A \approx B$.

Second case. $m(A)$ is infinite. Then A is the union of a sequence of disjoint measurable sets A_i of finite measure, and there is a subset B_1 of B with $A_1 \approx B_1$. The measure of $B - B_1$ being infinite, there is a subset B_2 of $B - B_1$ with $A_2 \approx B_2$. By continuing this reasoning one proves the existence of a measurable subset B' of B with $A \approx B'$. As $m(B)$ is infinite too, there is also a subset A' of A with $B \approx A'$. From $A \approx B' \subset B$ and $B \approx A' \subset B$ one deduces $A \approx B$ by replacing in the proof of F. Bernstein's equivalence theorem [**3**, p. 27] the equivalence relation by the relation $\approx$. This is legitimate as the transitivity of the relation $\approx$ is not difficult to prove.

5. Proof of Theorem 2.

Necessity. See §4.

Sufficiency. Let $m(X)$ and $n(X)$ be invariant measures and $m(Xg)$ an absolutely continuous function of the measurable set X for every

g in G. Let X_0 be a measurable set with $0<m(X_0)<\infty$ and with $c=n(X_0)/m(X_0)<\infty$.

(5.1) *$n(Z)=0$ implies $n(Z)=cm(Z)$.*

In fact, by (3.1) (with $A=Z, B=G$), $m(Zx^{-1})$ vanishes for some x, which implies $m(Z)=m[(Zx^{-1})x]=0$.

(5.2) *$0<n(Z)<\infty$ and $n(Z)=dm(Z)$ implies $d\leqq c$.*

Indeed, one sees from Lemma 2(b) that $m(Z)=0$ implies $n(Z)=0$. Therefore $n(Y)$ is an absolutely continuous additive function of the measurable set $Y\subset Z\cup X_0$. By the theorem of Radon-Nikodym [1, p. 36] there is, therefore, a function $f(y)$ such that we have $n(Y)=\int_Y f(y)dm(y)$ for every measurable set $Y\subset Z\cup X_0$.

Assume that c is less than d. Then there are numbers a, $b>a$ between c and d and the sets

$$A=X_0\cap \underset{y}{E}\,[f(y)\leqq a],\qquad B=Z\cap \underset{y}{E}\,[f(y)\geqq b]$$

are of positive measure. Therefore, there is (by Lemma 2(a)) an element g of G such that $m(g^{-1}A\cap B)>0$. As $g^{-1}A\cap B=C$ is a subset of B, we have

$$n(C)=\int_C f(y)dm(y)\geqq bm(C).$$

On the other hand, gC being a subset of A, we have

$$n(gC)=\int_{gC} f(y)\,dm(y)\leqq am(gC)=am(C)<bm(C).$$

Hence $n(gC)<n(C)$, although n is invariant.

(5.3) *$0<n(Z)<\infty$ implies $n(Z)=cm(Z)$.*

Indeed, by (5.2) the relations $n(X_0)=cm(X_0)$, $n(Z)=dm(Z)$ imply $d\leqq c$. By interchanging Z and X_0 we get $c=d$.

Finally, if Z is an arbitrary measurable set, it is the union of a sequence of disjoint sets Z_i, with $n(Z_i)<\infty$. By (5.1) and (5.3) we have $n(Z_i)=cm(Z_i)$ hence $n(Z)=cm(Z)$, which was to be proved.

6. Proof of Theorems 3 and 3*. Extension to transformation groups.

(6.1) *If the measure m satisfies the condition M_2 resp. M_2^*, then $A\in S$ implies the relation*

$$\int_G m(xA^{-1})dm(x)=\int_G m(yA)dm(y) \tag{6.2}$$

resp.

$$(6.2^*) \qquad \int_G m(A^{-1}x)dm(x) = \int_G m(Ay)dm(y).$$

Indeed, let H be the image of the set $A\times G$ by the transformation $[(x, y)\rightarrow(yx, y)]$ resp. $[(x, y)\rightarrow(xy, y)]$. The evaluation of the integral of the characteristic function of H over $G\times G$ by the generalized theorem of Fubini furnishes the above relations.

If m is an invariant measure which satisfies the condition M_2, then $m(A)=0$ implies $m(A^{-1})=0$ on account of (6.2). Hence we have $m(g^{-1}A^{-1})=m(A^{-1})=0$ and $m(Ag)=m[(g^{-1}A^{-1})^{-1}]=0$ for every g in G, which—according to Theorem 2—proves Theorem 3.

If m and n are two invariant measures, which both satisfy the condition M_2^*, then $A\in S$ implies $A^{-1}\in S$ (by (6.2*) and the fundamental assumption).[5] Then the functions of A, $m^*(A)=m(A^{-1})$, $n^*(A)=n(A^{-1})$, are readily seen to be measures invariant under the group operation $x^*y=yx$. As M_2^* is the condition M_2 for this group operation, n^* differs from m^* only by a multiplicative constant. Therefore n differs from m by the same constant.

(6.3) The above results can be extended to measures invariant under any transitive group G of transformations operating on a measure space M, by inducing either a measure structure in G or a group structure in M.

To do this one denotes by t_x the transformation which sends a fixed element x_0 in M into x in M. Then the images $t_x\{A\}$ of the measurable subsets A of M by the mapping t_x of M onto G form a σ-field, on which one defines a measure μ by the relation

$$\mu(t_x\{A\}) = m(A).^6$$

A group structure can be induced in M by putting either $xy=t_xy$ or $xy=t_yx$.

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[5] By (6.2*), $A^{-1}x$ is in S for almost all x, hence $A^{-1}=(A^{-1}x)x^{-1}$ is in S too.

[6] This is a generalization of an oral remark of Prof. H. Hadwiger, Bern, communicated to the author by Prof. H. Hopf, Zurich.

A NOTE ON LACUNARY POLYNOMIALS

MORRIS MARDEN

1. **Introduction.** In the present note we shall give an elementary derivation of some new bounds for the p smallest (in modulus) zeros of the polynomials of the lacunary type

$$(1.1)\quad f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k}, \qquad a_0 a_p \neq 0,\ 0 < p = n_0 < n_1 < \cdots < n_k.$$

This will be done by the iterated application, first, of Kakeya's Theorem[1] that, if a polynomial of degree n has p zeros in a circle C of radius R, its derivative has at least $p-1$ zeros in the concentric circle C' of radius $R' = R\phi(n, p)$; and, secondly, of the specific limits

$$(1.2)\qquad \phi(n, p) \leqq \csc\,[\pi/2(n - p + 1)],$$

$$(1.3)\qquad \phi(n, p) \leq \prod_{j=1}^{n-p} (n + j)/(n - j)$$

furnished by Marden[2] and Biernacki[3] respectively.

2. **Derivation of the bounds.** An immediate corollary to Kakeya's Theorem is:

THEOREM I. *If the derivative of an nth degree polynomial* $P(z)$ *has at most* $p-1$ *zeros in a circle* Γ *of radius* ρ, *then* $P(z)$ *has at most* p *zeros in the concentric circle* Γ' *of radius* $\rho' = \rho/\phi(n, p+1)$.

We shall use Theorem I to prove the following theorem.

THEOREM II. *If all the zeros of the polynomial*

$$(2.1)\quad \begin{aligned} f_0(z) = {} & n_1 n_2 \cdots n_k a_0 + (n_1 - 1)(n_2 - 1) \cdots (n_k - 1) a_1 z \\ & + \cdots + (n_1 - p)(n_2 - p) \cdots (n_k - p) a_p z^p \end{aligned}$$

lie in the circle $|z| \leqq R_0$, *at least* p *zeros of polynomial* (1.1) *lie in the circle*

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[1] S. Kakeya, Tôhoku Math. J. vol. 11 (1917) pp. 5–16.

[2] M. Marden, Trans. Amer. Math. Soc. vol. 45 (1939) pp. 335–368. See also M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, to be published as a volume of Mathematical Surveys.

[3] M. Biernacki, Bull. Soc. Math. France (2) vol. 69 (1945) pp. 197–203.

$$|z| \leqq R(p, k) = R_0 \prod_{i=1}^{k} \phi(n_i, n_i - p + 1).$$

For this purpose we define the sequence of polynomials

$$F_0(z) = z^{n_k} f(1/z), \tag{2.2}$$

$$F_j(z) = z^{1-n_{k-j+1}+n_{k-j}} F'_{j-1}(z), \qquad j = 1, 2, \cdots, k. \tag{2.3}$$

We may verify easily that

$$F_k(z) = z^p f_0(1/z). \tag{2.4}$$

All the zeros of $F_k(z)$ therefore lie outside the circle $|z| \geqq (1/R_0)$. By equation (2.3), the zeros of $F'_{k-1}(z)$ are the zeros of $F_k(z)$ and a zero of multiplicity $n_1 - p - 1$ at the origin and, hence, only the latter lies inside $|z| < 1/R_0$. By Theorem I, $F_{k-1}(z)$ has at most $n_1 - p$ zeros in

$$|z| < [R_0 \phi(n_1, n_1 - p + 1)]^{-1} = 1/R(p, 1).$$

Let us now assume, as already verified for $j = 1, 2, \cdots, s$, that $F_{k-j}(z)$ has at most $n_j - p$ zeros in the circle $|z| < 1/R(p, j)$. From equations (2.3) with j replaced by $k - s$, it follows then that $F'_{k-s-1}(z)$ has zeros of total multiplicity at most

$$(n_{s+1} - n_s - 1) + (n_s - p) = n_{s+1} - p - 1$$

in this circle. By Theorem I, therefore, $F_{k-s-1}(z)$ has at most $n_{s+1} - p$ zeros in the circle

$$|z| < [R(p, s)\phi(n_{s+1}, n_{s+1} - p + 1)]^{-1} = 1/R(p, s + 1).$$

By mathematical induction, it follows that $F_0(z)$ has at most $n_k - p$ zeros in the circle $|z| < 1/R(p, k)$.

By (2.2), $f(z)$ has therefore at most $n_k - p$ zeros outside the circle $|z| = R(p, k)$ and hence at least p zeros in or on this circle.

By using the limits (1.2) and (1.3), we now deduce from Theorem II the following corollary.

COROLLARY 1. *At least p zeros of polynomial* (1.1) *lie in each of the circles*

$$|z| \leqq R_0 \csc^k (\pi/2p), \tag{2.5}$$

$$|z| \leqq R_0 \prod_{i=1}^{k} \prod_{j=1}^{p-1} (n_i + j)/(n_i - j). \tag{2.6}$$

If it is known that all the zeros of the polynomial

$$h(z) = a_0 + a_1 z + \cdots + a_p z^p$$

lie in the circle $|z| \leqq R_1$, then the application of a theorem in a previous paper[4] permits us to take

$$R_0 \leqq [R_1 n_1 n_2 \cdots n_k/(n_1 - p)(n_2 - p) \cdots (n_k - p)] = R_2.$$

As (2.6) with R_0 replaced by R_2 is the bound furnished recently by Biernacki,[3] we see that the bound (2.6) is at least as good as his bound.

3. Application to lacunary series. We shall now use Corollary 1 to prove the following theorem.

THEOREM III. *Let* ρ_1, $0 < \rho_1 \leqq \infty$, *be the radius of convergence of the series*

$$g(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots,$$
$$a_0 a_p \neq 0, \qquad 1 \leqq p < n_1 < n_2 < \cdots.$$

Let the series $\sum(1/n_j)$ *be convergent, so that the product*

$$A(m) = \prod_{j=1}^{\infty} [1 - (m/n_j)]$$

is also convergent. Let ρ, *the radius of the circle* $|z| = \rho$ *containing all the zeros of the polynomial*

$$G(z) = A(0)a_0 + A(1)a_1 z + \cdots + A(p)a_p z^p,$$

be such that

$$\rho \prod_{j=1}^{p-1} A(-j)/A(j) = \rho_2 < \rho_1.$$

Then $g(z)$ *has at least* p *zeros in the circle* $|z| \leqq \rho_2$.

Let us consider equations (1.1) and (2.1) as defining the sequences of polynomials $f(z, k)$ and $f_0(z, k)$ respectively. When $k \to \infty$, the sequence $[f_0(z, k)/n_1 n_2 \cdots n_k]$ converges uniformly to $G(z)$ in $|z| \leqq \rho$. By Hurwitz' theorem, for any given positive ϵ, we may choose a positive k_1 so large that all the zeros of each $f_0(z, k)$, $k \geqq k_1$, lie in the circle $|z| \leqq \rho + \epsilon$. By Corollary 1, at least p zeros of the $f(z, k)$, $k \geqq k_1$, lie in the circle

$$|z| \leqq (\rho + \epsilon) \prod_{j=1}^{p-1} \prod_{i=1}^{k} (1 + j/n_i)/(1 - j/n_i) < \rho_2 + \epsilon(\rho_2/\rho) = \rho_2'.$$

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[4] M. Marden, Bull. Amer. Math. Soc. vol. 49 (1943) p. 97, Corollary.

Choosing ϵ so small that $\rho_2' + \epsilon < \rho_1$, we see that the $f(z, k)$ converge uniformly to $g(z)$ in $|z| \leqq \rho_2'$. Thus $g(z)$ has p zeros in the circle $|z| < \rho_2' + \epsilon$ and, since ϵ is arbitrary, in the circle $|z| \leqq \rho_2$.

As a corollary to Theorem III, we may prove that, if $g(z)$ is an entire function, it assumes every finite value an infinite number of times.

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A REFINEMENT OF PELLET'S THEOREM

MORRIS MARDEN

1. **Introduction.** S. Lipka[1] has recently announced a refinement of the classic theorem of Cauchy that all the zeros of the polynomial

$$(1.1)\qquad f(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_n \neq 0,$$

lie in the circle $|z| \leqq r$, where r is the positive root of the real equation

$$(1.2)\qquad F_n(z) = |a_0| + |a_1| z + \cdots + |a_{n-1}| z^{n-1} - |a_n| z^n = 0.$$

Lipka's refinement consists in replacing the circle $|z| = r$ by a curve $G(r_0, r; n, \alpha_0)$ which bounds a gear-wheel region. This region is formed by deleting from the circle $|z| \leqq r$ the points common to the annular ring $0 < r_0 < |z| \leqq r$ and to the n sectors

$$(1.3)\qquad \frac{\alpha_0}{n} - \frac{\pi}{2n} + \frac{2\pi k}{n} \leqq \arg z \leqq \frac{\alpha_0}{n} + \frac{\pi}{2n} + \frac{2\pi k}{n},$$

$k = 0, 1, \cdots, n-1$. In these formulas r_0 is the positive root of the equation

$$(1.4)\qquad \Phi_n(z) = |a_1| + |a_2| z + \cdots + |a_{n-1}| z^{n-2} - |a_n| z^{n-1} = 0$$

and $\alpha_0 = \arg a_0/a_n$.

Now, the Cauchy theorem is but a special case of the following theorem due to Pellet.[2]

PELLET'S THEOREM. *If the polynomial*

$$(1.5)\qquad f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n, \qquad a_p \neq 0,$$

is such that the real polynomial

$$(1.6)\qquad \begin{aligned} F_p(z) = {} & |a_0| + |a_1| z + \cdots + |a_{p-1}| z^{p-1} - |a_p| z^p \\ & + |a_{p+1}| z^{p+1} + \cdots + |a_n| z^n \end{aligned}$$

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[1] S. Lipka, Monatshefte für Mathematik und Physik vol. 50 (1944) pp. 209–221.

[2] A. Pellet, Bull. Sci. Math. vol. 5 (1881) pp. 393–395. The converse to this theorem was discussed by J. L. Walsh, Ann. of Math. vol. 26 (1924–1925) pp. 59–64 and A. Ostrowski, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 742–746. See also M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, chap. 7, to be published as a volume of Mathematical Surveys.

has two positive zeros r and R with $r<R$, then $f(z)$ has exactly p zeros in or on the circle $|z|\leq r$ and no zeros in the annular ring $r<|z|<R$.

It is Pellet's theorem which we propose to refine as indicated in the following theorem.

THEOREM 1.1. *Under the hypotheses of Pellet's Theorem the polynomial*

$$\begin{aligned}\Phi_p(z) = {} & |a_1| + |a_2|z + \cdots + |a_{p-1}|z^{p-2} - |a_p|z^{p-1} \\ & + |a_{p+1}|z^p + \cdots + |a_n|z^{n-1}\end{aligned} \tag{1.7}$$

has also two positive zeros r_0 and R_0 with

$$r_0 < r < R < R_0. \tag{1.8}$$

Furthermore, $f(z)$ has exactly p zeros in or on the curve $G(r_0, r; p, \alpha_0)$ where $\alpha_0 = \arg a_0/a_p$ and no zeros between the curves $G(r_0, r; p, \alpha_0)$ and $G(R, R_0; p, \alpha_0+\pi)$.

Theorem 1.1 will be proved in §2 and applied in §3 to the refinement of various known bounds on the zeros of a polynomial. Finally, the theorem will be generalized in §4, first by replacing the polynomial $\Phi_p(z)$ by the polynomial $\Phi_{kp}(z) = F_p(z) - |a_k|z^k$ and secondly by replacing the polynomial $f(z)$ by a power series.

2. **Proof of Theorem 1.1.** Let us first prove the existence of the roots r_0 and R_0 of equation $\Phi_p(z) = 0$ and the validity of inequality (1.8). Since r and R are the positive zeros of $F_p(z)$, it follows from (1.6) that, for any sufficiently small positive number ϵ,

$$F_p(\rho) < 0 \qquad \text{if } r+\epsilon \leq \rho \leq R-\epsilon. \tag{2.1}$$

In view of the equation

$$F_p(z) = |a_0| + z\Phi_p(z), \tag{2.2}$$

the zeros r and R of $F_p(z)$ satisfy the relations

$$\Phi_p(r) = -|a_0|/r < 0, \qquad \Phi_p(R) = -|a_0|/R < 0. \tag{2.3}$$

When taken together with the facts that

$$\Phi_p(0) > 0, \qquad \Phi_p(\infty) > 0, \tag{2.4}$$

the relations (2.3) imply the existence of two positive zeros r_0 and R_0 of $\Phi_p(z)$ and the validity of inequality (1.8), as well as the inequality

$$\Phi_p(\rho) < 0 \tag{2.5}$$

for $r_0+\epsilon \leq \rho \leq R_0 - \epsilon$.

Let us now set $z = \rho e^{i\theta}$ and

$$a_k/a_p = A_k e^{\alpha_k i}, \qquad k = 0, 1, \cdots, n. \tag{2.6}$$

In this notation, the real part of $\rho^p f(z)/a_p z^p$ is

$$\begin{aligned} \text{Re}\,[\rho^p f(z)/a_p z^p] = \sum_{j=0}^{p-1} A_j \rho^j \cos\,[(p-j)\theta - \alpha_j] + \rho^p \\ + \sum_{j=p+1}^{n} A_j \rho^j \cos\,[(j-p)\theta + \alpha_j] \end{aligned} \tag{2.7}$$

and the inequalities (2.1) and (2.5) become

$$\rho^p > A_0 + A_1\rho + \cdots + A_{p-1}\rho^{p-1} + A_{p+1}\rho^{p+1} + \cdots + A_n\rho^n \tag{2.8}$$

for $r+\epsilon \leqq \rho \leqq R-\epsilon$, and

$$\rho^p > A_1\rho + A_2\rho^2 + \cdots + A_{p-1}\rho^{p-1} + A_{p+1}\rho^{p+1} + \cdots + A_n\rho^n \tag{2.9}$$

for $r_0+\epsilon \leqq \rho \leqq R_0-\epsilon$.

On substituting from inequality (2.8) into (2.7), we find

$$\begin{aligned} \text{Re}\,(\rho^p f(z)/a_p z^p) > \sum_{j=0}^{p-1} A_j \rho^j \{\cos\,[(p-j)\theta - \alpha_j] + 1\} \\ + \sum_{j=p-1}^{n} A_j \rho^j \{\cos\,[(j-p)\theta + \alpha_j] + 1\} \geqq 0 \end{aligned} \tag{2.10}$$

for $r+\epsilon \leqq \rho \leqq R-\epsilon$. On substituting from inequality (2.9) into (2.7), we find

$$\begin{aligned} \text{Re}\,(\rho^p f(z)/a_p z^p) > A_0 \cos\,(p\theta - \alpha_0) \\ + \sum_{j=1}^{p-1} A_j \rho^j \{\cos\,[(p-j)\theta - \alpha_j] + 1\} \\ + \sum_{j=p-1}^{n} A_j \rho^j \{\cos\,[(j-p)\theta + \alpha_j] + 1\} \end{aligned} \tag{2.11}$$

for $r_0+\epsilon \leqq \rho \leqq R_0-\epsilon$. The right side of (2.11) is surely non-negative if θ is such that $\cos\,(p\theta-\alpha_0) \geqq 0$, that is, such that

$$-\frac{\pi}{2} + 2\pi k \leqq p\theta - \alpha_0 \leqq \frac{\pi}{2} + 2\pi k,$$

where k is an integer; that is, if

$$\frac{\alpha_0}{p} - \frac{\pi}{2p} + \frac{2\pi k}{p} \leqq \theta \leqq \frac{\alpha_0}{p} + \frac{\pi}{2p} + \frac{2\pi k}{p}, \qquad k = 0, 1, \cdots, p.$$

In other words,

$$\text{Re}\,(\rho^p f(z)/a_p z^p) > 0 \tag{2.12}$$

and hence $f(z)\neq 0$ at all points z between the curves $G(r_0, r; p, \alpha_0)$ and $G(R, R_0; p, \alpha_0+\pi)$.

Inequality (2.12) also may be used to show that in or on the curve $G(r_0, r; p, \alpha_0)$, there are exactly p zeros of $f(z)$. For, let us consider the net change $\Delta_{G_\epsilon} \arg w$ in the argument of the point $w = [\rho^p f(z)/a_p z^p]$ as z describes counterclockwise the curve $G_\epsilon = G(r_0+\epsilon, r+\epsilon; p, \alpha_0)$ where ϵ is a small positive number. Since Re $(w)>0$, w describes a closed curve entirely in the right-half w-plane. That is, $\Delta_{G_\epsilon} \arg w = 0$ on this curve. This means that the function w has as many zeros as poles in the curve G_ϵ and this, in turn, means that $f(z)$ has precisely p zeros in G_ϵ for every sufficiently small positive ϵ.

3. **Applications.** Let us first apply Theorem 1.1 to the class of polynomials

$$\begin{aligned} f(z) = {} & b_0 e^{i\phi_0} + (b_1 - b_0)e^{i\phi_1} z + \cdots \\ & + (b_m - b_{m-1})e^{i\phi_m} z^m - b_m e^{i\phi_{m+1}} z^{m+1} \end{aligned} \tag{3.1}$$

where the b_j are real numbers such that

$$b_{p-1} < b_{p-2} < \cdots < b_0 < 0 < b_m < b_{m-1} < \cdots < b_p. \tag{3.2}$$

The corresponding polynomials $F_p(z)$ and $\Phi_p(z)$ are

$$\begin{aligned} F_p(z) = {} & -b_0 + (b_0 - b_1)z + \cdots + (b_{p-2} - b_{p-1})z^{p-1} \\ & - (b_p - b_{p-1})z^p + (b_p - b_{p+1})z^{p+1} + \cdots \\ & + (b_{m-1} - b_m)z^m + b_m z^{m+1}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Phi_p(z) = {} & (b_0 - b_1) + \cdots + (b_{p-2} - b_{p-1})z^{p-2} \\ & - (b_p - b_{p-1})z^{p-1} + (b_p - b_{p+1})z^p + \cdots \\ & + (b_{m-1} - b_m)z^{m-1} + b_m z^m. \end{aligned} \tag{3.4}$$

On defining

$$g(z) = b_0 + b_1 z + \cdots + b_m z^m, \tag{3.5}$$

we may write

$$F_p(z) = (z-1)g(z), \qquad z\Phi_p(z) = b_0 + g(z)(z-1).$$

Clearly $F_p(1) = 0$. Since $F_p(1+\delta) = \delta g(1+\delta)$, then for δ sufficiently small $g(1) > 0$ implies that $F_p(1+\delta) > 0$ or < 0 according as $\delta > 0$ or < 0 and $g(1) < 0$ implies that $F_p(1+\delta) < 0$ or > 0 according as $\delta > 0$ or

<0. That is, using the notation of Theorem 1.1, we see that

$$\begin{aligned} r_0 < r < 1 = R < R_0 &\qquad \text{if } g(1) > 0, \\ r_0 < r = 1 < R < R_0 &\qquad \text{if } g(1) < 0, \\ \alpha_0 = \beta_0 - \beta_p + \pi. & \end{aligned}$$

We thereby conclude that the following is true.

THEOREM 3.1. *Let* $f(z)$, $\Phi_p(z)$ *and* $g(z)$ *denote the polynomials* (3.1), (3.4) *and* (3.5) *respectively. Then, if* $g(1)>0, f(z)$ *has exactly* p *zeros in the curve* $G(r_0, 1; p, \beta_0-\beta_p+\pi)$ *and* $g(z)$ *has* p *zeros in the curve* $G(r_0, 1; p, \pi)$. *If* $g(1)<0$, $f(z)$ *has* p *zeros in or on the curve* $G(r_0, 1; p, \beta_0-\beta_p+\pi)$ *and* $g(z)$ *has* $p-1$ *zeros in or on the curve* $G(r_0, 1; p, \pi)$.

An analogous result for $g(z)$ with, however, curve $G(r_0, 1; p, \pi)$ replaced by the circle $|z|=1$ was first stated by Berwald.[3] His result was a generalization of the Kakeya-Eneström[4] theorem that all the zeros of the real polynomial (3.5) with $0<b_0<b_1< \cdots <b_n$ lie in or on the unit circle $|z|=1$. Our analogy to the Kakeya-Eneström theorem will be included in the following theorem.

THEOREM 3.2. *Every polynomial of the form*

$$f(z) = \sum_{j=0}^{n} (b_j - b_{j-1}) e^{i\beta_j} z^j, \qquad b_{-1} = b_n = 0 < b_0 < b_1 < \cdots < b_{n-1},$$

has all of its zeros in or on the curve $G(r_0, 1; n, \beta_0-\beta_n+\pi)$ *where* r_0 *is the positive root of the equation*

$$\Phi_n = (b_1 - b_0) + (b_2 - b_1)z + \cdots + (b_{n-1} - b_{n-2})z^{n-2} - b_{n-1}z^{n-1} = 0.$$

Furthermore, every polynomial of the form

$$g(z) = b_0 + b_1 z + \cdots + b_{n-1}z^{n-1}, \qquad 0 < b_0 < b_1 < \cdots < b_{n-1},$$

has all of its zeros in or on the curve $G(r_0, 1; n, \pi)$.

This theorem may be derived from Theorem 3.1 indirectly by a limiting process or directly by the same methods as used for Theorem 3.1.

In our next application, we shall use Theorem 1.1 just in the case $p=n$. This restriction is made only to simplify the statement of results, since a similar application may be made when p is an arbitrary integer, $0<p\leqq n$. The result to be proved is the following.

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[3] L. Berwald, Math. Zeit. vol. 37 (1933) pp. 61–76.

[4] S. Kakeya, Tôhoku Math. J. vol. 2 (1912) pp. 140–142 and G. Eneström, Ibid. vol. 18 (1920) pp. 34–36.

THEOREM 3.3. *Let* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *and* $\mu_1, \mu_2, \cdots, \mu_{n-1}$ *be any two sets of numbers such that*

$$\sum_{j=1}^{n} (1/\lambda_j) = 1, \quad \sum_{j=1}^{n-1} (1/\mu_j) = 1; \qquad 0 < \mu_j \leqq \lambda_j,\ j = 1, 2, \cdots, n-1.$$

For the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, *let*

$$M = \max\, [\lambda_k |a_{n-k}| / |a_n|]^{1/k}, \qquad k = 1, 2, \cdots, n, \tag{3.6}$$

$$M_0 = \max\, [\mu_k |a_{n-k}| / |a_n|]^{1/k}, \qquad k = 1, 2, \cdots, n-1. \tag{3.7}$$

Then all the zeros of $f(z)$ *lie in or on the curve* $G(M_0, M; n, \alpha_0)$, *where* $\alpha_0 = \arg\,(a_0/a_n)$.

From (3.6) and (3.7), obviously $0 < M_0 < M$. Also,

$$\lambda_k |a_{n-k}| \leqq |a_n| M^k, \qquad \mu_k |a_{n-k}| \leqq |a_n| M_0^k$$

and thus

$$\sum_{k=1}^{n} |a_{n-k}| M^{n-k} \leqq \sum_{k=1}^{n} (1/\lambda_k) |a_n| M^n = |a_n| M^n, \tag{3.8}$$

$$\sum_{k=1}^{n-1} |a_{n-k}| M_0^{n-k} \leqq \sum_{k=1}^{n-1} (1/\mu_k) |a_n| M_0^n = |a_n| M_0^n. \tag{3.9}$$

An equality in (3.8) would imply that M is the positive root r of the equation (1.2) whereas an inequality in (3.8) would imply that $M > r$. Likewise, an equality in (3.9) would imply that M_0 is the positive root r_0 of the equation (1.4) whereas an inequality in (3.9) would imply that $M_0 > r_0$. Since by Theorem 1.1 all the zeros of $f(z)$ lie in or on the curve $G(r_0, r; n, \alpha_0)$, they surely all lie in or on the curve $G(M_0, M; n, \alpha_0)$.

Theorem 3.3 whose proof we have just completed is a refinement of the result due to Fujiwara[5] that all the zeros of $f(z)$ lie in or on the circle $|z| \leqq M$.

As a simple application of Theorem 3.3, let us take $\lambda_j = n$ for $j = 1, 2, \cdots, n$ and $\mu_j = n-1$ for $j = 1, 2, \cdots, n-1$. We obtain thereby the following corollary.

COROLLARY 3.3a. *For the polynomial* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ *let* $N = \max\, [n|a_{n-k}/a_n|]^{1/k}$, $k = 1, 2, \cdots, n$, *and* $N_0 = \max\, [(n-1)|a_{n-k}/a_n|]^{1/k}$, $k = 1, 2, \cdots, n-1$. *Then all the zeros of* $f(z)$ *lie in or on the curve* $G(N_0, N; n, \alpha_0)$ *where* $\alpha_0 = \arg\,(a_0/a_n)$.

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[5] M. Fujiwara, Tôhoku Math. J. vol. 10 (1916) pp. 167–171.

As another simple application of Theorem 3.3, let us take

$$\lambda_k = \sum_{j=0}^{n-1} | a_j | / | a_{n-k} |, \qquad k = 0, 1, 2, 3, \cdots, n,$$

$$\mu_k = \sum_{j=1}^{n-1} | a_j | / | a_{n-k} |, \qquad k = 0, 1, 2, \cdots, n - 1.$$

Clearly,

$$\sum_{j=1}^{n} 1/\lambda_j = 1, \qquad \sum_{j=1}^{n-1} 1/\mu_j = 1.$$

Here

$$M = \max \left[\sum_{j=0}^{n-1} | a_j | / | a_n | \right]^{1/k} = \lambda_0 \quad \text{or} \quad \lambda_0^{1/n}$$

according as $\lambda_0 > 1$ or < 1, and

$$M_0 = \max \left[\sum_{j=1}^{n-1} | a_j | / | a_n | \right]^{1/k} = \lambda_0 \quad \text{or} \quad \mu_0^{1/n}$$

according as $\mu_0 > 1$ or < 1. We thereby obtain the following corollary.

COROLLARY 3.3b. *For the polynomial* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, *let*

$$\lambda_0 = \sum_{j=0}^{n-1} | a_j | / | a_n | \quad \textit{and} \quad \mu_0 = \sum_{j=1}^{n-1} | a_j | / | a_n |.$$

Let $\gamma = \lambda_0$ *or* $\lambda_0^{1/n}$ *according as* $\lambda_0 > 1$ *or* < 1, *and let* $\delta = \mu_0$ *or* $\mu_0^{1/n}$ *according as* $\mu_0 > 1$ or < 1. *Then all the zeros of* $f(z)$ *lie in or on the curve* $G(\delta, \gamma; n, \alpha_0)$ *where* $\alpha_0 = \arg a_0/a_n$.

4. **Generalizations.** Let us define $\Psi_{kp}(z) = F_p(z) - |a_k| z^k$, $k \neq p$. Since $\Psi_{0p}(z) = z\Phi_p(z)$, the positive zeros of $\Phi_p(z)$ are also the positive zeros of $\Psi_{0p}(z)$. By modifying somewhat the details of proof of Theorem 1.1, we may prove the following generalization.

THEOREM 4.1. *Under the hypotheses of Pellet's Theorem the polynomial*

$$\Psi_{kp}(z) = F_p(z) - | a_k | z^k, \qquad k \neq p, \; a_k \neq 0,$$

has also two positive zeros r_k *and* R_k *with* $r_k < r < R < R_k$. *Furthermore* $f(z)$ *has exactly* p *zeros in or on the curve* $G(r_k, r; p-k, \alpha_k)$ *where* $\alpha_k = \arg (a_k/a_p)$ *and none between the curves* $G(r_k, r; p-k, \alpha_k)$ *and* $G(R, R_k; p-k, \alpha_k + \pi)$.

Our final generalization will consist in replacing the polynomial $f(z)$ of Theorem 4.1 by a power series.

THEOREM 4.2. *If the power series*

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots, \qquad a_k a_p \neq 0,$$

having a radius of convergence of ρ, $0 < \rho \leqq \infty$, *is such that each polynomial*

$$F_{np}(z) = |a_0| + |a_1| z + \cdots + |a_{p-1}| z^{p-1} - |a_p| z^p + |a_{p+1}| z^{p+1} + \cdots + |a_n| z^n$$

with $n \geqq N > p$ *has a positive zero* $r^{(n)}$, $r^{(n)} \leqq \rho_1 < \rho$, *then the function* $F_p(z) = \lim_{n\to\infty} F_{np}(z)$ *has a positive zero* $r < \rho$; *the function*

$$\Psi_{kp}(z) = F_p(z) - |a_k| z^k, \qquad k \neq p;$$

has a positive zero r_k, $r_k < r < \rho$, *and the function* $f(z)$ *has exactly* p *zeros in or on the curve* $G(r_k, r; p-k, \alpha_k)$ *and hence in the curve* $G(r_k, \rho; p-k, \alpha_k)$.

This theorem results from Theorem 4.1 on the use of the Hurwitz theorem that within its circle of convergence a non-constant power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$ has as zeros the limit points of the zeros of the polynomials $f_n(z) = \sum_{j=0}^{n} a_j z^j$.

If $F_{np}(z)$ has two positive zeros in $|z| < \rho$, we may choose $r^{(n)}$ as the smaller one. Letting

$$\Psi_{nkp}(z) = F_{np}(z) - |a_k| z^k,$$

we see that $\Psi_{nkp}(z)$ has a positive zero $r_k^{(n)}$, $r_k^{(n)} < r^{(n)}$. Clearly, the power series $F_p(z)$ and $\Psi_{kp}(z)$ have the same radius ρ of convergence and have respectively the positive zeros $r = \lim_{n\to\infty} r^{(n)}$ and $r_k = \lim_{n\to\infty} r_k^{(n)}$, with $r_k < r < \rho$. Now, given any small positive ϵ, we can find an $N > 0$ such that the circle of radius ϵ drawn about the point $z = r$ will contain $r^{(n)}$ for all $n \geqq N$ and the circle of radius ϵ drawn about $z = r_k$ will contain $r_k^{(n)}$ for all $n \geqq N$. This means that in or on the curve $G(r_k + \epsilon, r + \epsilon; p-k, \alpha_k)$, which for any sufficiently small positive ϵ is contained in the circle $|z| < \rho$, lie exactly p zeros of each polynomial $f_n(z)$ for all $n \geqq N$. Since a circle of radius ϵ about any zero of $f(z)$ in $|z| < \rho$ contains a zero of each $f_n(z)$, $n \geqq N$, it follows that in or on the curve $G(r_k + \epsilon, r + \epsilon; p-k, \alpha_k)$ lie exactly p zeros of $f(z)$. Since ϵ is an arbitrary, small positive number, it follows that exactly p zeros of $f(z)$ lie in or on the curve $G(r_k, r; p-k, \alpha_p)$ as stated in Theorem (4.2).

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ON THE FACTORISATION OF ORTHOGONAL TRANSFORMATIONS INTO SYMMETRIES

H. C. LEE

Consider the quadratic form

$$g_{11}(x^1)^2 + g_{22}(x^2)^2 + \cdots + g_{nn}(x^n)^2$$

where each g_{ii} is $+1$ or -1 (but fixed once for all). A homogeneous linear transformation in the variables $x^1, \cdots, x^n$ (or its matrix) is called *orthogonal* if it leaves this form invariant. If and only if all g_{ii} have the same sign we have orthogonality in the classical sense. In any sense it can be easily seen that the orthogonal transformations form a group and that they have determinants ± 1.

All quantities considered may be either in the real field or in the complex field (or in any field whatever).

Let $g_{ij} = 0$ $(i \neq j)$. Then, if s^i is a contravariant vector, its covariant is defined by $s_i = g_{ih}s^h$ (summation convention). The vector s is said to be *isotropic* if it has zero length, that is, if $s^h s_h = 0$. If s is not an isotropic vector, the matrix

$$s^i_k = \delta^i_k - 2s^i s_k / s^h s_h,$$

where $\delta^i_k = 1$ or 0 if $i = k$ or $i \neq k$, defines a special orthogonal transformation of determinant -1 which we call a *symmetry*.[1]

Now, by a series of arguments, E. Cartan has proved the following theorem.

THEOREM.[2] *Every orthogonal transformation is decomposable into the product of a number not greater than n of symmetries.*

I shall give a short proof[3] of this theorem below:

Let a^i_k be the matrix of an arbitrarily given orthogonal transformation. Form the product matrix

$$b^i_k = s^i_j a^j_k = a^i_k - 2s^i s_j a^j_k / s^h s_h$$

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[1] E. Cartan, *La théorie des spineurs*, vol. 1, p. 13.

[2] Cartan, loc. cit. pp. 13–17.

[3] The first half of the present proof is similar to a part of another simple proof given by W. Givens, *Factorisation and signatures of Lorentz matrices*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 82, but Givens' proof only establishes the weaker statement that the number of symmetries in the factorisation is not greater than $2n$.

and consider its first column b_1^i. We show that one can choose the vector s such that $b_1^i=\delta_1^i$. In fact, it suffices for this to take $s^i=\rho(a_1^i-\delta_1^i)$, $\rho\neq 0$. Then we find

$$s^k s_k = g_{ki} s^k s^i = 2\rho^2(g_{11} - g_{1i}a_1^i) = 2\rho^2 g_{11}(1 - a_1^1).$$

(1) If $a_1^1\neq 1$, we have $s^k s_k\neq 0$ and so s is not an isotropic vector. Then

$$b_1^i = a_1^i - 2s^i s_j a_1^j/2\rho^2 g_{11}(1 - a_1^1) = \delta_1^i.$$

This being the first column of the matrix b_k^i, the orthogonality between this column and each of the other columns implies that the matrix b has the form

$$\begin{pmatrix}1 & 0\\ 0 & c\end{pmatrix}$$

where c is evidently an orthogonal matrix of order $n-1$. Hence, by induction, the theorem is proved in this case.

(2) If $a_1^1=1$, we can evidently transform the given orthogonal transformation by another orthogonal transformation so as to render $a_1^1\neq 1$, and we have only to observe that the symmetries are transformed again into symmetries by an orthogonal transformation.

CAMBRIDGE, ENGLAND

NOTE ON DISCRIMINANTS OF BINARY QUADRATIC FORMS WITH A SINGLE CLASS IN EACH GENUS

J. DEAN SWIFT

When discriminants of binary quadratic forms have a single class in each genus, the number of representations function may be explicitly determined.[1,2] The completeness of the lists[1] of such discriminants is still an open question. Dickson,[1] Townes,[1] and Hall[2] have given extensive lists of tests based on the fundamental criterion:

A. If and only if the discriminant possesses a primitive reduced form, (a, b, c), $c>a>b>0$, there is more than a single class in each genus.

These tests are based on the prime decomposition of numbers defined recursively from the discriminant, and are thus rather difficult to apply in screening all possible discriminants up to a large N to check the completeness of the list. For this purpose, a simple linear congruence test capable of mechanical application would be far preferable. Accordingly, the following tests were developed:

B. A discriminant has more than a single class in each genus if:

a. $\Delta\equiv 7 \pmod 8$ unless $\Delta=7$ or 15. (Note: All odd Δ are congruent to 3 or 7 modulo 8.)

b. $\Delta\equiv 12 \pmod{16}$ unless $\Delta=12$, 28, or 60.

$\Delta\equiv 0 \pmod{16}$ unless $\Delta=16$, 48, 64, 112, 192, 240, 448, 960, or $\Delta\equiv 32 \pmod{64}$.

c. $\Delta\equiv 0 \pmod{p^2}$, p an odd prime, unless $\Delta=27$, 36, 72, 75, 99, 100, 147, 180, 288, 315.

d. Δ is a negative quadratic residue of an odd prime, p, such that $p^2<(\Delta+\beta^2)/4$, where β is the largest number less than p of the same parity as Δ.

Test (a) is proved by Dickson[1] and tests (b) and (c) by Hall.[2] For test (d) assume $\Delta\equiv -b^2 \pmod p$. Then $\Delta+b^2=4mp$, $m>p>b$. Thus the representation, (p, b, m), exists, satisfying A. Possible imprimitive forms are taken care of by test (c).

Test (d) is essentially similar to that used by Lehmer[4] in checking the completeness of the "class number one" tables to 5,000,000,000.

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[1] L. E. Dickson, *Introduction to the theory of numbers*, University of Chicago, 1929, chap. 5, pp. 63–90. The notation and terms used herein are as defined in this work.

[2] N. A. Hall, Amer. J. Math. vol. 62 (1940) pp. 589–598.

[3] N. A. Hall, Math. Zeit. vol. 44 (1938) pp. 85–90. Further references to this and associated problems will be found in these papers of Hall.

[4] D. H. Lehmer, Bull. Amer. Math. Soc. Abstract 39-5-188.

The T_k and S_k tests given by Dickson, Townes, and Hall are derivable from these tests by consideration of the requirements imposed by test (d) on the T_k and S_k.

Using a small linear congruence machine developed by D. H. Lehmer, and with the kind assistance of Prof. and Mrs. Lehmer, the author checked the possible discriminants to 10^7, verifying the following theorem.

THEOREM: *There are no discriminants with a single class in each genus*, $3315 < \Delta < 10{,}000{,}000$.

The largest prime necessary in this test was 79.

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ON FINITE EXTENDING GROUPS

ALBERT NEWHOUSE[1]

In his paper *Non-associative algebras*,[2] A. A. Albert defined extending groups $\mathfrak{G}$ for algebras $\mathfrak{A}$ with a unity element.[3] Such groups are merely finite multiplicative groups of nonsingular linear transformations on a linear space $\mathfrak{A}$ of order $n>1$ over a field $\mathfrak{F}$ defined so that all the transformations leave the unity element e of $\mathfrak{A}$ unaltered. With respect to the basis $(e, u_2, u_3, \cdots, u_n)$ of $\mathfrak{A}$ over $\mathfrak{F}$ these groups are then isomorphic to finite groups $\mathfrak{G}$ of n-rowed square matrices of the form

$$G = \begin{pmatrix} 1 & 0 \\ B & M \end{pmatrix},$$

where M is an $(n-1)$-rowed nonsingular square matrix and B a 1 by $n-1$ matrix.

In his paper Albert[4] has raised the question of the existence of such groups $\mathfrak{G}$ "such that no basis of $\mathfrak{A}$ exists for which $\mathfrak{G}$ may be regarded as a permutation group."

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[1] The author is indebted to the referee for his helpful comments.

[2] Ann. of Math. vol. 43 (1942) pp. 685–723.

[3] Ibid. p. 712.

[4] Ibid. Footnote, p. 722.

We shall prove that such groups exist for every algebra $\mathfrak{A}$ whose order $n>2$ over $\mathfrak{F}$ and shall completely settle the case $n=2$.

If $n>2$ and the characteristic of $\mathfrak{F}$ is different from 2 then the matrix

$$G = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where I_{n-2} is the identity matrix of order $n-2$, generates a cyclic group $\mathfrak{G}$ of order 2. The minimum function of G is x^2-1, its characteristic function is $(x-1)^{n-2}(x+1)^2$. This group is isomorphic to the permutation group of order 2, $\mathfrak{P}=[I_n, P]$, with P similar to

$$\begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The minimum function of P is x^2-1, its characteristic function is $(x-1)^{n-2}(x^2-1)=(x-1)^{n-1}(x+1)$. Thus G is not similar to P and $\mathfrak{G}$ is not a permutation group on any base of $\mathfrak{A}$.

Now let the characteristic of $\mathfrak{F}$ be 2. If n is not a power of 2 then there exists an integer m such that $2^m>n>2^{m-1}$. Let M_m be the companion matrix of $x^{2^{m-1}}+1$, a square matrix of 2^{m-1} rows. Now let

$$N_m = \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ 1 & & \\ 0 & & M_m \\ \vdots & & \\ 0 & & \end{pmatrix},$$

a square matrix of $2^{m-1}+1$ rows.

Then

$$G = \begin{pmatrix} I_{n-2^{m-1}-1} & 0 \\ 0 & N_m \end{pmatrix}$$

will generate a cyclic group of order 2^m since the characteristic and minimum function of N_m is $(x+1)^{2^{m-1}+1}$, a divisor of $(x+1)^{2^m}=x^{2^m}+1$ and not a divisor of $(x+1)^{2^{m-1}}=x^{2^{m-1}}+1$. Thus G is of order 2^m and G cannot be any permutation of n letters since one cycle would have to have $2^m>n$ letters.

If $n=2^m>4$, let

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

whose characteristic and minimum function is x^2+x+1. Then let

$$G = \begin{bmatrix} I_{2^{m-1}-2} & 0 & 0 \\ 0 & N_m & 0 \\ 0 & 0 & N \end{bmatrix},$$

an n-rowed square matrix. Its characteristic function is $(x+1)^{n-2}\cdot(x^2+x+1)$, its minimum function is $(x+1)^{2^{m-1}+1}(x^2+x+1)$ which is a divisor of $x^{3\cdot 2^m}+1=(x^3+1)^{2^m}=(x+1)^{2^m}(x^2+x+1)^{2^m}$. Thus G is of order $3\cdot 2^m$. No permutation on $n=2^m$ letters is of order $3\cdot 2^m$ since one cycle would have to have 3 letters and one cycle 2^m letters.

If $n=4$ let

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Its characteristic and minimum function is x^4+x^3+x+1, a divisor of $x^6+1=(x^4+x^3+x+1)(x^2+x+1)$. Thus G is of order 6 and not similar to a permutation matrix since the corresponding permutation matrix would have to have cycles of 3 and 2 letters each, and there are only 4 letters.

If $n=2$ the extending group $\mathfrak{G}$ consists of 2-rowed square matrices

$$G = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}.$$

Let m be the order of $\mathfrak{G}$, then $G^m=I_2$, but

$$G^m = \begin{pmatrix} 1 & 0 \\ a(1+b+\cdots+b^{m-1}) & b^m \end{pmatrix},$$

thus $b^m=1$ and b is an mth root of unity. Thus if $\mathfrak{F}$ contains a primitive mth root of unity b for $m>2$ then

$$G = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

generates a cyclic group of order m. The characteristic function of G is $x^2-(b+1)x+b$, different from the characteristic function of any permutation matrix on two letters.

If $\mathfrak{F}$ does not contain any roots of unity besides 1 and -1, b must be 1 or -1 and G has the form

$$G_1 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad \text{or} \quad G_2 = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix},$$

so that

$$G_1^m = \begin{pmatrix} 1 & 0 \\ ma & 1 \end{pmatrix}, \qquad G_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

If $\mathfrak{F}$ is non-modular $\mathfrak{G}$ can only contain elements of form G_2. Now let

$$S = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ b & -1 \end{pmatrix}, \qquad a \neq b, \qquad S^2 = T^2 = I,$$

then

$$ST = \begin{pmatrix} 1 & 0 \\ a - b & 1 \end{pmatrix}.$$

ST is of form G_1 and cannot be in $\mathfrak{G}$. Thus for $n=2$ and $\mathfrak{F}$ non-modular there exist finite extending groups only if $\mathfrak{F}$ contains a primitive mth root of unity for $m>2$.

If $\mathfrak{F}$ is of characteristic $p>2$,

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generates a cyclic group of order p. This group is not a permutation group on two letters since such a group has order two.

If $\mathfrak{F}$ is of characteristic 2 and contains an extension of the prime field $GF(2)$ then $\mathfrak{F}$ contains at least four elements 0, 1, a, $1+a$, $(a\neq 0, 1)$. Then

$$I, \qquad R = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 1+a & 1 \end{pmatrix},$$

$$RS = SR = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R^2 = S^2 = (RS)^2 = (SR)^2 = I,$$

form a group of order 4 not a permutation group on two letters, since two letters have only two permutations.

If $\mathfrak{F} = GF(2)$, the only nonsingular linear transformations of the prescribed form are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

However, the characteristic and minimum function of G is $(x+1)^2 = x^2+1$ and G is similar to the permutation matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus we have the following theorem.

THEOREM. *For every finite algebra* $\mathfrak{A}$ *over* $\mathfrak{F}$ *there exist finite extending groups* $\mathfrak{G}$ *which are not permutation groups on any basis of* $\mathfrak{A}$ *if the order of* $\mathfrak{A}$ *over* $\mathfrak{F}$ *is greater than* 2.

If the order of $\mathfrak{A}$ *over* $\mathfrak{F}$ *is* 2 *there exist such extending groups if and only if*

(a) $\mathfrak{F}$ *is non-modular and contains a primitive mth root of unity for* $m>2$,

(b) $\mathfrak{F}$ *is of characteristic p and contains more than two elements.*

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DECOMPOSITIONS OF A LOOP INTO CHARACTERISTIC FREE SUMMANDS

GRACE E. BATES

It is known[1] that if the group G is the free sum of proper subgroups G_1 and G_2 (or free product for multiplicative groups), then neither G_1 nor G_2 is normal in G. Clearly, then, there is no nontrivial decomposition of a group into *characteristic* free summands. In this paper we show that the above statement on normality holds for free summands of a loop, but that it is possible, nevertheless, with relatively few restrictions, to obtain decompositions of loops into characteristic free summands.[2]

We use definitions and theorems from a previous paper,[3] referred to throughout as [I], but the statements of these are here repeated for the convenience of the reader.

Notations. If J, K are subsets of a loop L, we denote by $J\cup K$ the set-theoretical sum of elements in J and K; by $J\cap K$ the cross-cut of J and K; and by $J+K$ the subloop of L which is generated by $J\cup K$. If α is a single-valued map of the subset J of L into some loop T, we denote by $J\alpha$ the image of J under α, and by $x\alpha$, the image of any element x in J.

A *half-loop* J is a set of elements with a rule of combination, $+$, defined for some ordered pairs of elements (a, b) of J, and subject to the following three conditions:

i. If $a+b=c$ and $a+b=d$, then $c=d$.

ii. If $x+a=y+a$, then $x=y$; if $b+x=b+y$, then $x=y$.

iii. There exists an element, 0, in J, such that $0+x=x+0=x$, for every x in J.

A *loop* L is a half-loop satisfying the added condition: If any two of the three elements a, b, c, are in L, then $a+b=c$ determines the third as an element of L. Every subset of a loop L which contains the 0-element of L is a half-loop and, since every half-loop is embeddable in a loop,[4] it follows that every half-loop is a subset of some loop.

By a *homomorphism* β of a half-loop J, we mean a single-valued

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[1] See Baer and Levi [1] and Kurosch [1]. The numbers in brackets refer to the bibliography at the end of this paper.

[2] Bruck pointed out the fact that a characteristic subloop need not be normal. See Bruck [1].

[3] See Bates [1].

[4] See Bates [1].

mapping of J *into some loop* T such that if $x+y$ is in J, then $(x+y)\beta = x\beta + y\beta$ in T. An *isomorphism* α of the half-loop J upon the half-loop K is a homomorphism of J upon K which is one-to-one and for which α^{-1} is a homomorphism of K upon J.

The half-loop J is *closed* in the half-loop K if $J \subset K$ and if, given any two of the elements a, b, c in J with $a+b=c$ in K, the third element is in J. We may then define generation as follows: The half-loop K is *generated* by its sub-half-loop J if no proper sub-half-loop of K containing J is closed in K.

The half-loop K is *free over* its sub-half-loop J if every homomorphism of J into any loop T may be extended to a homomorphism of K into the same loop T. In particular, a *free loop* F is one which is freely generated by a subset J (called a *free basis* of F) having the property: $a+b=c$ in J implies that at least one of the elements a, b is 0. The cardinal number of nonzero elements in a free basis of the free loop F is called the *rank* of F. A *free cyclic loop* is a free loop of rank one.

The loop L is the *free sum* of its subloops $L(v)$ (we write $L = \sum^* L(v)$), if $L(v) \cap L(w) = 0$ for $v \neq w$, and if L is freely generated by $\bigcup L(v)$. A free loop, then, is the free sum of free cyclic loops.

In [I] it is proved that if J is any half-loop, there exists one and essentially only one loop L which is freely generated by J, so that the existence of free loops and of free sums of loops is assured. In the appendix of [I] it is also shown that the above definitions are (relatively) analogous to the usual definitions of free groups and free sums of groups.

The following statements, here numbered for future reference, either follow immediately from the statements made above or have been proved in [I].

1°. If the loop L is freely generated by J, if the loop T is freely generated by K, and if there exists an isomorphism α of J upon K, then there exists one and only one isomorphism of L upon T which induces α in J.

2°. If $J \subset K$ are two sub-half-loops of the loop L,

(a) If L is freely generated by K and if K is freely generated by J, then L is freely generated by J, and:

(b) If L is freely generated by J and if K is generated by J, then L is freely generated by K.

3°. If $L = \sum^* L(v)$, then $L = L(w) * \sum^*_{v \neq w} L(v)$.

4°. If $L = A * B$, and if there exists an isomorphism α of L upon the loop T, then $T = A\alpha * B\alpha$.

Since our definition of free sum of loops is not a constructive one we need the following lemma:[5]

LEMMA A. *If* $L=A*B$, A, $B\neq 0$, *then*:

(a) *If* $0\neq x, y$ *are elements of* $A\cup B$ *for which* $x+y=y+x$ *in* L, *then* x, y *are both in* A *or* x, y *are both in* B.

(b) *If* $0\neq x$, y, z *are elements of* $A\cup B$ *for which* $(x+y)+z=x+(y+z)$ *in* L, *then* x, y, z *are all in* A, *or* x, y, z *are all in* B.

PROOF. (a) Suppose that $0\neq x$, y are elements of $A\cup B$ for which $x+y=y+x$ in L. Assume that x, y are not both in A and are not both in B. Then x is in A, y in B, *or* x is in B, y in A. Since these two cases are symmetric, we consider only the first case.

Since $A\cap B=0$, $x+y$ and $y+x$ are undefined in $A\cup B$. Consider the system S composed of elements in $A\cup B$ together with symbols $x\circ y$, $y\circ x$, with $x\circ y\neq y\circ x$. Define a rule of composition, $+$, in S as follows:

If s, s' are in A, let $s+s'$ be the uniquely defined sum of (s, s') in A.

If s, s' are in B let $s+s'$ be the uniquely defined sum of (s, s') in B.

For the given elements x, y, define $x+y=x\circ y$ and $y+x=y\circ x$.

Furthermore, we define: $(x+y)+0=x+y=0+(x+y)$ and $(y+x)+0=y+x=0+(y+x)$.

We require the above to be the only sums defined in S. In particular, then, $a+b$ for $0\neq a$ in A, $0\neq b$ in B, is defined only for $a=x$, $b=y$; and $b+a$, for $0\neq b$ in B, $0\neq a$ in A, is defined only for $b=y$ and $a=x$.

Then S is a half-loop under this composition and there exists one and essentially only one loop L' which is freely generated by S. But then L' is generated by $A\cup B$ since the subloop of L' which contains $A\cup B$ must contain S (since it must contain $x+y$ and $y+x$). Furthermore, if α is any homomorphism of $A\cup B$ into a loop T, then in T there exists one and only one z', and one and only one w', such that $x\alpha+y\alpha=z'$ and $y\alpha+x\alpha=w'$, so that α may be extended to a homomorphism β of S by mapping $x+y$ on z' and $y+x$ on w'. Since L' is free over S, there exists one and only one homomorphism γ of L' inducing β in S and α in $A\cup B$—that is, α may be extended to a homomorphism of L' into T and hence L' is free over $A\cup B$.

But L is freely generated by $A\cup B$, since $L=A*B$. Hence (by 1^0), there exists an $(A\cup B)$-isomorphism of L upon L' and therefore $x+y\neq y+x$ in L. But this is a contradiction since we chose x, y in $A\cup B$ such that $x+y=y+x$ in L. Hence our assumption that both

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[5] It is fairly obvious that a much more general proposition than Lemma A is valid but as its statement would be somewhat tedious the simpler form sufficient for the purposes of this paper is here preferred.

x, y were not in A and not in B is false, proving (a).

(b) Given $0 \neq x, y, z$ in $A \cup B$, with $(x+y)+z = x+(y+z)$ in L, suppose that x, y, z are not all in A and are not all in B. Then there are only two nonsymmetric possibilities: (1) x, y in A, but z in B, or (2) x, z in A, but y in B. We discuss these separately:

(1) x, y in A; z in B; then $x+y=a$, where a is a uniquely defined element in A, but $y+z$ and $x+(y+z)$ are undefined in $A \cup B$.

i. If $a=0$, then $a+z=z$. We define a system S consisting of the elements of $A \cup B$, together with distinct symbols $y \circ z$ and $x \circ (y \circ z)$, and define a composition, $+$, in S as follows: $s+s'$ is the uniquely defined sum of s, s', when this sum is defined in $A \cup B$ for s, s' in $A \cup B$; $y+z=y \circ z$; $x+(y+z)=x+(y \circ z)=x \circ (y \circ z)$; and $s+0 = 0+s=s$ for all s in S. Then S is a half-loop and the argument from here on goes through as in part (a).

ii. If $a \neq 0$, then $a+z$ is also undefined in $A \cup B$ and we adjoin to S in i a new symbol, $a \circ z$, and define in the extended system one more sum, $a+z=a \circ z$, before completing the argument.

(2) x, z in A, but y in B; then $x+y$, $y+z$, $x+(y+z)$, and $(x+y)+z$ are all undefined in $A \cup B$. Hence, as before, constructing a system S to include the elements of $A \cup B$, we adjoin now four distinct symbols, $x \circ y$, $y \circ z$, $x \circ (y \circ z)$, and $(x \circ y) \circ z$, and define an operation, $+$, in S as follows: For s, s' in $A \cup B$, with a (unique) sum defined in $A \cup B$ (hence s, s' both in A or both in B), we define $s+s'$ to be this sum; we define, for the given x, y, z, the following sums: $x+y=x \circ y$, $y+z=y \circ z$, $(x+y)+z=(x \circ y)+z=(x \circ y) \circ z$, and $x+(y+z)=x+(y \circ z)=x \circ (y \circ z)$. We also require that $0+s=s+0=s$, for every s in S. Then S is a half-loop, as is easily verified, and the argument proceeds as before.

We define a *normal* subloop of the loop L to be one which is the kernel of some homomorphism of L. We may now prove:

THEOREM 1. *A free summand of a loop L is normal if and only if the free summand is 0 or L (that is, nontrivial free summands are not normal).*

PROOF. Let $L=A*B$ (by 3^0 this is no loss in generality). If A (or B) is 0, then B (or A) is equal to L and hence A and B are both (trivial) normal subloops of L.

Suppose that B is a normal subloop of L. Then there exists a homomorphism α of L into a loop T such that B is the kernel of α. If $B=L$ or if $B=0$, the theorem is proved. Hence suppose that $B \neq 0$, $B \neq L$. Then $A \neq 0$ and there exists $a \neq 0$ in A and $b \neq 0$ in B. Let a' be the unique solution x in A to the equation $x+a=0$, and consider the element $c=a'+(a+b)$ in L.

Suppose that $c=c'+(a+b)$ is in B. Since $L=A*B$, there exists one (and only one) endomorphism β of L which is the extension of the identity map in B together with the null homomorphism of A. Then $c=c\beta=[a'+(a+b)]\beta=a'\beta+(a\beta+b\beta)=b\beta=b$. That is, if c is in B, then $c=a'+(a+b)=b$. But this is impossible since then $a'+(a+b)=b=(a'+a)+b$, which, by Lemma A, implies that a, a', b, are all in A or all in B. Hence c is not in B.

But, under the homomorphism α of L into T, we have $c\alpha=[a'+(a+b)]\alpha=a'\alpha+(a\alpha+b\alpha)=a'\alpha+a\alpha=(a'+a)\alpha=0$, so that c is in the kernel of α.

Hence B is not normal in L if $B\neq 0$, $B\neq L$, and similarly, A is not normal in L if $A\neq 0$, $A\neq L$. proving the theorem.

LEMMA B. *If $L=F*G$ is a nontrivial decomposition of the loop L into free summands F, G, where F is a cyclic free loop on generator f, then for any nonzero element g in G, $L=Z*G$, where Z is the cyclic (free) subloop of L generated by $z=f+g$.*

PROOF. L is generated by the half-loop, $z\cup G$, since the subloop of L which is generated by z and G contains $f+g$ and G, and therefore contains f and G.

Let β be a homomorphism of $z\cup G$ into a loop T. In T there exists one and only one element f' such that $f'+g\beta=z\beta$. We define a mapping α of $f\cup G$ into T as follows: Let $g\alpha=g\beta$ for every g in G, and define $f\alpha=f'$. This map is well-defined and is a homomorphism of $f\cup G$ into T. Since L is freely generated by $f\cup G$, α may be extended to a homomorphism γ of L into T. Then $z\gamma=(f+g)\gamma=f\gamma+g\gamma=f\alpha+g\alpha=f'+g\beta=z\beta$. Hence γ induces β on z. That is, the homomorphism β of $z\cup G$ into T has been extended to a homomorphism of L into T. Hence L is freely generated by $z\cup G$ and, therefore (by 2°), by $Z\cup G$. That is, $L=Z*G$.

We restate next a theorem proved in [1].

GENERAL SUBLOOP THEOREM FOR SUBLOOPS OF FREE SUMS. *If $L=\sum^* L(v)$, and if S is a subloop of L, then $S=F_S*\sum^*(S\cap L(v))$, where:*

(i) *F_S is a free loop whose rank is uniquely determined by S and the $L(v)$.*

(ii) *F_S has a free basis, F_0, with the property: If α is an automorphism of L such that $S\alpha=S$ and $L(v)\alpha=L(v)$ for every v, then $F_S\alpha=F_S$ and α effects a permutation of the elements of F_0.*

A corollary of this theorem is the statement that *every subloop of a free loop is free.*

Note. If L has a free summand, $F\neq 0$, then F is a free loop if and only if in every decomposition of L into free summands there is a free summand which has as a free summand a free loop $\neq 0$. For, suppose $L=A*B$ is any decomposition of L into free summands. Then $L=F*G$ is also a decomposition of L and $G\neq L$, since $F\neq 0$. By the subloop theorem, $A=F_A*(A\cap F)*(A\cap G)$ and $B=F_B*(B\cap F)*(B\cap G)$. Now if neither A nor B has a free loop not equal to 0 as a free summand, then $F_A=A\cap F=F_B=B\cap F=0$, since all these are free loops. But then $A=A\cap G$ and $B=B\cap G$, from which we see that $G\supset A\cup B$. Since L is (freely) generated by $A\cup B$, then $L=G$, which is a contradiction.

THEOREM 2. *If $L=A*B$, A, $B\neq 0$, then A is a characteristic subloop of L if and only if both the following conditions are satisfied:*

(i) *No free summand not equal to 0 of A is a free loop.*

(ii) *No free summand not equal to 0 of A is isomorphic to a free summand of B.*

PROOF. Suppose first that A is a characteristic subloop of L. We prove first the necessity of (i).

Assume that A has a free summand $\neq 0$ which is a free loop. Then we may represent A as $A=F*A_1$, where F is a free cyclic loop generated by $f\neq 0$ (see 3^0). Then $L=(F*A_1)*B=F*(A_1*B)$. Let b be any nonzero element in B and consider the unique element z of L satisfying $z=f+b$. Then z is *not* in A and *not* in B, since $A\cap B=0$. By Lemma B, we have $L=Z*(A_1*B)$, where Z is the cyclic subloop of L generated by z.

We define a mapping β of $f\cup(A_1*B)$ upon $z\cup(A_1*B)$ as follows: Let $x\beta=x$ for every x in A_1*B, and let $f\beta=z$. Then β is single-valued since f is not in A_1*B. Furthermore, β is an isomorphism of $f\cup(A_1*B)$ upon $z\cup(A_1*B)$. Since L is freely generated by $f\cup(A_1*B)$, β may be extended to an endomorphism α of L, but this endomorphism is actually an automorphism of L since L is also freely generated by $z\cup(A_1*B)$ (see 1^0). Since α is an extension of β, $F\alpha=Z$. But $A\alpha=(F*A_1)\alpha=F\alpha*A_1\alpha=F\alpha*A_1\beta=Z*A_1$, and hence $A\alpha\neq A$, since z is in $Z*A_1$, but z is not in A. This is impossible, however, since A is a characteristic subloop of L. Hence A can have no free summand $\neq 0$ which is free, proving the necessity of condition (i).

If A has a free summand not equal to 0 which is isomorphic to a free summand of B, we may represent A and B as follows: $A=A_1*A_2$; $B=B_1*B_2$ with β the given isomorphism of A_2 upon B_2. We then define a map α of $A_1\cup A_2\cup B_1\cup B_2=J$ into itself as follows: $x\alpha=x$ for every x in $A_1\cup B_1$; $A_2\alpha=A_2\beta=B_2$; and $B_2\alpha=B_2\beta^{-1}=A_2$. Then α is an automorphism of J and since L is freely generated by J, there is one and only one automorphism γ of L inducing α in J. But $A\gamma=(A_1*A_2)\gamma=A_1\gamma*A_2\gamma$ (by 4°) $=A_1\alpha*A_2\alpha=A_1*B_2$, and since $A\cap B=0$, $A_1*B_2\neq A$. Hence $A\gamma\neq A$, in contradiction to the fact that A is a characteristic subloop of L. That is, A can have no free summand not equal to 0 which is isomorphic to a free summand of B, which finishes the proof of the necessity of conditions (i) and (ii).

Now, assuming that conditions (i) and (ii) are satisfied, let α be any automorphism of L. Then $L=A*B=A\alpha*B\alpha$ (by 4°) and $A\alpha$ is isomorphic to A, $B\alpha$ is isomorphic to B. Using the Subloop Theorem we have: $A=(A\cap A\alpha)*(A\cap B\alpha)$, since by (i), A has no free summand not equal to 0 which is free. Also, $B\alpha=F_{B\alpha}*(B\alpha\cap A)*(B\alpha\cap B)$. But then $B\alpha\cap A$, a free summand of A, is isomorphic to a free summand of B and, by condition (ii), then $B\alpha\cap A=0$ and $A=A\cap A\alpha$.

We have, similarly, $B=F_B*(B\cap A\alpha)*(B\cap B\alpha)$ and $A\alpha=(A\alpha\cap A)*(A\alpha\cap B)$, since $A\alpha$ is isomorphic to A and A satisfies condition (i). Then $A\alpha\cap B$, a free summand of B, is isomorphic to a free summand of A. Hence $A\alpha\cap B=0$, and we have $A\alpha=A\cap A\alpha=A$, proving that A is characteristic.

COROLLARY. *If $L=\sum^*L(v)$ (each $L(v)\neq 0$) is a nontrivial decomposition of L into free summands, then the $L(v)$ are characteristic subloops of L if and only if no $L(v)$ has a free summand not equal to* 0 *which is free or which is isomorphic to a free summand of $L(w)$, for $w\neq v$.*

PROOF. The necessity of the above conditions is an immediate consequence of Theorem 2, since, for each $L(v)$, L may be represented in the form $L=L(v)*\sum^*_{w\neq v}L(w)$.

Suppose that the conditions of the corollary are satisfied for every v. Let $\sum^*_{w\neq v}L(w)=B$, for some fixed v; then $L=L(v)*B$. To apply Theorem 2 we must show that no free summand not equal to 0 of $L(v)$ is isomorphic to a free summand of B.

Assume that $B=B_1*B_2$; $L(v)=V_1*V_2$, with α an isomorphism such that $B_2\alpha=V_2$ ($B_2\neq 0$). Since $B_2\subset B=\sum^*_{w\neq v}L(w)$, then, $B_2=\sum^*_{w\neq v}(B_2\cap L(w))$, using the subloop theorem and the fact that B_2 contains no free summand not equal to 0 which is a free loop, since B_2 is isomorphic to V_2, a free summand of $L(v)$.

Then $V_2=B_2\alpha=\sum^*_{w\neq v}(B_2\cap L(w))\alpha$ and hence $B_2\cap L(w)$ is isomorphic to a free summand of V_2 and therefore of $L(v)$. But $L(w)\subset B=B_1*B_2$ and hence $L(w)=(L(w)\cap B_1)*(L(w)\cap B_2)$ so that

$B_v \cap L(w)$ is also a free summand of $L(w)$, $w \neq v$. This is impossible according to the hypotheses of our corollary and hence the assumption that a free summand not equal to 0 of $L(v)$ is isomorphic to a free summand of $B = \sum^*_{w\neq v} L(w)$ is false. Hence we have, for the decomposition of $L = L(v) * B$, the two conditions of Theorem 2 satisfied for $L(v)$, proving that $L(v)$ is a characteristic subloop of L.

THEOREM 3. *If* $L = \sum^* L(v)$ *is a decomposition of* L *into characteristic free summands, and if* S *is a characteristic subloop of* L, *then* $S = F * \sum^*(S \cap L(v))$, *where*

(1) F *is a free characteristic subloop of* L.

(2) *For every* v, $S \cap L(v)$ *is a characteristic subloop of* L *and of* $L(v)$.

(3) *There exists a free basis*, B, *of* F *such that every automorphism of* L *induces a permutation in* B.

The proof of this theorem is immediate upon application of the subloop theorem, since the $L(v)$ and S are characteristic in L.

THEOREM 4. *If* $L = \sum^* L(v) = \sum^* K(w)$ *are two decompositions of* L *into characteristic free summands, then* $L = \sum^*_{v,w}(L(v) \cap K(w))$, *where the* $L(v) \cap K(w)$ *are characteristic in* L.

PROOF. By Theorem 3, since $L(v)$ is a characteristic subloop of $L = \sum^* K(w)$, we have, for each v, $L(v) = \sum^*_w (K(w) \cap L(v))$, since $L(v)$ has no free summand not equal to 0 which is a free loop. Also, for each w, $K(w) \cap L(v)$ is a characteristic subloop $L = \sum^* L(v) = \sum^*_{v,w} (K(w) \cap L(v))$.

THEOREM 5. *If* L *is the free sum of subloops* $L(v)$, *then there exists one and only one decomposition of* L *into a free sum of characteristic subloops* $K(w)$ *such that*: (a) *every* $K(w)$ *is a free sum of some of the* $L(v)$; (b) *No refinement of the decomposition into the* $K(w)$ *has this property.*

PROOF. There exists one decomposition of L into characteristic free summands satisfying condition (a)—namely, the trivial one, $L = L$. Consider the cross-cuts of all free summands from any one decomposition of L into characteristic free summands satisfying (a), with *each* of those of every other such decomposition of L. By Theorem 4, then, L is the free sum of all such cross-cuts, the latter being characteristic subloops of L. Clearly this last decomposition of L satisfies also (b), proving the theorem.

A loop L is *free irreducible* if L has no decomposition into nonzero free summands other than the trivial decomposition, $L = L$. In particular, a free loop $F \neq 0$ is free irreducible if (and only if) it is cyclic. For, if $F = A * B$, then A, B are free and $1 = \text{rank } F = \text{rank } A + \text{rank } B$.

If the loop L has a decomposition into free irreducible summands

$L(v)$, none of which is a free loop, it follows immediately from the subloop theorem that such a decomposition is unique and any automorphism of L effects merely a permutation of the $L(v)$. Hence we have a stronger theorem in this case:

THEOREM 6. *If* $L=\sum^* L(v)$ *is a nontrivial decomposition of the loop* L *into free irreducible summands, then* L *has a nontrivial decomposition into characteristic free summands* $K(w)$ *if and only if the following two conditions are satisfied:* (i) *no* $L(v)$ *is a free loop and* (ii) *not all the* $L(v)$ *are isomorphic. This decomposition into the* $K(w)$ *will of necessity be such that each* $K(w)$ *is the free sum of some of the* $L(v)$ *and may be so carried out that no further refinement of the decomposition will yield a characteristic free decomposition of* L*—it will then be unique.*

PROOF. Suppose L has a nontrivial decomposition into characteristic free summands, $K(w)$. Then no $K(w)$ has a free summand not equal to 0 which is free or is isomorphic to a free summand of $K(w')$, for $w\neq w'$ (Corollary to Theorem 2). It follows from the remarks made above, then, that no $L(v)$ is a free loop and that not all the $L(v)$ are isomorphic.

Conversely, if conditions (i) and (ii) are satisfied, then L obviously has one nontrivial decomposition into characteristic free summands. For, there exists at least one pair of free summands, $L(v')$ and $L(v'')$, which are not isomorphic. Then $L=A*B$, where A is the free sum of all the $L(v)$ which are isomorphic to $L(v')$ and B is the free sum of the remaining $L(v)$, is such a decomposition. The decomposition consisting of the cross-cuts of all free summands from any one such decomposition with each of those of every other such decomposition will have the further properties of the theorem.

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RINGS WITH A POLYNOMIAL IDENTITY

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1. Introduction. In connection with his investigation of projective planes, M. Hall [2, Theorem 6.2][1] proved the following theorem: a division ring D in which the identity

$$(xy - yx)^2z = z(xy - yx)^2 \tag{1}$$

holds is either a field or a (generalized) quaternion algebra over its center F. In particular, D is finite-dimensional over F, something not assumed a priori. The main result (§2) in the present paper is the following: if D satisfies *any* polynomial identity it is finite-dimensional over F. There are connections with other problems which we note in §§3, 4.

2. Proof of finite-dimensionality. Let A be an algebra (no assumption of finite order) over a field F. We denote by $F[x_1, \cdots, x_r]$ the free algebra generated by r indeterminates over F. We say that A satisfies a polynomial identity if there exists a nonzero element f in $F[x_1, \cdots, x_r]$ such that $f(a_1, \cdots, a_r) = 0$ for all a_i in A.

LEMMA 1.[2] *If A satisfies any polynomial identity, then it satisfies a polynomial identity in two variables.*

PROOF. Suppose A satisfies the equation $f(x_1, \cdots, x_r) = 0$. Replacing x_i by u^iv we obtain the equation $g(u, v) = 0$, with g a polynomial which is not identically zero.

LEMMA 2. *If A satisfies any polynomial identity, it satisfies a polynomial identity which is linear in each variable.*

PROOF. Suppose A satisfies $f(x_1, \cdots, x_r) = 0$ and that f is not linear in x_1. Then

$$f(y + z, x_2, \cdots, x_r) - f(y, x_2, \cdots, x_r) - f(z, x_2, \cdots, x_r) = 0$$

is satisfied by A. This is a polynomial (in $r+1$ variables), not identically zero, and with degree in y and z lower than the degree of f in x_1. By successive steps of this kind we reach a polynomial linear in all variables.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Cf. [7, Satz 2].

LEMMA 3. *Suppose that A satisfies an identity $g=0$ where g is linear in all its variables, and let K be a field over F. Then the Kronecker product $A\times K$ also satisfies $g=0$.*

PROOF. We have to show that $g(x_1, \cdots, x_r)$ vanishes when the x's are replaced by elements $\sum a_i k_i$ of $A\times K(a_i\in A,\ k_i\in K)$. This follows from the linearity of g and the fact that the k's commute with everything.

LEMMA 4.[3] *If A satisfies an identity $f=0$ which is of degree not greater than n in each of its variables, and if F has at least $n+1$ elements, then A satisfies an identity which is homogeneous in each of its variables.*

PROOF. We write $f=\sum f_i$ where f_i is of degree i in x_1. Replacing x_1 by λx_1 where $\lambda\in F$, we find $\sum \lambda^i f_i=0$. We do this for n different nonzero scalars. Using the resulting Vandermonde determinant, we obtain $f_i=0$. We repeat this procedure with each of the variables.

LEMMA 5. *If $f(x, y)$ is any nonzero element of $F[x, y]$, we can find two matrices a, b with elements in F such that $f(a, b)\neq 0$.*

PROOF. A field of degree k over F may be represented by $k\times k$ matrices over F. Thus by taking matrices split into suitable smaller blocks, we may arrange for a scalar field of any desired (finite) size.

Suppose that on the contrary f vanishes for all matrices with elements in F. Then by Lemma 4 we may pass to a polynomial $g(x, y)$ which also vanishes for any matrices over F, and which is homogeneous in y, say of degree t. Take a to be a diagonal matrix: $a=\operatorname{diag}(u_1, \cdots, u_n)$, with n greater than the degree of f, and the u's as yet undefined. For b we choose an n by n matrix which permutes the elements of a diagonal matrix cyclically. Thus

$$b^i a b^{-i} = \operatorname{diag}(u_{i+1}, u_{i+2}, \cdots, u_n, u_1, \cdots, u_i). \tag{2}$$

We consider $g(a, b)b^{-t}$. Using the relation (2), we systematically push b to the right of a, and every such operation induces a cyclic permutation. Corresponding to the typical monomial $x^i y^j x^k y^l x^m \cdots$ in g, we obtain the term

$$(u_1)^i (u_{1+j})^k (u_{1+j+l})^m \cdots$$

in the upper left corner of $g(a, b)b^{-t}$. Moreover given any such term we can unambiguously reconstruct the monomial from which it arose (we use here the fact that n exceeds the degree of g, so that there is no overlapping of the u's). Thus the upper left entry of $g(a, b)b^{-t}$ is a polynomial $h(u_1, \cdots, u_n)$ (that is to say, an ordinary commutative polynomial), which is not identically zero. By enlarging the field of scalars, if again necessary, we can find values of the u's which make h different from 0. For this choice we have $g(a, b) \neq 0$, a contradiction.

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[3] Cf. [7, Satz 1].

We prove the main theorem of the paper not only for division rings, but more generally for primitive algebras in the sense of Jacobson.

THEOREM 1. *A primitive algebra satisfying a polynomial identity is finite-dimensional over its center.*

PROOF. First we assert that A is a matrix algebra of finite order over a division algebra. For if not we would have[4] that for every k, A has a sub-algebra homomorphic to D_k where D is a division algebra over F, and D_k is the algebra of matrices of order k over D. The polynomial identity satisfied by A is inherited by D_k for all k, and therefore holds for all matrices with coefficients in F. By Lemma 1 we may assume that the polynomial in question has two variables, and this contradicts Lemma 5.

The problem is thus reduced to the case where A is a division algebra, say with center Z. By Zorn's lemma, select a maximal subfield K of A, and form the Kronecker product $A \times K$ over Z. By a theorem of Nakayama and Azumaya [7], $A \times K$ is a dense algebra of linear transformations in a vector space over K. By Lemma 2, A has a linear polynomial identity, and by Lemma 3 this identity survives in $A \times K$. By Lemma 1 we pass to an identity in two variables in $A \times K$. Repetition of the argument of the preceding paragraph shows that $A \times K$ is finite-dimensional over K, hence A is finite-dimensional over Z.

The following further result is virtually a restatement of Lemma 5.

THEOREM 2. *The free algebra in any number of indeterminates has a complete set of finite-dimensional representations.*

PROOF. Let $f(x_1, \cdots, x_r)$ be an element in the free algebra. We make the replacement $x_i = u^i v$ of Lemma 1. For the resulting polynomial $g(u, v)$, we can find, by Lemma 5, matrices a and b such that $g(a, b) \neq 0$. The mapping induced by sending x_i into $a^i b$ $(i = 1, \cdots, r)$ and all other x's into 0 is a homomorphism of the free algebra into the algebra generated by a and b, and in this representation $f(x_1, \cdots, x_r)$ is not sent into 0.

3. **Algebraic algebras.** It is true conversely that any algebra of finite order satisfies a polynomial identity. For algebras over the real numbers, this was shown by Wagner [8, pp. 531–532]; his identity

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[4] [3, Th. 3]. Jacobson has since shown that a unit element is not necessary for this theorem (oral communication).

for n by n matrices has degree $(n^2-n+2)(n^2-n+4)/4$. E. R. Kolchin has remarked that any algebra of order n satisfies an identity of degree $n+1$:

$$\sum \pm x_1x_2 \cdots x_{n+1} = 0$$

where the sign is positive or negative according as the permutation is even or odd. Incidentally the existence of these identities furnishes a brief proof of Malcev's result [6, Theorem 9] that the free algebra in two or more indeterminates cannot be faithfully represented by matrices of finite order.

A wider class of algebras than those of finite order is covered by the following result.[5]

THEOREM 3. *An algebraic algebra of bounded degree satisfies a polynomial identity.*

PROOF. To avoid complications of notation, we give the proof for the case where the degrees of the elements are bounded by 3. Thus every element x satisfies an equation

$$x^3 + \alpha x^2 + \beta x = 0 \tag{3}$$

where α, β are scalars which of course depend on x. (We do not assume a unit element, and so no constant appears in (3).) We take the commutator with y, obtaining

$$[x^3, y] + \alpha[x^2, y] + \beta[x, y] = 0$$

where $[x, y]$ denotes $xy-yx$. Then we take the commutator with $[x, y]$ and finally with $[[x, y], [x^2, y]]$. The result is:

$$[[[x, y], [x^2, y]], [[x, y], [x^3, y]]] = 0,$$

a polynomial of degree 11. The polynomial moreover is not identically zero, since for example there is only one term

$$xyx^2yxyx^3y.$$

In the general case where the degree is bounded by n, we get an identity of degree $2^{n+1}-2^{n-1}-1$. The case $n=2$ yields an identity equivalent to (1), with z replaced by x.

Combining Theorems 1 and 3, we obtain a new proof of a theorem due to Jacobson [3, Theorems 5 and 7].

THEOREM 4. *A primitive algebraic algebra of bounded degree is finite-dimensional over its center.*

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[5] I am indebted to Prof. Jacobson for Theorem 3 and its proof.

An outstanding question in the theory of algebraic algebras is Kurosch's analogue of Burnside's problem: is an algebraic algebra necessarily locally finite?[6] Affirmative answers have been contributed in two cases: (1) algebraic algebras of bounded degree [3, 4, and 5], and (2) representable[7] algebraic algebras [6]. A third (trivial) instance where the answer is affirmative is the commutative case. A hypothesis that is weaker than any of these is the assumption of a polynomial identity. It thus seems natural to try Kurosch's problem next for algebraic algebras with a polynomial identity. We shall now contribute an affirmative answer for the case of a nil algebra; in the light of [3, Theorem 15] the question is thus reduced to the semi-simple case.

THEOREM 5. *A nil algebra satisfying a polynomial identity is locally finite.*

PROOF. We use the terminology and results of Levitzki (cf. [5] and the references given there). Exactly as in [5] we reduce to the case where A is semi-regular. We choose an element $a \neq 0$ such that $a^2 = 0$. The algebra Aa is also semi-regular. We convert our identity into one, say $f(x_1, \cdots, x_r) = 0$, which is linear in each variable (Lemma 2). Suppose the variable x_1 actually appears first in at least one of the monomials comprising f. We gather all such terms and write

$$f(x_1, \cdots, x_r) = x_1 g(x_2, \cdots, x_r) + h(x_1, \cdots, x_r). \tag{4}$$

Each monomial of h has a factor x_1 which appears later than the first term. It follows that if we substitute a for x_1, and any elements of Aa for $x_2, \cdots, x_r$, we introduce a factor a^2 in each term of h and thus make h vanish. Thus $ag(x_2, \cdots, x_r)$ must also vanish for $x_2, \cdots, x_r$ in Aa. This makes g a right annihilator of Aa. Since Aa is semi-regular it has no right annihilator except 0. Hence Aa satisfies the identity $g = 0$. By induction on the degree of the identity we have that Aa is locally finite, which contradicts its semi-regularity.

4. **Further remarks.** (a) Wagner's main theorem in [8] asserts that an ordered algebra over the reals satisfying a polynomial identity is necessarily commutative. Our results furnish a short proof of a special case: *any ordered primitive algebra satisfying a polynomial identity is commutative.* This is an immediate consequence of Theorem

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[6] An algebra is locally finite if every finitely generated sub-algebra is of finite order.

[7] An algebra is representable if it is isomorphic to a ring of matrices of finite order over some extension field.

1 and Albert's theorem [1] to the effect that any ordered algebra of finite order is a field.

(b) The argument of Theorem 1 can be made to yield an explicit upper bound for the order of the algebra over its center, in terms of the degree and number of variables of the identity; the bound however is far too generous. In a special case like Hall's theorem there is no difficulty in getting the precise bound. Specifically it is only necessary to produce 3 by 3 matrices violating the linearized form of (1). One may in this way also verify the following result: a semi-simple algebra satisfying a polynomial identity of degree not greater than 3 is commutative.

(c) We have for simplicity given all the results in this paper for algebras, but they may be extended to rings as follows. Assume that the polynomials in question have as coefficients operators α such that $\alpha x=0$ implies $x=0$. It is to be observed that this holds in particular if the coefficients are ± 1. Thus we may assert that any primitive ring satisfying (1) is an algebra of finite order; and Theorem 5 thus extended subsumes Levitzki's theorem [5] that a finitely generated nil-ring of bounded index is nilpotent.

Added in proof (May 1, 1948). I am indebted to Dr. Harish-Chandra for the following brief proof of Lemma 5. Let n be the degree of $f(x, y)$ and I the ideal $F[x, y]$ generated by monomials of degree not less than $n+1$. Then $F[x, y]/I$ is an algebra, which can be faithfully represented by matrices since it is of finite order. This gives us matrices for which $f\neq 0$.

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PROOF OF A FORMULA OF LIOUVILLE

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1. Introduction. In his summary of the formulas stated without proof in Liouville's famous series of articles on the theory of numbers, Dickson [1][1] remarked that no proof had been published for (Q) of the Sixth Article [2]. This formula, however, was later derived by Bell [3], who obtained it by paraphrase, a general method which he applied to various identities between elliptic, abelian, and theta functions to obtain some of Liouville's formulas and others of new kinds.

Bell noted that among all of Liouville's, formula (Q) is unique, as "it is the only one which depends immediately upon the addition theorems for the elliptic functions." For this reason, together with the fact that it is the only elementary proof so far published, the derivation of formula (Q) presented below may be of interest.

2. Notation. Throughout this paper $(d_1, \delta_1, d_2, \delta_2, d_3, \delta_3)$ and $(d_1', \delta_1', d_2', \delta_2', d_3', \delta_3')$ denote integer solutions for n, a fixed positive integer, in the forms

$$n = d_1\delta_1 + d_2\delta_2 + d_3\delta_3, \tag{1}$$

$$n = d_1'\delta_1' + d_2'\delta_2' + d_3'\delta_3' \tag{2}$$

respectively, each of the d_i, δ_i, d_i', δ_i' $(i=1, 2, 3)$ being positive unless otherwise specified. Further, $F(x, y)$ denotes a function which takes a single definite value for each integral pair (x, y), satisfies the conditions

$$\begin{aligned} F(-x, y) &= -F(x, y) = F(x, -y) = F(y, x), \\ F(0, y) &= 0 = F(x, 0), \end{aligned} \tag{3}$$

but is otherwise general in the widest sense.

For the sake of brevity, certain expressions are denoted by single letters with subscripts. These are as follows:

$$\begin{aligned} S_1 = d_3\{&F(d_1 + d_2 + d_3, d_1 + d_2) - F(d_1 - d_2 - d_3, d_1 - d_2)\} \\ &+ \delta_2\{F(\delta_1 - \delta_3, \delta_1 + \delta_2 - \delta_3) - F(\delta_1 + \delta_3, \delta_1 - \delta_2 + \delta_3)\}, \\ S_2 = (d_3 - d_2)\{&F(d_1 + d_3, d_1 + d_2) - F(d_1 - d_3, d_1 - d_2)\} \\ &+ (\delta_2 + \delta_3)\{F(\delta_1 - \delta_3, \delta_1 + \delta_2) - F(\delta_1 + \delta_3, \delta_1 - \delta_2)\}, \end{aligned}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$\begin{aligned} S_3 = {} & d_3\{F(d_1+d_2+d_3,\, d_1+d_2) - F(d_1+d_2,\, d_1+d_2-d_3)\} \\ & + \delta_3\{F(\delta_2-\delta_1,\, \delta_2-\delta_1+\delta_3) - F(\delta_2-\delta_1-\delta_3,\, \delta_2-\delta_1)\}, \\ S_4 = {} & d_3\{F(d_1+d_3,\, d_1) - F(d_1,\, d_1-d_3)\} \\ & + \delta_3\{F(\delta_2,\, \delta_2+\delta_3) - F(\delta_2-\delta_3,\, \delta_2)\}. \end{aligned}$$

When each d_i, δ_i is replaced by the corresponding d_i', δ_i' the resulting expressions are denoted by S_1', S_2', S_3', S_4' respectively, and when $F(x, y)$ is replaced by $\phi(x, y)$, to be defined later, they are likewise denoted by $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$, $\bar{S}_4$.

The symbol $\sum$ indicates that the expression S_i, S_i' which it precedes is to be summed over integer solutions of n in the form (1), (2) respectively. Any restrictions placed on these solutions are indicated in a square bracket immediately following the summand.

3. **Fundamental formulas.** It is necessary to derive two fundamental formulas.

Because of the one-one correspondences between (1) and (2) given by

$$(4)\quad d_1 = d_1',\ \delta_1 = \delta_1',\ d_2 = d_2',\ \delta_2 = \delta_2',\ d_3 = d_3',\ \delta_3 = \delta_3',$$

$$(5)\quad d_1' = d_1,\ \delta_1' = \delta_1,\ d_2' = d_2,\ \delta_2' = \delta_2+\delta_3,\ d_3' = d_3-d_2,\ \delta_3' = \delta_3,$$

it is clear that

$$(6)\quad \begin{aligned} & \sum S_1[d_2 \equiv \delta_3 \pmod 2,\ \delta_2 \equiv d_3 \pmod 2] \\ & \quad = \sum S_1'\,[d_2' \equiv \delta_3' \pmod 2,\ \delta_2' \equiv d_3' \pmod 2] \\ & \quad = \sum S_2[d_2 \equiv \delta_3 \pmod 2,\ \delta_2 \equiv d_3 \pmod 2, \\ & \qquad\qquad\qquad\qquad\qquad d_3 > d_2,\ \delta_2+\delta_3 > 0], \end{aligned}$$

where, in the last sum, δ_2 is not restricted to positive values.

As a consequence of (3), S_2 in the final sum of (6) is invariant for $\delta_2>0$, when d_2, δ_2, d_3, δ_3 are replaced by d_3, δ_3, d_2, δ_2 respectively. Hence

$$\begin{aligned} & \sum S_2[d_2 \equiv \delta_3 \pmod 2,\ \delta_2 \equiv d_3 \pmod 2,\ d_3 > d_2,\ \delta_2+\delta_3 > 0,\ \delta_2 > 0] \\ & \quad = \sum S_2[d_2 \equiv \delta_3 \pmod 2,\ \delta_2 \equiv d_3 \pmod 2,\ d_2 > d_3,\ \delta_2+\delta_3 > 0,\ \delta_3 > 0] \\ & \quad = 2^{-1} \sum \{S_2[d_2 \equiv \delta_3 \pmod 2,\ \delta_2 \equiv d_3 \pmod 2] \\ & \qquad\qquad - S_2[d_2 \equiv \delta_2 \equiv d_3 \equiv \delta_3 \pmod 2,\ d_2 = d_3]\}. \end{aligned}$$

Similarly, S_2 in the final sum of (6) for $\delta_2<0$ goes into $-S_2$, with no change in the restrictions, when d_1, δ_1, d_2, δ_2, d_3, δ_3 are replaced by δ_1, d_1, $-\delta_2$, $-d_2$, δ_3, d_3 respectively. For $\delta_2<0$, the final sum in (6) is therefore zero. Hence it follows from (6) that

$$
\begin{aligned}
&2\sum S_1[d_2 \equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 \equiv d_3 \ (\mathrm{mod}\ 2)] \\
(7)\qquad &\quad = \sum \{S_2[d_2 \equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 \equiv d_3 \ (\mathrm{mod}\ 2)] \\
&\qquad - S_2[d_2 \equiv \delta_2 \equiv d_3 \equiv \delta_3 \ (\mathrm{mod}\ 2),\ d_2 = d_3] \\
&\qquad + 2S_2[d_2 \equiv \delta_3 \ (\mathrm{mod}\ 2),\ d_3 \equiv 0 \ (\mathrm{mod}\ 2),\ d_3 > d_2,\ \delta_2 = 0]\}.
\end{aligned}
$$

This is the first fundamental formula.

As a consequence of the correspondences (4) and

$$(8)\quad d_1' = d_1 + d_2,\ \delta_1' = \delta_1,\ d_2' = d_2,\ \delta_2' = \delta_2 - \delta_1,\ d_3' = d_3,\ \delta_3' = \delta_3,$$

it is evident that

$$
\begin{aligned}
&\sum S_4[d_1 \not\equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 \not\equiv d_3 \ (\mathrm{mod}\ 2)] \\
(9)\qquad &\quad = \sum S_4' [d_1' \not\equiv \delta_3' \ (\mathrm{mod}\ 2),\ \delta_2' \not\equiv d_3' \ (\mathrm{mod}\ 2)] \\
&\quad = \sum S_3[d_1 + d_2 \not\equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 - \delta_1 \not\equiv d_3 \ (\mathrm{mod}\ 2),\ \delta_2 > \delta_1, \\
&\qquad\qquad d_1 + d_2 > 0],
\end{aligned}
$$

where, in the last sum, d_1 is not restricted to positive values. As in the case of (6), S_3 in the final sum of (9) is invariant for $d_1>0$, except that $\delta_2>\delta_1$ becomes $\delta_1>\delta_2$, when d_1, δ_1, d_2, δ_2 are replaced by d_2, δ_2, d_1, δ_1, respectively; and for $d_1<0$, S_3 goes into $-S_3$ when d_1, δ_1, d_2, δ_2, d_3, δ_3 are replaced by $-\delta_1$, $-d_1$, δ_2, d_2, δ_3, d_3 respectively. Because of the form of the expression S_4 it is also obvious that the first sum in (9) is zero. From these results we obtain the second fundamental formula

$$
\begin{aligned}
&\sum S_3[d_1 + d_2 \not\equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 - \delta_1 \not\equiv d_3 \ (\mathrm{mod}\ 2)] \\
(10)\qquad &\quad = \sum \{S_3[\delta_3 \not\equiv d_1 + d_2 \ (\mathrm{mod}\ 2),\ d_3 \not\equiv 0 \ (\mathrm{mod}\ 2),\ \delta_1 = \delta_2] \\
&\qquad - 2S_3[d_2 \not\equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 - \delta_1 \not\equiv d_3 \ (\mathrm{mod}\ 2),\ \delta_2 > \delta_1,\ d_1 = 0]\}.
\end{aligned}
$$

4. **Formula** (Q). In each of (7) and (10) let us choose $n=m$ (odd), $F(x, y)=\phi(x, y)$ for $x\equiv y\equiv 0$ (mod 2), and $F(x, y)=0$ otherwise. It is then clear that $\phi(x, y)$ is subject to the same conditions as $F(x, y)$ in (3) and, after slight changes in notation, that the left members of (7) and (10) are equal. Elimination of this common sum gives

$$
\begin{aligned}
&\sum \bar{S}_2[d_i \equiv \delta_i \equiv 1 \ (\mathrm{mod}\ 2)\ (i = 1, 2, 3)] \\
(11)\qquad &\quad = \sum \{\bar{S}_2[d_i \equiv \delta_i \equiv 1 \ (\mathrm{mod}\ 2)\ (i = 1, 2, 3),\ d_2 = d_3] \\
&\qquad - 2\bar{S}_3[d_2 \not\equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 - \delta_1 \not\equiv d_3 \ (\mathrm{mod}\ 2), \\
&\qquad\qquad \delta_2 > \delta_1,\ d_1 = 0]\}.
\end{aligned}
$$

In (11) and later expressions, it is to be remembered that x and y in $\phi(x, y)$ are both even integers.

It will now be proved that the sum in the right member of (11) is zero. On placing $\Delta_2=\delta_2+\delta_3$ we have

$$\begin{aligned}\sum \bar{S}_2[d_i \equiv \delta_i \equiv 1 \ (\mathrm{mod}\ 2)\ (i = 1, 2, 3),\ d_2 = d_3]\\ = 2\sum{}'\sum{}_J\Delta_2\phi(\delta_1 - J,\ \delta_1 + \Delta_2 - J)[d_1 \equiv \delta_1 \equiv d_2\\ \equiv J \equiv 1 \not\equiv \Delta_2 \ (\mathrm{mod}\ 2),\ \Delta_2 > J],\end{aligned}$$

where $\sum_J$ indicates the summation over all positive odd integers J, in this case less than Δ_2, and $\sum'$ indicates the summation over positive integral solutions $(d_1, \delta_1, d_2, \Delta_2)$ for m in the form $m=d_1\delta_1+d_2\Delta_2$, with $d_1\equiv\delta_1\equiv d_2\not\equiv\Delta_2\equiv 0$ (mod. 2). Now

$$\begin{aligned}&2\sum{}'\sum{}_J\Delta_2\phi(\delta_1 - J,\ \delta_1 + \Delta_2 - J)[\Delta_2 > J]\\ &\quad= 2\sum{}'\sum{}_J\{\Delta_2\phi(\delta_1 - J,\ \delta_1 + \Delta_2 - J)\\ (12)\qquad &\qquad\qquad - \Delta_2\phi(\delta_1 - J,\ \delta_1 + \Delta_2 - J)[\Delta_2 < J]\}\\ &\quad= 2\sum{}'\sum{}_J\Delta_2\{\phi(\delta_1 - J,\ \delta_1 + \Delta_2 - J)\\ &\qquad\qquad - \phi(\delta_1 - \Delta_2 - J,\ \delta_1 - J)\},\end{aligned}$$

the last term being obtained from that for which Δ_2 is less than J by first replacing J by Δ_2+k and then changing k into J.

In case $\delta_1>J$ the final sum in (12) is equal to

$$2\sum \bar{S}_2[d_2 \not\equiv \delta_3 \ (\mathrm{mod}\ 2),\ \delta_2 - \delta_1 \not\equiv d_3 \ (\mathrm{mod}\ 2),\ \delta_2 > \delta_1,\ d_1 = 0].$$

In case $\delta_1<J$ let us replace J by δ_1+2k, k a positive integer. Then the final sum in (12) becomes

$$\begin{aligned}(13)\qquad &2\sum{}'\sum{}_k\Delta_2\{\phi(-2k,\ \Delta_2 - 2k) - \phi(-\Delta_2 - 2k,\ -2k)\}\\ &\quad= 2\sum{}'\sum{}_k\Delta_2\{\phi(2k,\ \Delta_2 + 2k) + \phi(\Delta_2 - 2k,\ 2k)\}.\end{aligned}$$

Now, for $\Delta_2>2k$ and hence for $2k=\Delta_2-2i$, i a positive integer,

$$\begin{aligned}\sum{}'\sum{}_k\Delta_2\phi(\Delta_2 - 2k,\ 2k)[\Delta_2 > 2k]\\ = -\sum{}'\sum{}_i\Delta_2\phi(\Delta_2 - 2i,\ 2i)[\Delta_2 > 2i],\end{aligned}$$

so that

$$\sum{}'\sum{}_k\Delta_2\phi(\Delta_2 - 2k,\ 2k)[\Delta_2 > 2k] = 0.$$

On the other hand, for $\Delta_2<2k$ and hence for $2k=\Delta_2+2i$,

$$\begin{aligned}\sum{}'\sum{}_k\Delta_2\phi(\Delta_2 - 2k,\ 2k)[\Delta_2 < 2k]\\ = \sum{}'\sum{}_i\Delta_2\phi(-2i,\ \Delta_2 + 2i)[i > 0]\\ = -\sum{}'\sum{}_k\Delta_2\phi(2k,\ \Delta_2 + 2k)[k > 0].\end{aligned}$$

Since the right-hand sum in (13) is thus zero, it follows from (11) that

$$\sum \bar{S}_2[d_i \equiv \delta_i \equiv 1 \pmod 2)\ (i = 1, 2, 3)] = 0. \tag{14}$$

As a consequence of (3), formula (14) reduces to

$$\begin{aligned}\sum d_3\{\phi(d_1 + d_3, d_1 + d_2) + \phi(d_1 - d_3, d_1 + d_2)\}\\ = \sum d_3\{\phi(d_1 + d_3, d_1 - d_2) + \phi(d_1 - d_3, d_1 - d_2)\},\end{aligned}$$

where the sums are taken over all solutions of m in the form $d_1\delta_1 + d_2\delta_2 + d_3\delta_3$ with $d_i \equiv \delta_i \equiv 1 \pmod 2$ for $i = 1, 2, 3$, and where $\phi(x, y)$ satisfies the same conditions as $F(x, y)$ in (3). This is Liouville's (Q).

5. A more general formula. Formula (Q) is only one of several arithmetical identities which we have established. Each of (7) and (10), for example, is also an identity of the Liouville type. Further formulas are obtained from (7) and (10) by deleting the congruence relations from the restrictions in the square brackets.

In §2 let us re-define the notation by placing

$$\begin{aligned}S_1 &= F(d_3, d_1 + d_2 + d_3, d_1 + d_2) - F(d_3, d_1 - d_2 - d_3, d_1 - d_2)\\ &\quad + F(\delta_2, \delta_1 - \delta_3, \delta_1 + \delta_2 - \delta_3) - F(\delta_2, \delta_1 + \delta_3, \delta_1 - \delta_2 + \delta_3),\\ S_2 &= F(d_3 - d_2, d_1 + d_3, d_1 + d_2) - F(d_3 - d_2, d_1 - d_3, d_1 - d_2)\\ &\quad + F(\delta_3 + \delta_2, \delta_1 - \delta_3, \delta_1 + \delta_2) - F(\delta_2 + \delta_3, \delta_1 + \delta_3, \delta_1 - \delta_2),\\ S_3 &= F(d_3, d_1 + d_2 + d_3, d_1 + d_2) - F(d_3, d_1 + d_2, d_1 + d_2 - d_3)\\ &\quad + F(\delta_3, \delta_2 - \delta_1, \delta_2 - \delta_1 + \delta_3) - F(\delta_3, \delta_2 - \delta_1 - \delta_3, \delta_2 - \delta_1),\\ S_4 &= F(d_3, d_1 + d_3, d_1) - F(d_3, d_1, d_1 - d_3)\\ &\quad + F(\delta_3, \delta_2, \delta_2 + \delta_3) - F(\delta_3, \delta_2 - \delta_3, \delta_2),\end{aligned}$$

where

$$\begin{aligned}F(w, -x, y) &= -F(w, x, y) = F(w, x, -y)\\ &= F(w, y, x) = F(-w, x, y),\\ F(0, x, y) &= 0 = F(w, 0, y) = F(w, x, 0).\end{aligned} \tag{15}$$

It is not difficult to see that (7) and (10) still hold. If, further, we let $n = m$ (odd) in each of these formulas, choose $F(w, x, y) = \psi(w, x, y)$ for $w \equiv x \equiv y \equiv 0 \pmod 2$ and $F(w, x, y) = 0$ otherwise, and follow the procedure outlined in §4, we obtain the more general formula

$$\begin{aligned}&\sum \{\psi(d_3 - d_2, d_1 + d_3, d_1 + d_2) + \psi(d_3 + d_2, d_1 - d_3, d_1 + d_2)\}\\ &= \sum \{\psi(d_3 + d_2, d_1 + d_3, d_1 - d_2) + \psi(d_3 - d_2, d_1 - d_3, d_1 - d_2)\},\end{aligned} \tag{16}$$

where $\psi(w, x, y)$ is subject to the same conditions as $F(w, x, y)$ in (15).

Liouville's (Q) is a special case of (15) for the choice $\psi(w, x, y) = w\phi(x, y)$.

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SET FUNCTIONS AND SOUSLIN'S HYPOTHESIS

DOROTHY MAHARAM

1. **Introduction.** It is known[1] that Souslin's hypothesis[2] is *implied* by the existence of a nontrivial outer measure on every field of sets satisfying certain conditions. We shall here prove that Souslin's hypothesis is *equivalent* to the existence, on a wide class of fields of sets, of set-functions of a certain type. The axiom of choice is assumed, but not the continuum hypothesis.

Instead of working with fields of sets, it is more convenient to use the equivalent notion of a (finitely additive) Boolean algebra, E.[3] We say that $x, y \in E$ are *disjoint* if $x \wedge y = o$, and that they *intersect* otherwise. A set S of elements of E will be called a *Souslin system* if it satisfies the following three postulates:

(1) $S \not\ni o$, and whenever $s, s' \in S$, then either $s \wedge s' = o$, or $s \geqq s'$, or $s' \geqq s$.

(2) If $A \subset S$ consists of pairwise disjoint elements only, then A is (at most) countable.

(3) If $A \subset S$ is such that every two of its elements intersect, then A is countable.

Souslin's hypothesis is known to be equivalent to the assertion that every Souslin system is countable.[4]

THEOREM. *Souslin's hypothesis is true if and only if there exists, on each non-atomic Boolean algebra E satisfying the countable chain condition, a real-valued function f such that* (i) $x \geqq y \rightarrow f(x) \geqq f(y)$, (ii) $f(x) = 0 \leftrightarrow x = o$, *and* (iii) *given* $x \in E-(o)$ *and* $\epsilon > 0$, *there exists* $y \in E-(o)$ such that $y < x$ and $f(y) < \epsilon$.

2. "If." Suppose an uncountable Souslin system exists. Then, as easily follows from [2, §7], there exists a complete Boolean algebra E, satisfying the countable chain condition, and an uncountable Souslin system $S \subset E$ having the following additional properties:

(4) $S = \bigcup S_\alpha$, where α ranges over all countable ordinals, and the elements of each S_α are pairwise disjoint.

(5) If $\alpha < \beta$, then for each $s_\beta \in S_\beta$ there exists an s_α ($\in S_\alpha$) such that $s_\alpha > s_\beta$.

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[1] See [2]; in the case of a measure, the result is due to K. Gödel. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Souslin, Fund. Math. vol. 1 (1920) p. 223.

[3] See [2] for notations, and so on.

[4] This follows from [3], together with some results in [1].

(6) If $\alpha<\beta$, $s_\alpha = \bigvee\{s_\beta | s_\beta < s_\alpha\}$.

(7) For each $x\in E$, there exists an α such that

$$x = \bigvee\{s_\alpha | s_\alpha \leqq x\}.$$

(In the notation of [2], we have only to take $E=(D+_sN)/N$, and the elements s_α are the equivalence classes c_α mod N. The notation s_α is intended to imply that $s_\alpha \in S_\alpha$, and so on.)

Clearly E is non-atomic, so by hypothesis there exists on E a real-valued function f having the properties (i)–(iii) of the theorem.

Now, for a given positive integer n, each $s\in S$ for which $f(s)<1/n$ is contained in a *maximal* such element of S, say $m(s, n)$; in fact, if $s=s_\beta$, we have that for each $\alpha \leqq \beta$ there is a (unique) $s_\alpha \geqq s_\beta$, and we take $m(s, n)=s_\alpha$ for the smallest α for which $f(s_\alpha)<1/n$. Let M_n denote the set of all elements $m(s, n)$ (for fixed n); clearly the elements of M_n are pairwise disjoint, so that each M_n is countable. The desired contradiction is now obtained by showing that S is countable after all; and this will follow (in virtue of (3)) once it is established that:

(8) Each $s\in S$ is greater than or equal to some $m(s', n)$.

But, given s, we have from (ii) that $f(s)>1/n$ for some n. By (iii), there is an $x\in E$ such that $o<x<s$ and $f(x)<1/n$. From (7) there exists an $s'\in S$ such that $s'\leqq x$; hence $m(s', n)$ exists and is greater than or equal to s'. Now $m(s', n)$ and s are not disjoint (for both are greater than or equal to s'); and $m(s', n)\not\geq s$, for $f(s)>1/n>f(m(s', n))$. Hence $s\geqq m(s', n)$, by (1).

3. "**Only if.**" Let E be a non-atomic Boolean algebra satisfying the countable chain condition. By Zorn's lemma (or transfinite induction) there exists a maximal subset $S\subset E$ satisfying (1); and the countable chain condition ensures that (2) and (3) hold also. Hence, by Souslin's hypothesis, S can be enumerated as $\{s_n\}$ ($n=1, 2, \cdots$ to ∞; it is easy to see that S is necessarily infinite), where for convenience we may suppose that the unit element e (which necessarily belongs to S) is s_1. We assert:

(9) Given s_k, there exists $s_m<s_k$.

For, since E is non-atomic, there exists $x\in E$ such that $o<x<s_k$. If $x\in S$, there is nothing to prove. If not, since S is maximal, there must be an s_m such that $s_m\wedge x\neq o$, and s_m and x are incomparable. But then s_m intersects s_k, so either $s_k\leqq s_m$—which implies $x\leqq s_m$ and so is excluded—or $s_k>s_m$, q.e.d.

Let ϵ_i denote either 1 or -1, and write $\epsilon_i s_i$ to denote s_i if $\epsilon_i=1$, and the complement $-s_i$ if $\epsilon_i=-1$. For each finite sequence, $\epsilon_1\epsilon_2\cdots\epsilon_n$ of ± 1's, we write $\bigwedge_1^n(\epsilon_i s_i)=t(\epsilon_1\epsilon_2\cdots\epsilon_n)$.

Now let $\{t^i\}$ be any infinite sequence of elements $t^i = t(\epsilon_1^i \epsilon_2^i \cdots \epsilon_{n(i)}^i)$ such that $1 \leqq n(1) < n(2) < \cdots$. We shall show that:

(10) If $a \in E$ is such that a is less than or equal to each t^i, then $a = o$.

For suppose $a \neq o$. Then clearly ϵ_j^i is independent of i (provided only that $n(i) \geqq j$), so that we may write $\epsilon_j^i = \epsilon_j$, and have that, for every j,

$$\epsilon_j s_j \geqq a. \tag{11}$$

Hence $a \in S$ (else a could be adjoined to S without violating (1), and S is maximal); say $a = s_k$. From (9), $s_k > s_m$ for some m. But (11) gives $\epsilon_m s_m \geqq s_k$—a contradiction.

Now, given any $x \in E-(o)$, (10) shows that there will be a greatest n, say $n(x)$, for which there exists an element $t(\epsilon_1 \epsilon_2 \cdots \epsilon_n) \geqq x$. (Note that $t(1) = e$, so that $n(x)$ is always defined.) We put $f(x) = 1/n(x)$, and complete the definition by setting $f(o) = 0$. Properties (i) and (ii) are immediate. To verify (iii), suppose that $x \in E-(o)$ and $\epsilon > 0$ are given. Choose $n > \max(1/\epsilon, 1/f(x))$, and consider the 2^n (not necessarily distinct) elements $t(\epsilon_1 \epsilon_2 \cdots \epsilon_n)$ for all possible choices of $\epsilon_i = \pm 1$. The $\bigvee$ of these elements is e, so at least one of them, say t, intersects x. Write $y = t \wedge x$; thus $o < y \leqq x$, and $n(y) \geqq n$, so that $f(y) < \min(\epsilon, f(x))$, and (iii) is established.

4. Further remarks. (a) Let E be a non-atomic Boolean algebra satisfying the countable chain condition. It does *not* follow that the existence of a function f on E *alone*, satisfying conditions (i)–(iii) of the theorem, implies that every Souslin system *in* E is countable. In fact, this is *false*—unless Souslin's hypothesis is true. For if Souslin's hypothesis is false, there will be a Boolean algebra E_1 satisfying our conditions and containing an uncountable Souslin system S. (Cf. §2.) Let E_2 be (say) the algebra of measurable sets modulo null sets on the unit interval. We can regard E_1 and E_2 as the algebras of open-closed subsets of their respective representation spaces R_1 and R_2. The "product" algebra $E = E_1 \times E_2$ can now be defined to consist of all finite unions of open-closed "rectangles" in the topological product $R_1 \times R_2$. For each $x \in E$, say $x = \bigcup_1^n (x_1^i \times x_2^i)$ $(x_2^i \neq o)$ we define $f(x) =$ measure of $\bigcup x_1^i$. It is easy to see that E and f fulfil all the requirements. Yet E contains an uncountable Souslin system S^*, formed by the cylinder sets on S.

It can however be shown that the countability of every Souslin system in E (where E is, as hitherto, non-atomic and satisfies the countable chain condition) is equivalent to the existence, *for every non-atomic subalgebra F of E,* of a function f (depending on F, in general), defined on F and satisfying postulates (i)–(iii) for F. In one direction this is an immediate consequence of §3; the proof of the other implication, while using the same ideas as §2, is more complicated.

(b) The arguments of §§2 and 3 also readily give purely algebraic properties equivalent to Souslin's hypothesis. We have, for example:

Souslin's hypothesis is true if and only if each non-atomic Boolean σ-algebra satisfying the countable chain condition contains a double sequence of elements t_{ni} such that (α) $\bigvee_i t_{ni} = e$ *and* (β) *for every function $i(n)$ of n,* $\bigwedge_n t_{ni(n)} = o$.

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ON UNIVERSAL MAPPINGS AND FREE TOPOLOGICAL GROUPS

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It has been observed[1] that constructions so apparently different as Kronecker products, extension of the ring of operators of a module, field of quotients of an integral domain, free groups, free topological groups, completion of a uniform space, Čech compactification enter in the same frame. We intend in this paper to explain a rather general process of construction which may be applied to most of the examples quoted above.

This paper will proceed axiomatically. In fact the problem under question (problem of a "universal mapping") can be only stated after a certain number of axioms. When the method of construction has been explained we shall illustrate it by the classical example of the completion of a uniform space. For more examples the reader is referred to a forthcoming book of N. Bourbaki. The same method gives also necessary and sufficient conditions for many imbedding problems. Both topological and algebraic examples will be given. In the last part of the paper our method of construction will be applied to Markoff's theory of free topological groups.[2]

1. **Problems of universal mappings.** Given a set E it is possible to define on it certain kinds of structures, that is structure of ring, field, topological space.[3] We shall denote by S or T certain kinds of structures. A set with a structure T will be called a *T-set*: if T is the structure of group the T-sets are the groups. An isomorphism for the structure T will be called a T-isomorphism:

T-mappings. Induced structures. Given a kind of structure T it happens very often that, for every pair E_1E_2 of T-sets, there has been defined a family of mappings of E_1 into E_2 satisfying the following axioms:

A$_1$. *Every T-isomorphism is a T-mapping.*

A$_2$. *If $f:E_1\rightarrow E_2$ and $g:E_2\rightarrow E_3$ are T-mappings, then the composite*

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[1] Unpublished manuscripts of N. Bourbaki.

[2] Markoff, Bull. Acad. Sci. USSR. vol. 9 (1945) pp. 3–64.

[3] For precise definitions of the words "structure," "kind of structure," "isomorphism" see N. Bourbaki, *Theorie des ensembles* (*Résultats*), Part 10, Paris, Herrmann, 1939.

mapping $g \circ f: E_1 \to E_3$ *is a* T*-mapping.*

A$_3$. *A necessary and sufficient condition for a one-to-one mapping* f *of* E_1 *onto* E_2 *to be a* T*-isomorphism is that* f *and* f^{-1} *be* T*-mappings.*

EXAMPLE. If T is the structure of group the T-mappings are the homomorphisms; if T is the structure of topological space the T-mappings are the continuous ones.

Let now σ and σ' be two structures T defined on E and $E' \subset E$ respectively. We shall say that σ' is *induced* by σ when:

I$_1$. *The injection of* E' *into* E *is a* T*-mapping.*

I$_2$. *If* $f: F \to E$ *is a* T*-mapping and if* $f(F) \subset E'$, *then* f *considered as mapping of* F *into* E' *is a* T*-mapping.*

I$_1$, I$_2$ and A$_3$ show immediately the *uniqueness* of the induced structure.

If $E' \subset E$ is capable of an induced structure we shall say that E' is T*-closed.* We suppose that the following axioms hold:

S$_1$. *A subset of* E *composed of all the elements where a family of* T*-mappings takes the same value is* T*-closed.*

S$_2$. *Any intersection of* T*-closed sets is* T*-closed.*

S$_2$ is a consequence of S$_1$ if we allow mappings which are not everywhere defined: The notion of T-closed set is obviously transitive. S$_2$ allows us to define the T*-closure* $\overline{E}'$ of a subset $E' \subset E$ as the intersection of all T-closed subsets containing E'. We shall suppose:

S$_3$. Cardinal $(\overline{E}') \leqq$ certain function of cardinal (E'), a function which depends only on the structure T.

In most cases the function "$2^{2^{\mathrm{card}(E')}}$" will be sufficient.

Axioms for the cartesian products. In many important cases it is possible, given a family (E_α) of T-sets, to define on the cartesian product $\prod_\alpha E_\alpha$ a structure T which satisfies the following conditions:

P$_1$. *The projections (on the components) are* T*-mappings.*

P$_2$. *If the* $f_\alpha: E \to E_\alpha$ *are* T*-mappings, the product mapping* $f: E \to \prod_\alpha E_\alpha$ *(defined by* $f(x) = (f_\alpha(x))$*) is a* T*-mapping.*

As easy consequences: the projections on the partial products are T-mappings; these partial products are T-closed; the given structure of the "coordinate axis" is induced by the product structure. Applying A$_3$ we see also that the product structure is *unique.*

Remark. The example of the structure of topological space shows easily that the axioms A, S, P are independent.

Statement and solution of the problem of universal mappings. Given two kinds of structures S and T, suppose we have defined the T-mappings, and also mappings of S-sets into T-sets, called the (S-T-*mappings*, denoted by greek letters, and satisfying:

(S-T)$_1$. *The composite mapping* $f \circ \phi$ *of an* (S-T)*-mapping* ϕ *and of* *a T-mapping f is an (S-T)-mapping.*

(S-T)$_2$. *The product mapping of a family of (S-T)-mappings is an (S-T)-mapping.*

The structure T is supposed to satisfy the axioms A, S and P.

The problem we have in view ("problem of universal mappings" or "problem U") is the following: given any S-set E to find a T-set F_0 and an $(S$-$T)$-mapping ϕ_0 of E into F such that:

(U$_1$). *Every (S-T)-mapping ϕ of E into any T-set F has the form $\phi = f \circ \phi_0$ where f is a T-mapping of F_0 into F.*

It is clear that, if such an F_0 exists, the T-closure of $\phi_0(E)$ in F_0 will also satisfy (U$_1$). Therefore (S$_1$) there will exist a pair (F_0, ϕ_0) such that:

(U$_2$). *Two T-mappings of F_0 into F which coincide on $\phi_0(E)$ are identical.*

We deduce immediately from A$_2$ that a pair (F_0, ϕ_0) satisfying (U$_1$) and (U$_2$) is *uniquely determined* up to isomorphisms.

We now come to the *construction* of a pair (F_0, ϕ_0) satisfying (U$_1$). Consider the set of all $(S$-$T)$-mappings of E into all T-sets whose cardinal does not exceed the one indicated in S$_3$ ($2^{2^{\text{card}\,E}}$ in most cases). Let $\{\phi_\alpha\}$ be this set, ϕ_α mapping E into F_α. Let $F_0 = \prod_\alpha F_\alpha$, and ϕ_0 be the product mapping $x \rightarrow (\phi_\alpha(x))$ of E into F_0. F_0 is a T-set (P), and ϕ_0 an (S-T)-mapping ($(S$-$T)_2$). Let ϕ be an $(S$-T-)mapping of E into F, F' the T-closure of $\phi(E)$ in F, i the injection of F' in F. By S$_3$, I$_2$ and (S-T) the contraction ϕ' of ϕ, mapping of E into F, is among the ϕ_α, say ϕ_{α_0}. Let p_0 be the projection of $\prod_\alpha F_\alpha$ onto F_{α_0}. We may write $\phi = i \circ \phi' = i \circ \phi_{\alpha_0} = i \circ p_0 \circ \phi_0$. Since $i \circ p_0$ is a T-mapping of $\prod_\alpha F_\alpha$ into F, the pair (F_0, ϕ_0) satisfies (U$_1$). Q.E.D.

Example. The preceding construction applies to all the examples quoted in the introduction except the field of quotients of an integral domain (a product of fields being not a field). For these examples the reader is referred to a forthcoming book of N. Bourbaki.

We shall give only the example of the completion of a uniform space.[4] S is the structure of separated uniform space, T the structure of complete separated uniform space. The T- and $(S$-$T)$-mappings are the uniformly continuous ones. All our axioms are verified. Therefore, given a separated uniform space E, the preceding construction provides with a complete space F_0 and a uniformly continuous mapping ϕ_0 of E into F such that every uniformly continuous mapping of E is "induced" by a uniformly continuous mapping of F_0.

We shall prove that, in this case, ϕ_0 is a uniform structural isomorphism of E onto $\phi_0(E)$. In fact consider the family $\mathfrak{C}$ of all uniformly continuous distances (not necessarily satisfying the separation axiom "$d(x, y)=0$ implies $x=y$") on E. Since the real line is a complete space the function f defined by fixing one of the arguments in the distance d is among the f_α. Since the uniform structure of E may be defined by the family $\mathfrak{E}$,[3] we see immediately—denoting by $\{f_\beta\}$ the subset of $\{f_\alpha\}$ composed of all mappings deduced from distances, and by π the projection of $\prod_\alpha F_\alpha$ onto $\prod_\beta F_\beta$—that $\pi \circ \phi_0$ is an isomorphism. Hence ϕ_0 is one-to-one and $\phi_0^{-1}=(\pi \circ \phi_0)^{-1} \circ \pi$ is uniformly continuous. Hence ϕ_0 is an isomorphism.

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[4] N. Bourbaki, *Topologie générale*, chap. 2, Paris, Hermann, 1940.

By definition of the T-closure (smallest complete, that is, closed subspace), we see that $\phi_0(E)$ is dense in F. We have therefore proved the existence and the uniqueness of the completion of a uniform space (provided we have defined the real line without completion, for example by the cuts process). If E is not separated one verifies easily that $\phi_0(E)$ is the associated separated space.

2. **Imbedding problems.** It often happens that the structure T is "richer" than the structure S, that is, that there exists a canonical process for giving a structure S to a T-set. In an example where $\phi_0(E)$ is, in the S-set F_0, capable of an induced structure S, arises, with the problem (U), an "imbedding problem": may we consider E as a subset of a set F, subset whose structure S is induced by the structure S of F canonically deduced from a structure T? We shall suppose that a T-mapping is also an S-mapping for the deduced structures S. As a consequence (A_3 for S) the operations "induced structure" and "deduced structure" *commute*.

If $\phi_0(E)$ is capable of an induced structure S, and if ϕ_0 is an S-isomorphism, the imbedding problem is solved by (F_0, ϕ_0). Let conversely (F, ϕ) be a solution of the imbedding problem, ϕ being an $(S\text{-}T)$-mapping. We can write $\phi=f \circ \phi_0$, f being a T-mapping of F_0 into F. Since ϕ is an S-isomorphism, and since $f|\phi_0(E)$ and ϕ_0 are S-mappings, it follows from A_2 and A_3 (for S) that $f|\phi_0(E)$ and ϕ_0 are S-isomorphisms. Therefore (F_0, ϕ_0) gives also a solution of the imbedding problem.

We may therefore consider F as a set of equivalence classes in F_0, the sets $\{a\}$ $(a \in \phi_0(E))$ being equivalence classes. The fact that this identification is not always trivial (and that the imbedding problem does not admit a unique solution) is shown by the example of the compactifications of a uniform space (F_0 being the Čech compactification). Therefore: *A necessary and sufficient condition for the imbedding problem of E to be possible is that ϕ_0 be an S-isomorphism.*

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[3] N. Bourbaki, *Topologie générale*, chap. 9, part 1, Theorem 1, 1948.

EXAMPLES. (1) *Characterization of uniformizable spaces.*[6] The structures S and T are the structures of topological space and of compact space respectively. The S-, T-, (S-T)-mappings are the continuous ones. Let $\mathfrak{T}$ be the given topology on E, $\mathfrak{T}'$ the one induced on $\phi_0(E)$ by the product space topology. $\mathfrak{T}'$ and $\mathfrak{T}_0 = \phi_0^{-1}(\mathfrak{T}')$ are uniformizable. In general $\mathfrak{T}$ is finer than $\mathfrak{T}_0$. A necessary and sufficient condition for $\mathfrak{T}$ to be uniformizable is that $\mathfrak{T} = \mathfrak{T}_0$. Then F_s is the Čech compactification of E. Using the fact that every compact space may be imbedded in some, finite or transfinite, cube, one shows easily that the condition "$\mathfrak{T} = \mathfrak{T}_0$" is equivalent with the complete regularity of E.

(2) *Alexandroff's T_1-space.*[7] S is the structure of topological space, T the structure of T_1-space, the S-, T-, (S-T)-mappings are the continuous ones. Then $\phi_0(E)$ is the "greatest possible" identification space of E which is a T_1-space.

Remark. It may happen, in some cases where our axioms are not all fulfilled (in particular S_2 relative to intersections of T-closed sets), that one can however construct ϕ_0, while F_s is "too big."

(1). Let S be the structure of ring, T the structure of semi-simple ring. The S-, T-, (S-T)-mappings are the homomorphisms *into*. In this case $\phi_0^{-1}(\{0\})$ is the *extension radical* $\mathfrak{R}$ of the ring E.[8] If $\mathfrak{R} = 0$, E may be imbedded in a semi-simple ring and conversely. The structure of $\mathfrak{R}$ may be obtained by studying a sufficient number of homomorphisms of E into semi-simple rings. Let $\mathfrak{a}$ be the two-sided ideal $T \cap \bigcap_p pE$ (T: ideal of the elements of finite order in the additive group of E; p: prime number). Every semi-simple ring being contained in a product of full matrix rings over *s*-fields, it is clear that $\phi(\mathfrak{a}) = 0$ for every homomorphism ϕ of E into a semi-simple ring. Hence $\mathfrak{a} \subset \mathfrak{R}$. On the other hand the rings E/pE and the Kronecker product $E/T \otimes Q$ of E/T by the rational field Q are algebras over the prime fields. But every algebra is a subalgebra of a full matrix algebra (adjoin a unit element, and consider the left regular representation). Hence $\mathfrak{R} \subset pE$, $\mathfrak{R} \subset T$. Therefore $\mathfrak{R} = \mathfrak{a}$.

(2) If we restrict the (S-T)-mappings to be the homomorphisms *onto* semi simple rings, it is easily seen that $\phi_0^{-1}(\{0\})$ is the *radical* of E.[9]

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[6] P. Samuel, *Ultrafilters and compactification of uniform spaces*, Princeton thesis, 1947.

[7] P. Alexandroff, *Bikompakte Erweiterungen von Räumen*, Rec. Math. (Mat. Sbornik) N.S. (1939).

[8] O. Goldman. *Semi-simple extensions of rings*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1028–1032.

[9] N. Jacobson, *Radical and semi simplicity for arbitrary rings*, Amer. J. Math. vol. 67 (1945) pp. 300–320.

3. Free topological groups. As a more elaborate example of our general method of construction, we shall give a sketchy treatment of Markoff's theory of free topological groups. Only very easy proofs have been omitted.

S is the structure of topological space, T the structure of separated topological group (abelian, or precompact, if desired). The (S-T)-mappings are the continuous ones, the T-mappings are the continuous homomorphisms. All the axioms A, S, P are satisfied. Applying the general construction to the topological space E, we get for F_0 a separated topological group (abelian, or precompact, if desired) denoted $G(E)$, ($GA(E)$, $GC(E)$). By construction $G(E)$ is such that:

(1) *For every continuous mapping ϕ of E into a topological group G, there exists a continuous homomorphism g of $G(E)$ into G such that $\phi = g \circ \phi_0$.*

(2) *Two continuous homomorphisms of $G(E)$ into a topological group H which coincide on $\phi_0(E)$ are identical; $\phi_0(E)$ generates $G(E)$; $G(E)$ is uniquely determined by E.*

Similar properties hold for $GA(E)$ and $GC(E)$. The uniqueness shows that $GA(E) \approx G(E)/C$, C being the closure of the commutator subgroup of $G(E)$. If E is discrete, $G(E)$ and $GA(E)$ are the classical free group and free abelian group generated by the set E.

THEOREM 1. *A necessary and sufficient condition for ϕ_0 to be a homeomorphism is that E be completely regular. The uniform structures induced on the $\phi_0(E)$ are the universal uniform structure (for $G(E)$ and $GA(E)$), and the Čech uniform structure (for $GC(E)$).*

The necessity is clear. The sufficiency comes easily from the fact that one has sufficiently many mappings continuous of E into the real line R or the unit circle T.

THEOREM 2. *If E' is a subspace of the completely regular space E, $G(E')$, $GA(E')$ and $GC(E')$ are algebraically isomorphic to subgroups of $G(E)$, $GA(E)$ and $GC(E)$, and have finer topologies.*

If we extend the injection $i: E' \rightarrow E \rightarrow G(E)$, we get a continuous homomorphism $g: G(E') \rightarrow G(E)$. g is shown to be one-to-one by proving that, given a word $\prod_i a_i^{n_i}$ ($a_i \in E'$), there exists a topological group G and a continuous mapping f of E' into G such that $\prod_i f(a_i)^{n_i} \neq I$. One may take for G the unimodular orthogonal group $O_3[R]$ which contains free groups with as many generators as desired, and which is arcwise connected. For $GA(E')$ an R^n is good enough.

As corollaries, we have:

(a) *$G(E)$ and $GC(E)$ (resp. $GA(E)$) have the algebraic structure of* *the free (resp. free abelian) group generated by the underlying set of* E.

(b) *The commutator subgroup* C *of* $G(E)$ *is closed*; $GA(E) \approx G(E)/C$.

(c) $\phi_0(E)$ *is a closed subset of* $G(E)$ (*resp.* $GA(E)$, $GC(E)$).

((c) is proved by imbedding E in a compact space.)

(d) *Every free group is algebraically isomorphic with a subgroup of a compact group.*

(e) *The topology of* $G(E)$ *is finer than the topology of* $GC(E)$.

When E is connected, the subgroup A of $G(E)$ (resp. $GA(E)$, $GC(E)$) composed of all the words of total degree zero is the connected component of the identity. If E is compact, A is a countable union of compact sets.

If $E' \subset E$, $G(E')$ is topologically isomorphic with a subgroup of $G(E)$ in the following cases (where continuous mappings of E' into sufficiently many groups can be extended to E):

(a) E' is a retract of E.

(b) E is the completion of E' for the universal uniform structure.

(c) E is normal and E' is a closed subset of E.

A counter-example is the following: E' is a discrete noncountable space, and E is its Čech compactification (consider the subgroup composed of the words of total degree zero).

The Markoff's schemes. By a ***scheme*** is meant a set $\mathfrak{S}$ of words formed from elements of E. We shall consider the continuous mappings $\{f_\gamma\}$ of E into topological groups such that $\prod_i f_\gamma(a_i)^{n_i} = 1$ for every word $\prod_i a_i^{n_i} \in \mathfrak{S}$. Such an f_γ is called a ***realization of the scheme*** $\mathfrak{S}$.

We apply the general construction to the mappings f_γ (with the usual restriction on cardinal numbers): we form the partial product $\prod_\gamma G_\gamma$ of $\prod_\alpha G_\alpha$, and construct the subgroup $G\mathfrak{S}(E)$ generated by the elements $(f_\gamma(x))$ $(x \in E)$. Let $\phi_{\mathfrak{S}}$ be the mapping $x \to (f_\gamma(x))$. It is clear that every realization of the scheme $\mathfrak{S}$ may be written $f = g \circ \phi_{\mathfrak{S}}$, where g is a continuous homomorphism of $G\mathfrak{S}(E)$. On the other hand such a pair $(G\mathfrak{S}(E), \phi_{\mathfrak{S}})$ is unique. If H denotes the closure of the invariant subgroup of $G(E)$ generated by $\mathfrak{S}$, $G\mathfrak{S}(E)$ is isomorphic with $G(E)/H$ (because of the uniqueness).

EXAMPLES. (a) $G(E)$ is defined by the empty scheme, $GA(E)$ by the set of commutators.

(b) If E has a structure of topological group, we may take for $\mathfrak{S}$ the "multiplication table" of E. A pair having the properties of $(G\mathfrak{S}(E), \phi_{\mathfrak{S}})$ being (E, identity), we deduce from the uniqueness that:

Every topological group E *is isomorphic with a factor group of the free topological group generated by the underlying topological space of* E.

(c) Taking for E the topological sum of two topological groups (with identity elements identified), and for $\mathfrak{S}$ the union of their multiplication tables, one gets the *free topological product* of the two groups.

The referee has pointed out to me that S. Kakutani (*Free topological groups and infinite direct products of topological groups*, Proc. Imp. Acad. Tokyo vol. 20 (1944) pp. 595–598) gives substantially the same proof of Markoff's theorem as I do.

Notice also that Nakayama's results (*A note on free topological groups*, Proc. Imp. Acad. Tokyo vol. 19 (1943) pp. 471–475) can be obtained by our method: his topology for $G(E)$, deduced from the continuous representations, is that of our $GC(E)$. If E is a uniform space, Nakayama's uniform free topological group is obtained by taking, as (S-T)-mappings, the uniformly continuous ones.

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